EMPossible

Computational Science:
Introduction to Finite-Difference Time-Domain

Review of Electromagnetics and Introduction to FDTD

Lecture Outline

- Review
  - Maxwell's equations
  - Physical boundary conditions
  - The constitutive relations
  - Parameter relations
  - *see Balanis Chapter 1*
- Introduction to FDTD
  - Flow of Maxwell's equations
  - Finite-difference approximations
  - The update equation
  - The FDTD algorithm…for now ☺

Review of Maxwell's Equations and Electromagnetics

GOVERNING EQUATIONS FOR CLASSICAL ELECTROMAGNETICS

Pioneering 21st Century Electromagnetics and Photonics

http://emlab.utep.edu

	Integral Form	Differential Form	Name
Time-Domain	$Q_e(t)=\oint\!\!\oint_S \vec{D}(t)\bullet d\vec{s}=\iiint_V \rho_v(t)\,dv$	$\nabla\bullet\vec{D}(t)=\rho_v(t)$	Gauss' Law
	$\oint\!\!\oint_S \vec{B}(t)\bullet d\vec{s}=0$	$\nabla\bullet\vec{B}(t)=0$	No Magnetic Charge
	$V_{\text{emf}}(t)=\oint_L \vec{E}(t)\bullet d\vec{\ell}=-\iint_S\left[\frac{\partial\vec{B}(t)}{\partial t}\right]\bullet d\vec{s}$	$\nabla\times\vec{E}(t)=-\frac{\partial\vec{B}(t)}{\partial t}$	Faraday's Law
	$I(t)=\oint_L \vec{H}(t)\bullet d\vec{\ell}=\iint_S\left[\vec{J}(t)+\frac{\partial\vec{D}(t)}{\partial t}\right]\bullet d\vec{s}$	$\nabla\times\vec{H}(t)=\vec{J}(t)+\frac{\partial\vec{D}(t)}{\partial t}$	Ampere's Circuit Law
	$\oint\!\!\oint_S \vec{J}\bullet d\vec{s}=-\frac{\partial Q_e}{\partial t}$	$\nabla\bullet\vec{J}=-\frac{\partial\rho_v}{\partial t}$	Continuity of Current
	$\vec{D}(t)=[\varepsilon(t)]*\vec{E}(t)$ $\vec{B}(t)=[\mu(t)]*\vec{H}(t)$	Electric Response Magnetic Response	Constitutive Relations
Frequency-Domain	$Q_e=\oint\!\!\oint_S \vec{D}\bullet d\vec{s}=\iiint_V \rho_v\,dv$	$\nabla\bullet\vec{D}=\rho_v$	Gauss' Law
	$\oint\!\!\oint_S \vec{B}\bullet d\vec{s}=0$	$\nabla\bullet\vec{B}=0$	No Magnetic Charge
	$V_{\text{emf}}=\oint_L \vec{E}\bullet d\vec{\ell}=-\iint_S\left[j\omega\vec{B}\right]\bullet d\vec{s}$	$\nabla\times\vec{E}=-j\omega\vec{B}$	Faraday's Law
	$I=\oint_L \vec{H}\bullet d\vec{\ell}=\iint_S\left[\vec{J}+j\omega\vec{D}\right]\bullet d\vec{s}$	$\nabla\times\vec{H}=\vec{J}+j\omega\vec{D}$	Ampere's Circuit Law
	$\oint\!\!\oint_S \vec{J}\bullet d\vec{s}=-j\omega Q_e$	$\nabla\bullet\vec{J}=-j\omega\rho_v$	Continuity of Current
	$\vec{D}=[\varepsilon]\vec{E}$ $\vec{B}=[\mu]\vec{H}$	Electric Response Magnetic Response	Constitutive Relations

Parameter Definitions

Electric Field Intensity, E (V/m)

Electric Flux Density, D (C/m^2)

Magnetic Field Intensity, H (A/m)

Magnetic Flux Density, B (Wb/m^2)

Electric Current Density, J (A/m^2)

Volume Charge Density, ρ_v (C/m^3)

Permittivity, ε (F/m)

Permeability, μ (H/m)

Electrical Conductivity, $\sigma\,(1/\Omega\cdot\text{m})$

Constants

Permittivity: $[\varepsilon]=\varepsilon_0[\varepsilon_r]$
$\varepsilon_0=8.8541878176\times10^{-12}\ (\text{F/m})$

Permeability: $[\mu]=\mu_0[\mu_r]$
$\mu_0\approx4\pi\times10^{-7}\ (\text{H/m})$
$\mu_0=1.2566370614\times10^{-6}\ (\text{H/m})$

Impedance: $\eta_0\approx120\pi\ (\Omega)$
$\eta_0=376.73031346177\ (\Omega)$

Speed of Light: $c_0=299{,}792{,}458\ (\text{m/s})$

Lorentz Force Law

$$\vec{F}=q\vec{E}+q\left(\vec{v}\times\vec{B}\right)$$

Four Field Terms in Maxwell's Equations

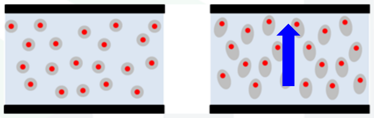
$$\nabla \times \vec{H} = \vec{J} + \partial \vec{D} / \partial t \qquad \nabla \times \vec{E} = -\partial \vec{B} / \partial t$$

Electric Field Quantities

$\vec{E}$ (V/m)

Electric field intensity
Initial electric push

+



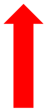
Displaced Charges

$$= \vec{D}$$

$(\mathrm{C/m^2})$

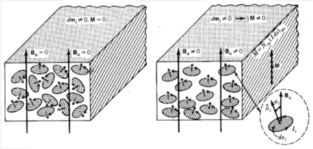
Magnetic Field Quantities

$\vec{H}$ (A/m)



Magnetic field intensity
Initial magnetic push

+



Tilted Magnetic Dipoles

$$= \vec{B}$$

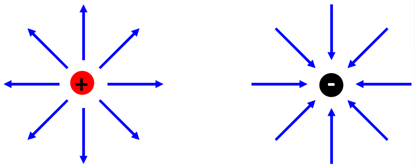
$(\mathrm{Wb/m^2})$

Gauss's Law

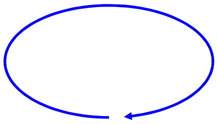
$$\boxed{\nabla \bullet \vec{D} = \rho_v}$$

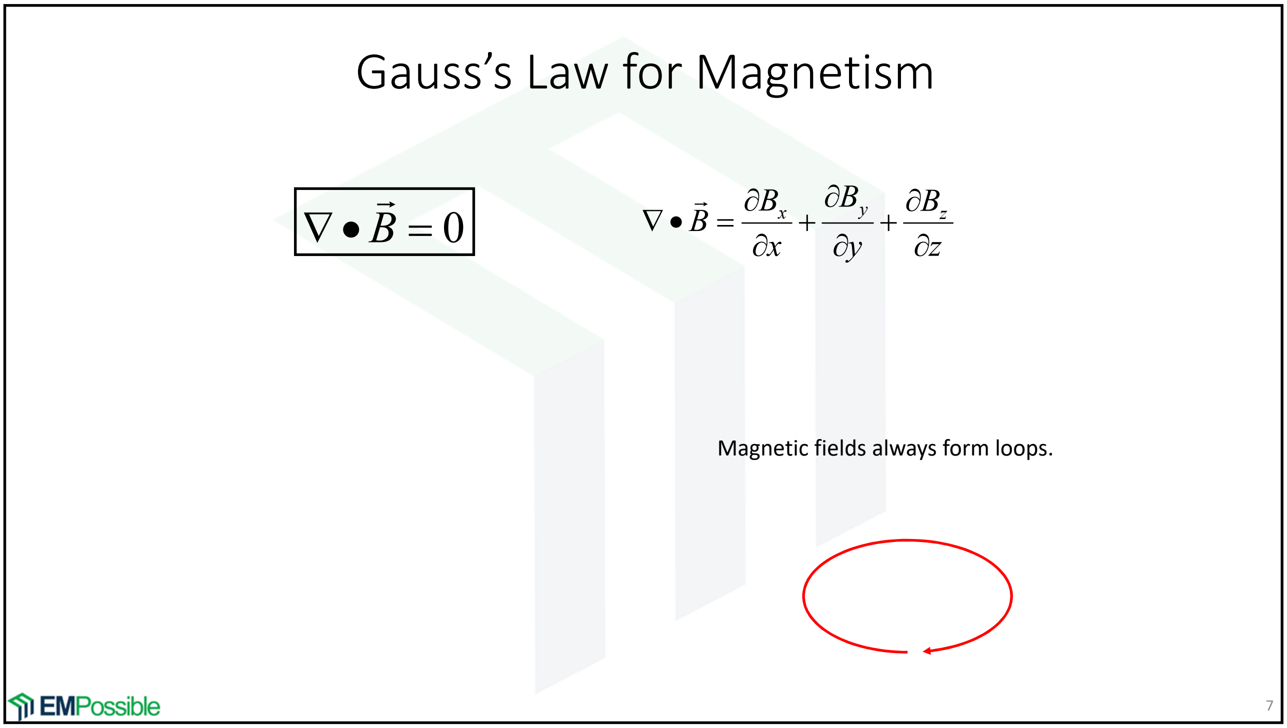
$$\nabla \bullet \vec{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}$$

Electric fields diverge from positive charges and converge on negative charges.



