



Computational Science:
Introduction to Finite-Difference Time-Domain

Enhancing One-Dimensional FDTD

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Lecture Outline

- Convergence
- Pure sine wave source
- Grid dispersion
- Incorporating loss
- Frequency dependent materials
- 1D anisotropic FDTD

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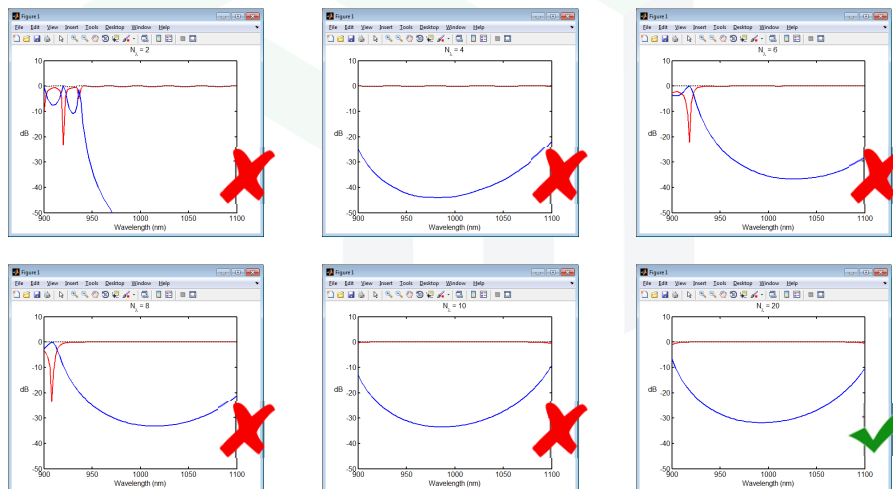
Convergence

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Convergence (1 of 2)

Convergence is the tendency of a numerical model to asymptotically approach a constant answer as the grid resolution (Δx , Δy , and Δz) and/or the time step Δt is reduced.



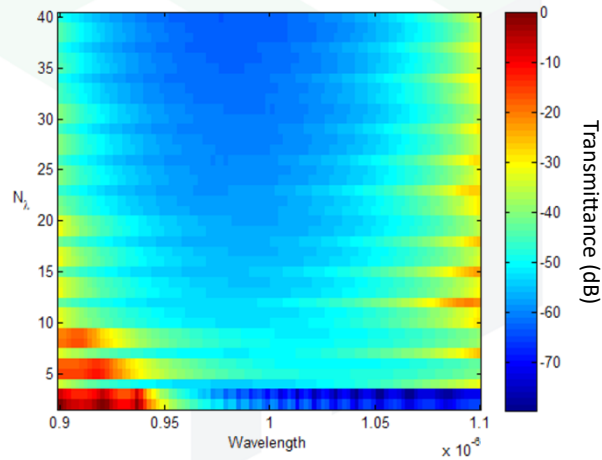
EMPossible

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Convergence (2 of 2)

This is a better way to view and judge convergence, although it is more computationally intensive to calculate.



Conclusions on Convergence

- The equations and procedures provided in this course are only “rules of thumb” that will give you a reasonable first guess at grid resolution and time step.
- The real way to determine sufficient grid resolution and sufficiently small time step is to check for convergence.

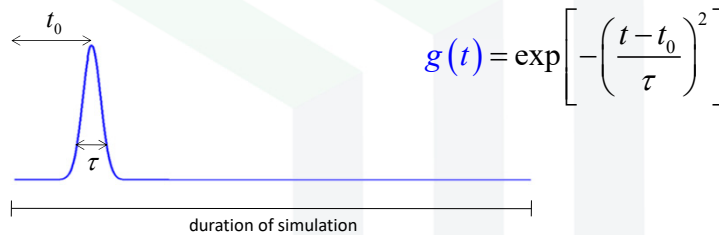
Pure Sine Wave Source

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The Gaussian Pulse Source

The Gaussian pulse source is a single pulse that contains energy at all frequencies from DC up to some maximum frequency. It is not easy to make conclusions on the steady-state response at a particular frequency by watching this pulse propagate through a device.



$$\tau \cong \frac{0.5}{B}$$

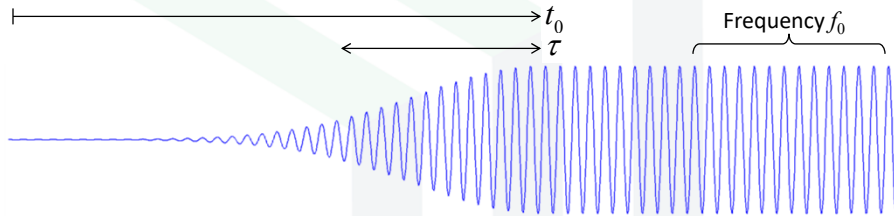
$$t_0 \approx 6\tau$$

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The Pure Frequency Source

FDTD can also be excited by a “pure” sine wave of frequency f_0 . For a robust simulation, the amplitude of the wave should be tapered from zero to unity amplitude. Using a single frequency source eliminates many of the benefits of FDTD, but visualizing the field is more intuitive.



$$g(t) = A(t) \sin(2\pi f_0 t)$$

$$A(t) = \begin{cases} \exp\left[-\left(\frac{t-t_0}{\tau}\right)^2\right] & t < t_0 \\ 1 & t \geq t_0 \end{cases}$$

$$\tau \cong \frac{3}{f_0}$$

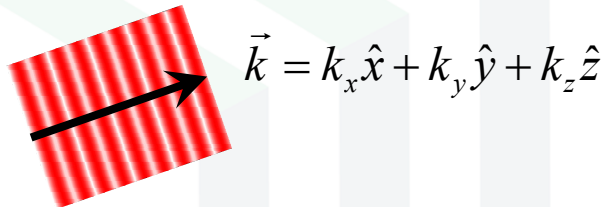
$$t_0 \cong 3\tau$$

Grid Dispersion

The Wave Vector

The wave vector is a vector quantity that conveys two pieces of information at the same time:

1. **Direction** – The direction of the wave is perpendicular to the wave fronts.