If there are no charges,
electric fields must form loops.



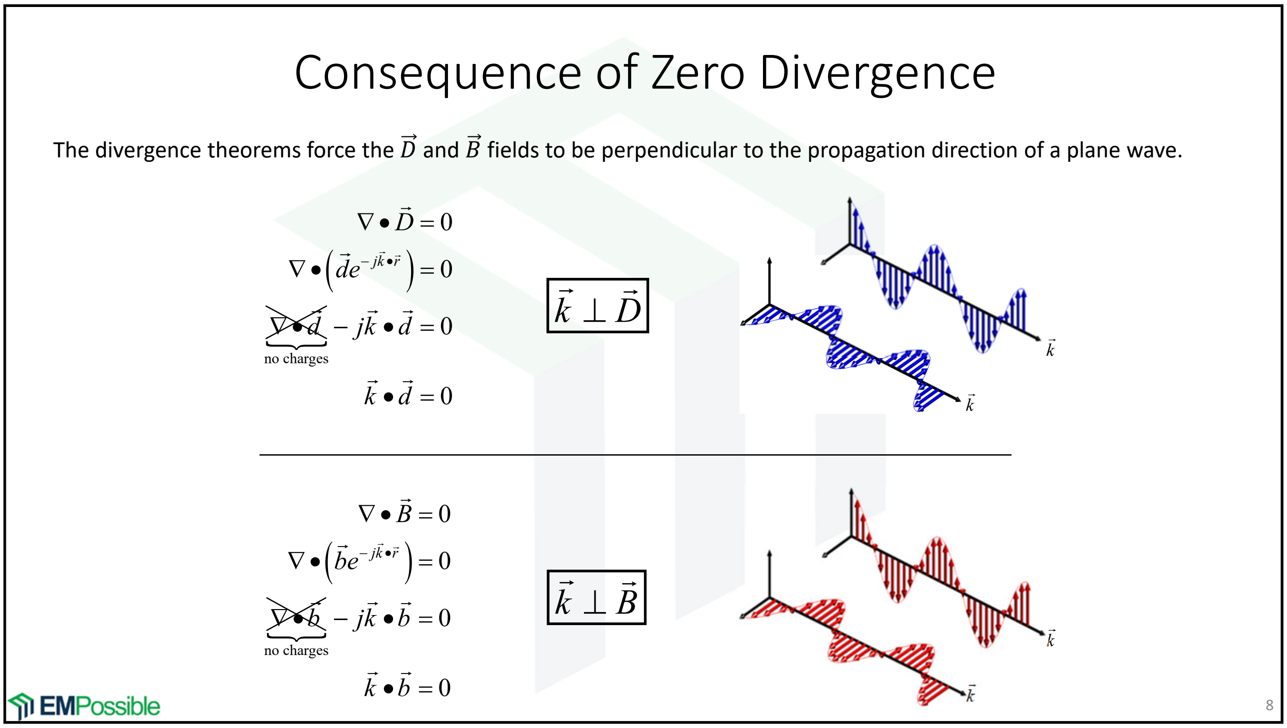


Gauss's Law for Magnetism

$$\boxed{\nabla \bullet \vec{B} = 0}$$

$$\nabla \bullet \vec{B} = \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z}$$

Magnetic fields always form loops.



Consequence of Zero Divergence

The divergence theorems force the $\vec{D}$ and $\vec{B}$ fields to be perpendicular to the propagation direction of a plane wave.

$$\nabla \bullet \vec{D} = 0$$

$$\nabla \bullet \left(\vec{d} e^{-j\vec{k} \bullet \vec{r}} \right) = 0$$

$$\underbrace{\cancel{\nabla \bullet \vec{d}}}_{\text{no charges}} - j\vec{k} \bullet \vec{d} = 0$$

$$\vec{k} \bullet \vec{d} = 0$$

$$\boxed{\vec{k} \perp \vec{D}}$$

$$\nabla \bullet \vec{B} = 0$$

$$\nabla \bullet \left(\vec{b} e^{-j\vec{k} \bullet \vec{r}} \right) = 0$$

$$\underbrace{\cancel{\nabla \bullet \vec{b}}}_{\text{no charges}} - j\vec{k} \bullet \vec{b} = 0$$

$$\vec{k} \bullet \vec{b} = 0$$

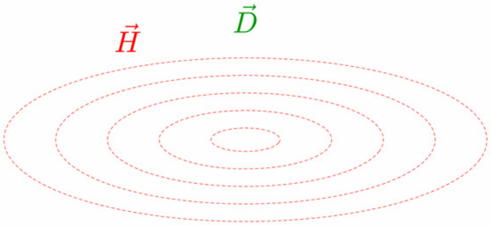
$$\boxed{\vec{k} \perp \vec{B}}$$

Ampere’s Law with Maxwell’s Correction

$$\boxed{\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}}$$

$$\nabla \times \vec{H} = \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} \right) \hat{a}_x + \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} \right) \hat{a}_y + \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right) \hat{a}_z$$

Circulating magnetic fields induce currents and/or time varying electric fields. Currents and/or time varying electric fields induce circulating magnetic fields.

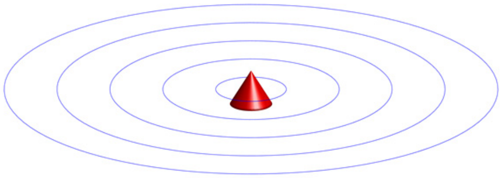


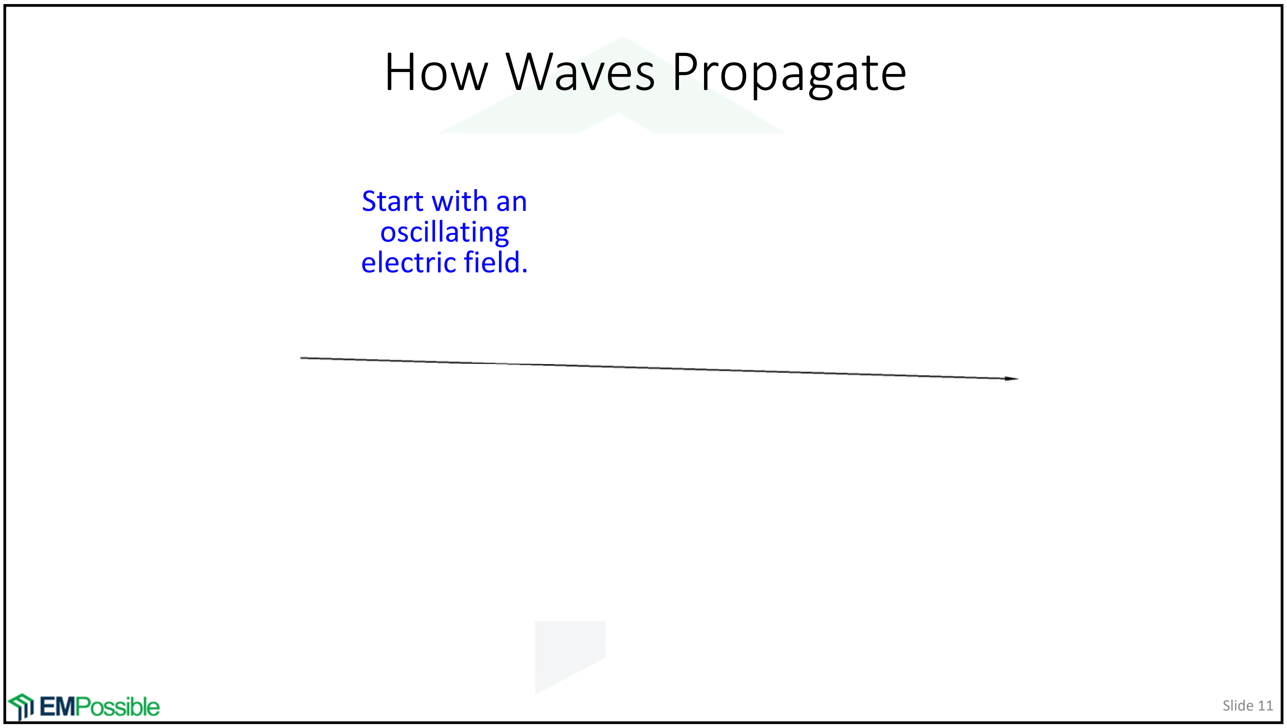
Faraday’s Law of Induction

$$\boxed{\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}}$$

$$\nabla \times \vec{E} = \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} \right) \hat{a}_x + \left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} \right) \hat{a}_y + \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right) \hat{a}_z$$

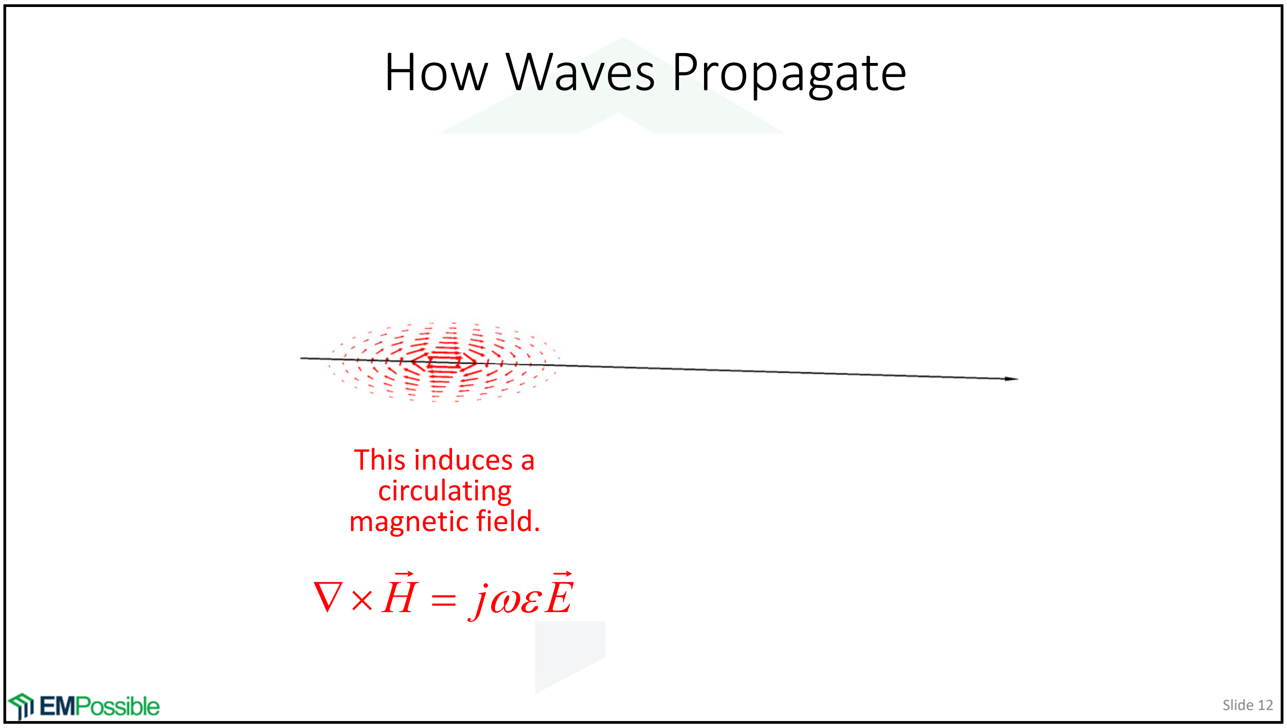
Circulating electric fields induce time varying magnetic fields.
Time varying magnetic fields induce circulating electric fields.





How Waves Propagate

Start with an oscillating electric field.



How Waves Propagate

This induces a circulating magnetic field.

$$\nabla \times \vec{H} = j\omega\varepsilon\vec{E}$$

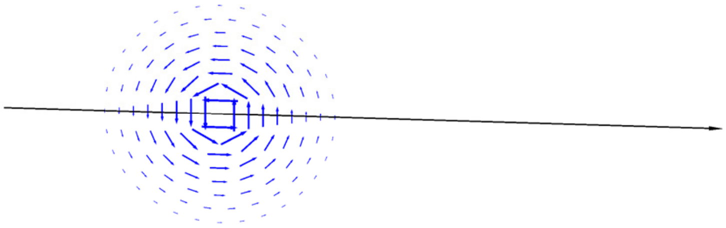
How Waves Propagate

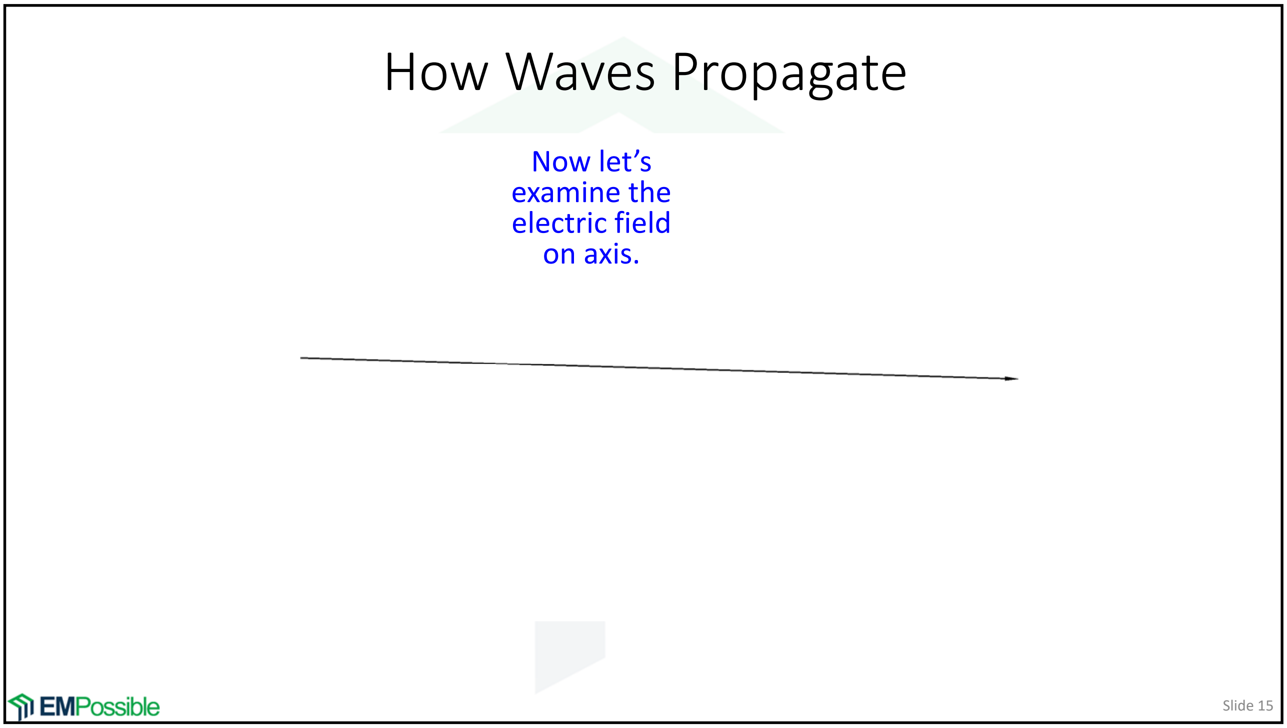
Now let's examine the magnetic field on axis.

How Waves Propagate

This induces a circulating electric field.

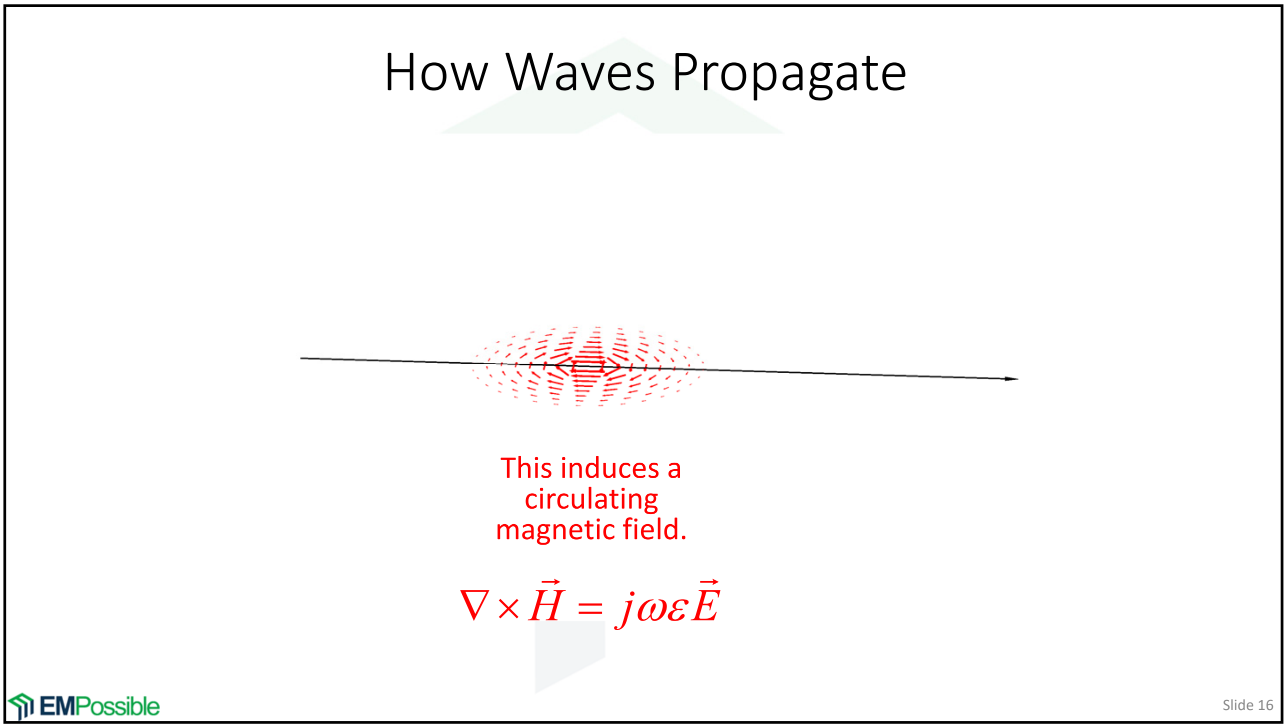
$$\nabla \times \vec{E} = -j\omega\mu\vec{H}$$





How Waves Propagate

Now let's examine the electric field on axis.



How Waves Propagate

This induces a circulating magnetic field.

$$\nabla \times \vec{H} = j\omega\varepsilon\vec{E}$$

How Waves Propagate

...and so on...

Starting Point for Electromagnetic Analysis

Divergence Equations

$$\nabla \bullet \vec{B} = 0$$

$$\nabla \bullet \vec{D} = \rho_v$$

Curl Equations

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

What produces fields

Constitutive Relations

$$\vec{D}(t) = \left[\varepsilon(t)\right] * \vec{E}(t)$$

$$\vec{B}(t) = \left[\mu(t)\right] * \vec{H}(t)$$

$*$ means convolution

[ ] means tensor

How fields interact with materials

Maxwell’s Equations in Cartesian Coordinates (1 of 4)

Vector Terms

$$\vec{E} = E_x\hat{a}_x + E_y\hat{a}_y + E_z\hat{a}_z \qquad \vec{H} = H_x\hat{a}_x + H_y\hat{a}_y + H_z\hat{a}_z \qquad \vec{J} = J_x\hat{a}_x + J_y\hat{a}_y + J_z\hat{a}_z$$

$$\vec{D} = D_x\hat{a}_x + D_y\hat{a}_y + D_z\hat{a}_z \qquad \vec{B} = B_x\hat{a}_x + B_y\hat{a}_y + B_z\hat{a}_z$$

Divergence Equations

$$\nabla \bullet \vec{D} = 0 \qquad \nabla \bullet \vec{B} = 0$$

$$\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = 0 \qquad \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0$$

Maxwell’s Equations in Cartesian Coordinates (2 of 4)

Constitutive Relations

$$\vec{D} = [\varepsilon]\vec{E}$$

$$D_x\hat{a}_x + D_y\hat{a}_y + D_z\hat{a}_z = \left(\varepsilon_{xx}E_x + \varepsilon_{xy}E_y + \varepsilon_{xz}E_z\right)\hat{a}_x + \left(\varepsilon_{yx}E_x + \varepsilon_{yy}E_y + \varepsilon_{yz}E_z\right)\hat{a}_y + \left(\varepsilon_{zx}E_x + \varepsilon_{zy}E_y + \varepsilon_{zz}E_z\right)\hat{a}_z$$

$$D_x = \varepsilon_{xx}E_x + \varepsilon_{xy}E_y + \varepsilon_{xz}E_z$$
$$D_y = \varepsilon_{yx}E_x + \varepsilon_{yy}E_y + \varepsilon_{yz}E_z$$
$$D_z = \varepsilon_{zx}E_x + \varepsilon_{zy}E_y + \varepsilon_{zz}E_z$$

$$\vec{B} = [\mu]\vec{H} \longrightarrow \begin{aligned} B_x &= \mu_{xx}H_x + \mu_{xy}H_y + \mu_{xz}H_z \\ B_y &= \mu_{yx}H_x + \mu_{yy}H_y + \mu_{yz}H_z \\ B_z &= \mu_{zx}H_x + \mu_{zy}H_y + \mu_{zz}H_z \end{aligned}$$

Maxwell’s Equations in Cartesian Coordinates (3 of 4)

Curl Equations

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right)\hat{a}_x + \left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right)\hat{a}_y + \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)\hat{a}_z = -\frac{\partial}{\partial t}\left(B_x\hat{a}_x + B_y\hat{a}_y + B_z\hat{a}_z\right)$$

$$\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right)\hat{a}_x + \left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right)\hat{a}_y + \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)\hat{a}_z = -\frac{\partial B_x}{\partial t}\hat{a}_x - \frac{\partial B_y}{\partial t}\hat{a}_y - \frac{\partial B_z}{\partial t}\hat{a}_z$$

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -\frac{\partial B_x}{\partial t}$$

$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -\frac{\partial B_y}{\partial t}$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -\frac{\partial B_z}{\partial t}$$

Maxwell’s Equations in Cartesian Coordinates (4 of 4)

Curl Equations

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

$$\left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right)\hat{a}_x + \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right)\hat{a}_y + \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)\hat{a}_z = \left(J_x\hat{a}_x + J_y\hat{a}_y + J_z\hat{a}_z\right) + \frac{\partial}{\partial t}\left(D_x\hat{a}_x + D_y\hat{a}_y + D_z\hat{a}_z\right)$$

$$\left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right)\hat{a}_x + \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right)\hat{a}_y + \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)\hat{a}_z = \left(J_x + \frac{\partial D_x}{\partial t}\right)\hat{a}_x + \left(J_y + \frac{\partial D_y}{\partial t}\right)\hat{a}_y + \left(J_z + \frac{\partial D_z}{\partial t}\right)\hat{a}_z$$

$$\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = J_x + \frac{\partial D_x}{\partial t}$$

$$\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = J_y + \frac{\partial D_y}{\partial t}$$

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = J_z + \frac{\partial D_z}{\partial t}$$

Alternative Form of Maxwell’s Equations in Cartesian Coordinates (1 of 2)

Alternate Curl Equations

$$\nabla \times \vec{H} = [\varepsilon]\frac{\partial \vec{E}}{\partial t}$$

$$\left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right)\hat{a}_x + \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right)\hat{a}_y + \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)\hat{a}_z = \left(\varepsilon_{xx}\frac{\partial E_x}{\partial t} + \varepsilon_{xy}\frac{\partial E_y}{\partial t} + \varepsilon_{xz}\frac{\partial E_z}{\partial t}\right)\hat{a}_x$$

$$+\left(\varepsilon_{yx}\frac{\partial E_x}{\partial t} + \varepsilon_{yy}\frac{\partial E_y}{\partial t} + \varepsilon_{yz}\frac{\partial E_z}{\partial t}\right)\hat{a}_y$$

$$+\left(\varepsilon_{zx}\frac{\partial E_x}{\partial t} + \varepsilon_{zy}\frac{\partial E_y}{\partial t} + \varepsilon_{zz}\frac{\partial E_z}{\partial t}\right)\hat{a}_z$$

$$\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = \varepsilon_{xx}\frac{\partial E_x}{\partial t} + \varepsilon_{xy}\frac{\partial E_y}{\partial t} + \varepsilon_{xz}\frac{\partial E_z}{\partial t}$$

$$\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = \varepsilon_{yx}\frac{\partial E_x}{\partial t} + \varepsilon_{yy}\frac{\partial E_y}{\partial t} + \varepsilon_{yz}\frac{\partial E_z}{\partial t}$$

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = \varepsilon_{zx}\frac{\partial E_x}{\partial t} + \varepsilon_{zy}\frac{\partial E_y}{\partial t} + \varepsilon_{zz}\frac{\partial E_z}{\partial t}$$

Alternative Form of Maxwell’s Equations in Cartesian Coordinates (2 of 2)

Alternate Curl Equations

$$\nabla \times \vec{E} = -[\mu]\frac{\partial \vec{H}}{\partial t}$$

$$\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right)\hat{a}_x + \left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right)\hat{a}_y + \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)\hat{a}_z = -\left(\mu_{xx}\frac{\partial H_x}{\partial t} + \mu_{xy}\frac{\partial H_y}{\partial t} + \mu_{xz}\frac{\partial H_z}{\partial t}\right)\hat{a}_x$$

$$-\left(\mu_{yx}\frac{\partial H_x}{\partial t} + \mu_{yy}\frac{\partial H_y}{\partial t} + \mu_{yz}\frac{\partial H_z}{\partial t}\right)\hat{a}_y$$

$$-\left(\mu_{zx}\frac{\partial H_x}{\partial t} + \mu_{zy}\frac{\partial H_y}{\partial t} + \mu_{zz}\frac{\partial H_z}{\partial t}\right)\hat{a}_z$$

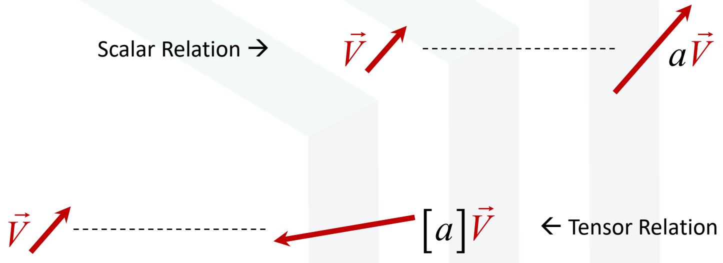
$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -\mu_{xx}\frac{\partial H_x}{\partial t} - \mu_{xy}\frac{\partial H_y}{\partial t} - \mu_{xz}\frac{\partial H_z}{\partial t}$$

$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -\mu_{yx}\frac{\partial H_x}{\partial t} - \mu_{yy}\frac{\partial H_y}{\partial t} - \mu_{yz}\frac{\partial H_z}{\partial t}$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -\mu_{zx}\frac{\partial H_x}{\partial t} - \mu_{zy}\frac{\partial H_y}{\partial t} - \mu_{zz}\frac{\partial H_z}{\partial t}$$

Tensors

Tensors are a generalization of a scaling factor where the direction of a vector can be altered in addition to its magnitude.



$$[a]\vec{V} = \begin{bmatrix} a_{xx} & a_{xy} & a_{xz} \\ a_{yx} & a_{yy} & a_{yz} \\ a_{zx} & a_{zy} & a_{zz} \end{bmatrix} \begin{bmatrix} V_x \\ V_y \\ V_z \end{bmatrix}$$

The Constitutive Relations

Linear, isotropic and non-dispersive materials:

$$\vec{D}(t) = \varepsilon \vec{E}(t)$$

This will be used throughout this course.

Dispersive materials:

$$\vec{D}(t) = \varepsilon(t) * \vec{E}(t)$$

Anisotropic materials:

$$\vec{D}(t) = [\varepsilon]\vec{E}(t)$$

The point is that all of the complexities associated with modeling strange materials are incorporated into this single equation. The implementation of Maxwell's equations in FDTD will never change as different materials are introduced. Keeping this as a separate step will make the FDTD code more modular and easier to modify. It may not be as efficient as it could be though.

Nonlinear materials:

$$D(t) = \varepsilon_0 \left[1 + \chi_{\text{e}}^{(1)} E(t) + \chi_{\text{e}}^{(2)} E^2(t) + \chi_{\text{e}}^{(3)} E^3(t) + \cdots \right]$$

Anisotropic Materials

A generalized tensor for permittivity is written as

$$\vec{D}(t)=\begin{bmatrix}\varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz}\\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz}\\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz}\end{bmatrix}\vec{E}(t)$$

$\varepsilon_{ij} \equiv$ how much of E_j contributes to D_i

We see that E and d can be in different directions when the permittivity is anisotropic.

It greatly simplifies a finite-difference method to consider only diagonal tensors. That is, all of the off-diagonal terms will be set to zero.

$$\vec{D}(t)=\begin{bmatrix}\varepsilon_{xx} & 0 & 0\\ 0 & \varepsilon_{yy} & 0\\ 0 & 0 & \varepsilon_{zz}\end{bmatrix}\vec{E}(t) \quad \rightarrow \quad \begin{aligned}D_x(t)&=\varepsilon_{xx}E_x(t)\\ D_y(t)&=\varepsilon_{yy}E_y(t)\\ D_z(t)&=\varepsilon_{zz}E_z(t)\end{aligned}$$

Special Note: There are only three degrees of freedom for the tensor components. The nine elements cannot be chosen arbitrarily. It is always possible to choose a coordinate system that makes the tensor diagonal. The off diagonal terms only arise when the chosen coordinate system does not match the crystal axes of the anisotropic material. The simplification above restricts us to only be able to model anisotropic materials that align perfectly with our x, y, and z axes. ☹

Simplifying Maxwell's Equations

1. Assume no charges or current sources: $\rho_v = 0$ $\vec{J} = 0$

$$\nabla \bullet \vec{B} = 0 \qquad \nabla \times \vec{H} = \partial \vec{D}/\partial t \qquad \vec{D}(t) = [\varepsilon(t)] * \vec{E}(t)$$

$$\nabla \bullet \vec{D} = 0 \qquad \nabla \times \vec{E} = -\partial \vec{B}/\partial t \qquad \vec{B}(t) = [\mu(t)] * \vec{H}(t)$$

2. Assume linear, isotropic, and non-dispersive materials:

$$\nabla \bullet \vec{B} = 0 \qquad \nabla \times \vec{H} = \partial \vec{D}/\partial t \qquad \vec{D}(t) = \varepsilon \cdot \vec{E}(t)$$

$$\nabla \bullet \vec{D} = 0 \qquad \nabla \times \vec{E} = -\partial \vec{B}/\partial t \qquad \vec{B}(t) = \mu \cdot \vec{H}(t)$$

Convolution becomes simple multiplication

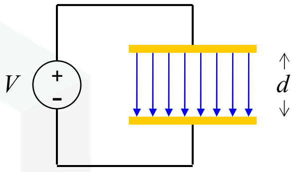
3. Sometimes the constitutive relations are substituted into Maxwell's equations:

$$\nabla \bullet \left[\mu \vec{H}(t)\right] = 0 \qquad \nabla \times \vec{H} = \varepsilon \frac{\partial \vec{E}}{\partial t}$$

$$\nabla \bullet \left[\varepsilon \vec{E}(t)\right] = 0 \qquad \nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t}$$

Note: It is helpful to retain μ and ε and not replace them with refractive index *n*.

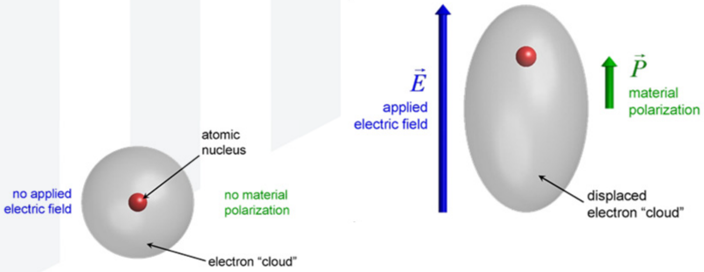
Physical Interpretation of $\vec{E}$ and $\vec{D}$



$$E = \frac{V}{d}$$

$\vec{E}$ – Electric Field
- A disturbance produced around **charges** or in the presence of a **time-varying magnetic field**.
- Think of $\vec{E}$ as a "push"
- Units are volts per meter (V/m)

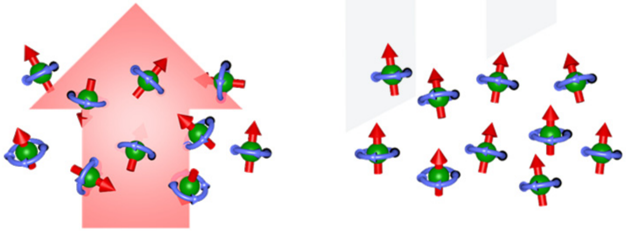
$\vec{D}$ – Electric Displacement Field
- "*D*" stands for displacement
- Includes $\vec{E}$, but also accounts for displaced charges in a material (material polarization)
- Equivalent to flux density
- Think of $\vec{D}$ as displaced charge
- Units are dipole moments per unit volume (C·m/m^3), or just (C/m^2)
- We can make $\vec{E}$ look like an equivalent displaced charge through $\vec{D} = \varepsilon_0 \vec{E}$.



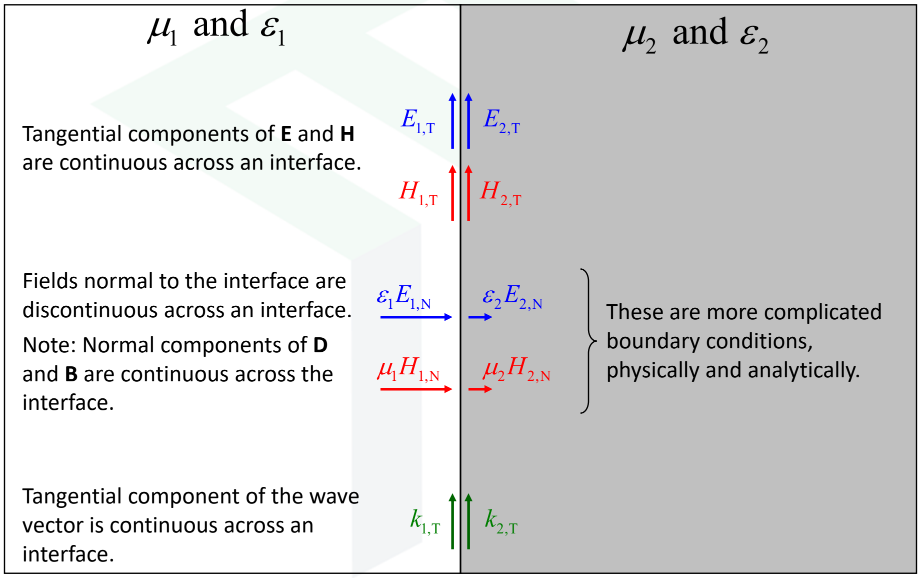
Physical Interpretation of $\vec{H}$ and $\vec{B}$

$\vec{H}$ – Magnetic Field
- A disturbance produced around **currents** or in the presence of **a time-varying electric field**.
- Think of $\vec{H}$ as a magnetic "push"
- Units are amperes per meter (A/m)

$\vec{B}$ – Magnetic Displacement Field
- Includes $\vec{H}$, but also accounts for tilted magnetic dipoles in a material (magnetization)
- Equivalent to flux density
- Think of $\vec{B}$ as reoriented magnetic dipoles
- Units are magnetic dipole moments per unit volume (W·m/m^3), or just (W/m^2)
- We can make $\vec{H}$ look like an equivalent reoriented dipole through $\vec{B} = \mu_0 \vec{H}$.



Physical Boundary Conditions



Tangential components of **E** and **H** are continuous across an interface.

Fields normal to the interface are discontinuous across an interface.

Note: Normal components of **D** and **B** are continuous across the interface.

These are more complicated boundary conditions, physically and analytically.

Tangential component of the wave vector is continuous across an interface.

Parameter Relations

The Dielectric Constant, ε_r

The permittivity ε is a measure of how well a material stores electric energy. A circulating magnetic field induces an electric field at the center of the circulation in proportion to the permittivity.

$$\nabla \times \vec{H} = \varepsilon \frac{\partial \vec{E}}{\partial t}$$

The dielectric constant ε_r of a material is its permittivity relative to the permittivity of free space ε_0.

$$\varepsilon = \varepsilon_0 \varepsilon_r$$

$$\varepsilon_0 = 8.854187817 \times 10^{-12} \quad \text{F/m}$$

$$1 \le \varepsilon_r \le \infty$$

ε_r is the relative permittivity or dielectric constant

Table of Dielectric Constants

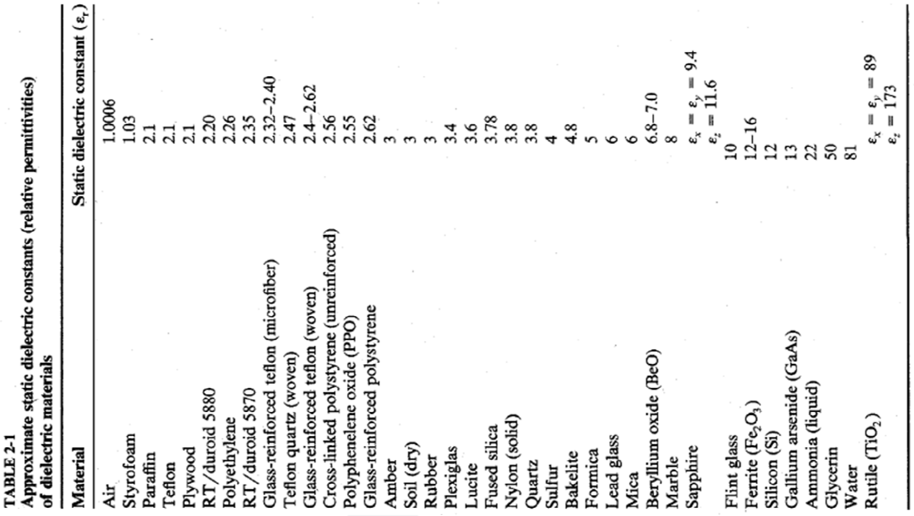


TABLE 2-1
Approximate static dielectric constants (relative permittivities) of dielectric materials

Material	Static dielectric constant (ε_r)
Air	1.0006
Styrofoam	1.03
Paraffin	2.1
Teflon	2.1
Plywood	2.1
RT/duroid 5880	2.20
Polyethylene	2.26
RT/duroid 5870	2.35
Glass-reinforced teflon (microfiber)	2.32–2.40
Teflon quartz (woven)	2.47
Glass-reinforced teflon (woven)	2.4–2.62
Cross-linked polystyrene (unreinforced)	2.56
Polyphenelene oxide (PPO)	2.55
Glass-reinforced polystyrene	2.62
Amber	3
Soil (dry)	3
Rubber	3
Plexiglas	3.4
Lucite	3.6
Fused silica	3.78
Nylon (solid)	3.8
Quartz	3.8
Sulfur	4
Bakelite	4.8
Formica	5
Lead glass	6
Mica	6
Beryllium oxide (BeO)	6.8–7.0
Marble	8
Sapphire	$\varepsilon_x = \varepsilon_y = 9.4$ $\varepsilon_z = 11.6$
Flint glass	10
Ferrite (Fe_2O_3)	12–16
Silicon (Si)	12
Gallium arsenide (GaAs)	13
Ammonia (liquid)	22
Glycerin	50
Water	81
Rutile (TiO_2)	$\varepsilon_x = \varepsilon_y = 89$ $\varepsilon_z = 173$

Constantine A. Balanis, Advanced Engineering Electromagnetics, Wiley, 1989.

The Relative Permeability, μ_r

The permeability μ is a measure of how well a material stores magnetic energy. A circulating electric field induces a magnetic field at the center of the circulation in proportion to the permeability.

$$\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t}$$

The relative permeability μ_r of a material is its permeability relative to the permeability of free space μ_0.

$$\mu = \mu_0 \mu_r$$

$$\mu_0 = 1.256637061 \times 10^{-6} \quad \text{H/m}$$

$$1 \le \mu_r \le \infty$$

μ_r is the relative permeability

Table of Permeabilities

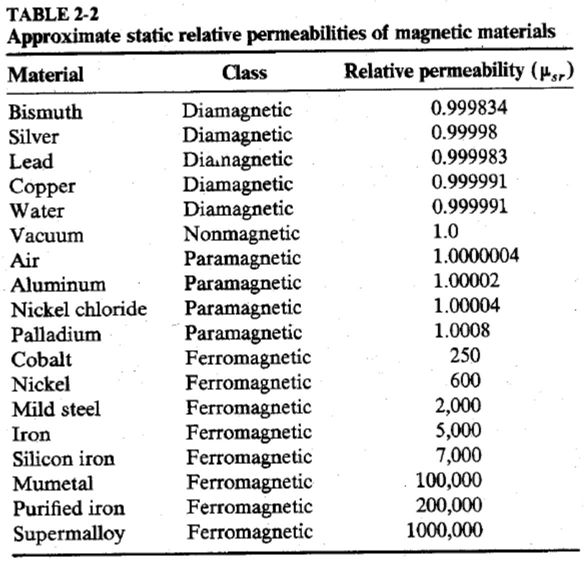


TABLE 2-2
Approximate static relative permeabilities of magnetic materials

Material	Class	Relative permeability (μ_{sr})
Bismuth	Diamagnetic	0.999834
Silver	Diamagnetic	0.99998
Lead	Diamagnetic	0.999983
Copper	Diamagnetic	0.999991
Water	Diamagnetic	0.999991
Vacuum	Nonmagnetic	1.0
Air	Paramagnetic	1.0000004
Aluminum	Paramagnetic	1.00002
Nickel chloride	Paramagnetic	1.00004
Palladium	Paramagnetic	1.0008
Cobalt	Ferromagnetic	250
Nickel	Ferromagnetic	600
Mild steel	Ferromagnetic	2,000
Iron	Ferromagnetic	5,000
Silicon iron	Ferromagnetic	7,000
Mumetal	Ferromagnetic	100,000
Purified iron	Ferromagnetic	200,000
Supermalloy	Ferromagnetic	1000,000

Constantine A. Balanis, Advanced Engineering Electromagnetics, Wiley, 1989.

The Refractive Index

The permittivity and permeability appear in Maxwell's equations so they are the most fundamental material properties. However, it is difficult to determine physical meaning from them in terms of how waves propagate (i.e. speed, loss, etc.). In this case, the refractive index is a more meaningful quantity.

$$n = \sqrt{\mu_{\mathrm{r}} \varepsilon_{\mathrm{r}}}$$

Most materials exhibit a negligible magnetic response and the refractive index and dielectric constant are related through

$$n^2 = \varepsilon_{\mathrm{r}}$$

Hint: one of the most common mistakes made in this course is using values of refractive index directly as permittivity.

Material Impedance

The impedance η of a material quantifies the relation between the electric and magnetic field of a wave travelling through that material. It is the most fundamental quantity that causes reflections and scattering.

$$\eta \simeq \left|\vec{E}\right| \Big/ \left|\vec{H}\right|$$

The impedance can be written relative to the free space impedance as

$$\eta = \eta_0 \sqrt{\frac{\mu_r}{\varepsilon_r}} \qquad \eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 376.73031346177 \ \Omega$$

This shows that the electric field is around two to three orders of magnitude larger than the magnetic field.

ω versus f

ω is the angular frequency measured in radians per second. It relates more directly to phase and k. Think $\cos(\omega t)$.

f is the ordinary frequency measured in cycles per second.
It relates most directly to time t. Think $\cos(2\,\pi f t)$ and $\tau = 1/f$.

$$\boxed{\omega = 2\pi f}$$

Wavelength and Frequency

The frequency f and free space wavelength λ_0 are related through

$$c_0 = f\lambda_0 \qquad c_0 = 299792458\ \tfrac{\text{m}}{\text{s}} \equiv \text{speed of light in vacuum}$$

Inside a material, the wave slows down according to the refractive index as follows.

$$v = \frac{c_0}{n}$$

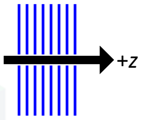
The frequency is the most fundamental parameter because it is fixed. Inside a material, the wave slows down so the wavelength is reduced.

$$v = f\lambda$$

The free space wavelength λ_0 is often used interchangeably with frequency f. This is most common in optics.

Sign Convention

How do you define forward wave propagation?



Sign convention for this course

Quantity	$-\beta z$	$+\beta z$
Wave Solution	$\vec{E}=\vec{E}_0 e^{j(\omega t-kz)}$	$\vec{E}=\vec{E}_0 e^{-j(\omega t-kz)}$
Dielectric Function	$\tilde{\varepsilon}=\varepsilon'-j\varepsilon''$	$\tilde{\varepsilon}=\varepsilon'+j\varepsilon''$
Refractive Index	$N=n-j\kappa$	$N=n+j\kappa$

Summary of Parameter Relations

Permittivity

$\varepsilon=\varepsilon_0\varepsilon_r$

$\varepsilon_0=8.854187817\times10^{-12}\quad\mathrm{F/m}$

Permeability

$\mu=\mu_0\mu_r$

$\mu_0=1.256637061\times10^{-6}\quad\mathrm{H/m}$

Refractive Index

$n=\sqrt{\mu_r\varepsilon_r}$

Impedance

$\eta=\eta_0\sqrt{\mu_r/\varepsilon_r}$

$\eta_0=\sqrt{\mu_0/\varepsilon_0}=376.73031346177\ \ \Omega$

Wave Velocity

$$v=\frac{c_0}{n}$$

$c_0=299792458\quad\mathrm{m/s}$

Exact

Frequency and Wavelength

$\omega=2\pi f$

$c_0=f\lambda_0$

Wave Number

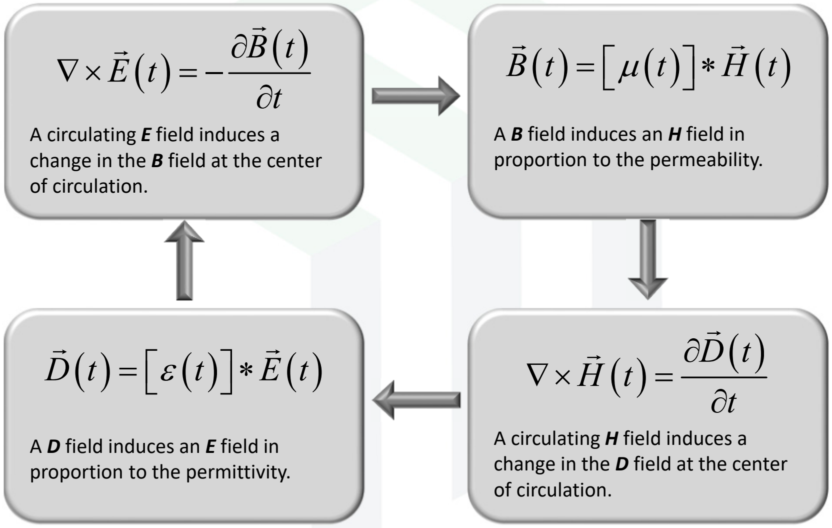
$$k_0=\frac{2\pi}{\lambda_0}$$

Duality Between E-D and H-B

Electric Field	Magnetic Field
E	H
D	B
P	M
ε	μ

Introduction to Finite-Difference Time-Domain

Flow of Maxwell's Equations



$$\nabla \times \vec{E}(t) = -\frac{\partial \vec{B}(t)}{\partial t}$$

A circulating ***E*** field induces a change in the ***B*** field at the center of circulation.

$$\vec{B}(t) = [\mu(t)] * \vec{H}(t)$$

A ***B*** field induces an ***H*** field in proportion to the permeability.

$$\nabla \times \vec{H}(t) = \frac{\partial \vec{D}(t)}{\partial t}$$

A circulating ***H*** field induces a change in the ***D*** field at the center of circulation.

$$\vec{D}(t) = [\varepsilon(t)] * \vec{E}(t)$$

A ***D*** field induces an ***E*** field in proportion to the permittivity.

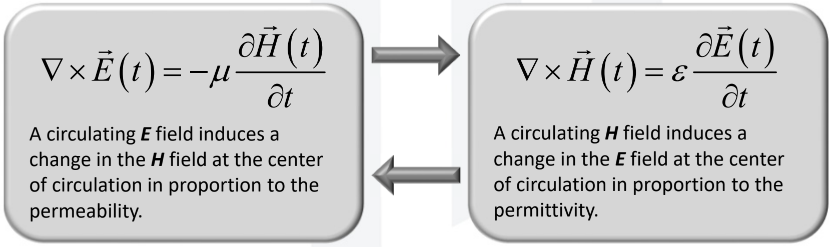
Note: In reality, this all happens simultaneously. In FDTD, it follows this flow.

Flow of Maxwell's Equations Inside Linear, Isotropic and Non-Dispersive Materials

In materials that are linear, isotropic and non-dispersive we have

$$[\mu(t)] * = \mu \cdot \qquad [\varepsilon(t)] * = \varepsilon \cdot$$

In this case, the flow of Maxwell's equations reduces to



$$\nabla \times \vec{E}(t) = -\mu \frac{\partial \vec{H}(t)}{\partial t}$$

A circulating ***E*** field induces a change in the ***H*** field at the center of circulation in proportion to the permeability.

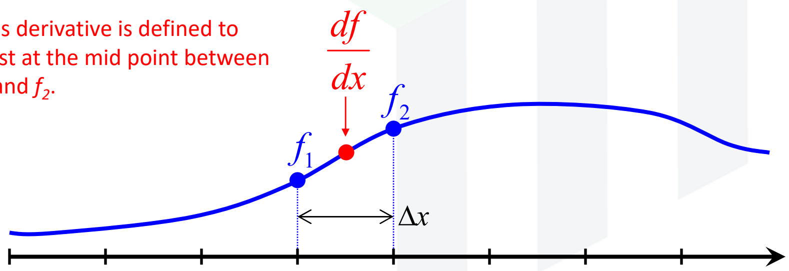
$$\nabla \times \vec{H}(t) = \varepsilon \frac{\partial \vec{E}(t)}{\partial t}$$

A circulating ***H*** field induces a change in the ***E*** field at the center of circulation in proportion to the permittivity.

Finite-Difference Approximations

$$\boxed{\frac{df_{1.5}}{dx} \approx \frac{f_2 - f_1}{\Delta x}}$$

second-order accurate first-order derivative



This derivative is defined to exist at the mid point between f_1 and f_2.

Stable Finite-Difference Equations

Each term in a finite-difference equation must exist at the same point in time and space.

Example: $$\frac{\partial f(x)}{\partial x} + f(x) = 0 \qquad \text{Given } f(0), f(\Delta x), f(2\Delta x), \ldots$$

Exists at $x + \Delta x/2$ Exists at x

$$\boxed{\frac{f(x+\Delta x) - f(x)}{\Delta x}} + \boxed{f(x)} = 0$$

Your simulation will be unstable (i.e. explode).

Exists at $x + \Delta x/2$ Exists at $x + \Delta x/2$

$$\boxed{\frac{f(x+\Delta x) - f(x)}{\Delta x}} + \boxed{f\left(x + \frac{\Delta x}{2}\right)} = 0$$

$f(x)$ is only known at integer multiples of Δx. How do we calculate $f(x+\Delta x/2)$?

Exists at $x + \Delta x/2$ Exists at $x + \Delta x/2$

$$\boxed{\frac{f(x+\Delta x) - f(x)}{\Delta x}} + \boxed{\frac{f(x+\Delta x) + f(x)}{2}} = 0$$

Approximating the Time Derivative (1 of 3)

An intuitive first guess at approximating the time derivatives in Maxwell's equations is:

$$\nabla\times\vec{E}(t)=-\mu\frac{\partial\vec{H}(t)}{\partial t}\Rightarrow\overset{\text{Exists at }t}{\boxed{\nabla\times\vec{E}(t)}}\cong-\mu\overset{\text{Exists at }t+\Delta t/2}{\boxed{\frac{\vec{H}(t+\Delta t)-\vec{H}(t)}{\Delta t}}}$$

$$\nabla\times\vec{H}(t)=\varepsilon\frac{\partial\vec{E}(t)}{\partial t}\Rightarrow\overset{\text{Exists at }t}{\boxed{\nabla\times\vec{H}(t)}}\cong\varepsilon\overset{\text{Exists at }t+\Delta t/2}{\boxed{\frac{\vec{E}(t+\Delta t)-\vec{E}(t)}{\Delta t}}}$$

This is an unstable formulation.

Approximating the Time Derivative (2 of 3)

We adjust the finite-difference equations so that each term exists at the same point in time.

$$\nabla\times\vec{E}(t)=-\mu\frac{\partial\vec{H}(t)}{\partial t}\Rightarrow\nabla\times\vec{E}(t)\cong-\mu\frac{\frac{\vec{H}(t)+\vec{H}(t+\Delta t)}{2}-\frac{\vec{H}(t)+\vec{H}(t-\Delta t)}{2}}{\Delta t}$$

$$\nabla\times\vec{H}(t)=\varepsilon\frac{\partial\vec{E}(t)}{\partial t}\Rightarrow\frac{\vec{H}(t)+\vec{H}(t+\Delta t)}{2}\cong\varepsilon\frac{\vec{E}(t+\Delta t)-\vec{E}(t)}{\Delta t}$$

This works, but we will be doing more calculations than are necessary.

Is there a simpler approach?

Approximating the Time Derivative (3 of 3)

Stagger $\vec{E}$ and $\vec{H}$ in time so that $\vec{E}$ exists at integer time steps $(0, \Delta t, 2\Delta t, \ldots)$ and $\vec{H}$ exists at half time steps $(\Delta t/2, t+\Delta t/2, 2t+\Delta t/2, \ldots)$.

$$\nabla\times\vec{E}(t)=-\mu\frac{\partial\vec{H}(t)}{\partial t} \quad\Rightarrow\quad \nabla\times\vec{E}\Big|_{t}\cong-\mu\frac{\vec{H}\Big|_{t+\Delta t/2}-\vec{H}\Big|_{t-\Delta t/2}}{\Delta t}$$

$$\nabla\times\vec{H}(t)=\varepsilon\frac{\partial\vec{E}(t)}{\partial t} \quad\Rightarrow\quad \nabla\times\vec{H}\Big|_{t+\Delta t/2}\cong\varepsilon\frac{\vec{E}\Big|_{t+\Delta t}-\vec{E}\Big|_{t}}{\Delta t}$$

We will handle the spatial derivatives in ∇× next lecture in a very similar manner.

Derivation of the Update Equations

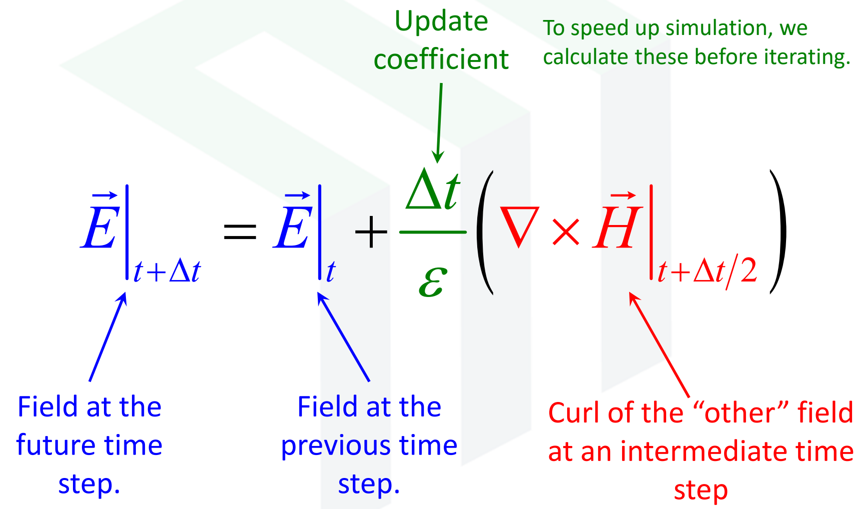
The "update equations" are the equations used inside the main FDTD loop to calculate the field values at the next time step.

They are derived by solving our finite-difference equations for the fields at the future time values.

$$\nabla\times\vec{E}\Big|_{t}=-\mu\frac{\boxed{\vec{H}\Big|_{t+\Delta t/2}}-\vec{H}\Big|_{t-\Delta t/2}}{\Delta t} \quad\Rightarrow\quad \boxed{\vec{H}\Big|_{t+\Delta t/2}}=\vec{H}\Big|_{t-\Delta t/2}-\frac{\Delta t}{\mu}\left(\nabla\times\vec{E}\Big|_{t}\right)$$

$$\nabla\times\vec{H}\Big|_{t+\Delta t/2}=\varepsilon\frac{\boxed{\vec{E}\Big|_{t+\Delta t}}-\vec{E}\Big|_{t}}{\Delta t} \quad\Rightarrow\quad \boxed{\vec{E}\Big|_{t+\Delta t}}=\vec{E}\Big|_{t}+\frac{\Delta t}{\varepsilon}\left(\nabla\times\vec{H}\Big|_{t+\Delta t/2}\right)$$

Anatomy of the FDTD Update Equation



Update coefficient

To speed up simulation, we calculate these before iterating.

$$\vec{E}\Big|_{t+\Delta t} = \vec{E}\Big|_{t} + \frac{\Delta t}{\varepsilon}\left(\nabla \times \vec{H}\Big|_{t+\Delta t/2}\right)$$

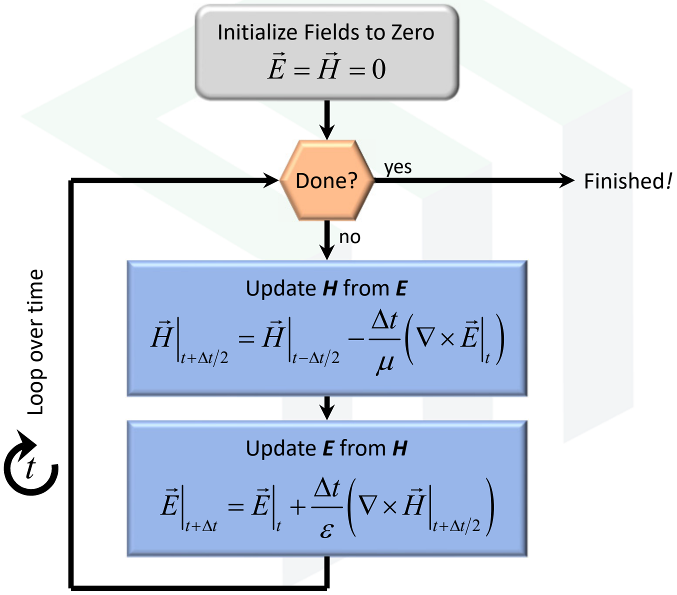
Field at the future time step.

Field at the previous time step.

Curl of the “other” field at an intermediate time step

EMPossible

The FDTD Algorithm…for now ☺



$\vec{H}\Big|_{t-\frac{\Delta t}{2}} = 0 \qquad \vec{E}\Big|_{t} = 0$

$$\begin{aligned}
\vec{H}\Big|_{t+\frac{\Delta t}{2}} &\Leftarrow \vec{E}\Big|_{t} \\
\vec{E}\Big|_{t+\Delta t} &\Leftarrow \vec{H}\Big|_{t+\frac{\Delta t}{2}} \\
\vec{H}\Big|_{t+\frac{3\Delta t}{2}} &\Leftarrow \vec{E}\Big|_{t+\Delta t} \\
\vec{E}\Big|_{t+2\Delta t} &\Leftarrow \vec{H}\Big|_{t+\frac{3\Delta t}{2}} \\
\vec{H}\Big|_{t+\frac{5\Delta t}{2}} &\Leftarrow \vec{E}\Big|_{t+2\Delta t} \\
\vec{E}\Big|_{t+3\Delta t} &\Leftarrow \vec{H}\Big|_{t+\frac{5\Delta t}{2}} \\
&\vdots
\end{aligned}$$

EMPossible