2. **Wavelength and Refractive Index** – The magnitude of the wave vector tells us the spatial period (wavelength) of the wave inside the material. Therefore, $|\vec{k}|$ also conveys the material's refractive index n .

$$|\vec{k}| = \frac{2\pi}{\lambda} = \frac{2\pi n}{\lambda_0} \quad \lambda_0 \equiv \text{free space wavelength}$$

Dispersion Relations

The dispersion relation for a material relates the wave vector to frequency. Essentially, it tells us the refractive index as a function of direction through a material.

It is derived by substituting a plane wave solution into the wave equation.

For an ordinary linear, homogeneous, and isotropic (LHI) material, the dispersion relation is:

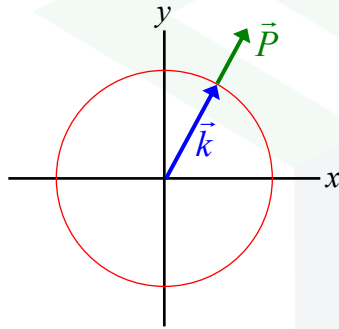
$$k_x^2 + k_y^2 + k_z^2 = k_0^2 n^2 = \left(\frac{\omega}{v} \right)^2$$

$v = \frac{c_0}{n} \equiv \text{velocity of physical wave}$

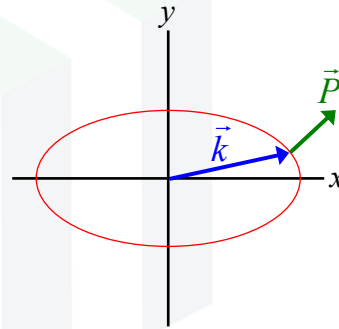
Index Ellipsoids

Index ellipsoids are a map of the refractive index as a function of direction through a material. They are constructed by plotting the dispersion relation.

Isotropic Materials



Anisotropic Materials



Phase propagates in the direction of k . Therefore, the refractive index derived from $|k|$ is best described as the phase refractive index. Velocity here is the phase velocity.

Energy propagates in the direction of P which is always normal to the surface of the index ellipsoid. From this, we can define a group velocity and a group refractive index.



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Derivation of Numerical Dispersion Relation

We derive the numerical dispersion relation by substituting a plane wave solution into our finite-difference equations representing Maxwell's equations on a Yee grid.

$$\vec{E} = \begin{bmatrix} E_{x0} \\ E_{y0} \\ E_{z0} \end{bmatrix} e^{j(2\pi fT\Delta t - \tilde{k}_x I \Delta x - \tilde{k}_y J \Delta y - \tilde{k}_z K \Delta z)}$$

$$\vec{H} = \begin{bmatrix} \tilde{H}_{x0} \\ \tilde{H}_{y0} \\ \tilde{H}_{z0} \end{bmatrix} e^{j(2\pi fT\Delta t - \tilde{k}_x I \Delta x - \tilde{k}_y J \Delta y - \tilde{k}_z K \Delta z)}$$

- $T = 1, 2, 3, \dots$ time step
- $I = 1, 2, 3, \dots$ grid index along x direction
- $J = 1, 2, 3, \dots$ grid index along y direction
- $K = 1, 2, 3, \dots$ grid index along z direction

$$\frac{E_z^{i,j,k+1} - E_z^{i,j,k}}{\Delta y} - \frac{E_y^{i,j,k+1} - E_y^{i,j,k}}{\Delta z} = -\frac{\mu_{xx}^{i,j,k}}{c_0} \frac{\tilde{H}_x^{i,j,k} - \tilde{H}_x^{i,j,k-1}}{\Delta t}$$

$$\frac{E_x^{i,j,k+1} - E_x^{i,j,k}}{\Delta z} - \frac{E_z^{i,j,k+1} - E_z^{i,j,k}}{\Delta x} = -\frac{\mu_{yy}^{i,j,k}}{c_0} \frac{\tilde{H}_y^{i,j,k} - \tilde{H}_y^{i,j,k-1}}{\Delta t}$$

$$\frac{E_y^{i,j,k+1} - E_y^{i,j,k}}{\Delta x} - \frac{E_x^{i,j,k+1} - E_x^{i,j,k}}{\Delta y} = -\frac{\mu_{zz}^{i,j,k}}{c_0} \frac{\tilde{H}_z^{i,j,k} - \tilde{H}_z^{i,j,k-1}}{\Delta t}$$

$$\frac{\tilde{H}_z^{i,j,k} - \tilde{H}_z^{i,j,k-1}}{\Delta y} - \frac{\tilde{H}_y^{i,j,k} - \tilde{H}_y^{i,j,k-1}}{\Delta z} = \frac{\epsilon_{xx}^{i,j,k}}{c_0} \frac{E_x^{i,j,k} - E_x^{i,j,k-1}}{\Delta t}$$

$$\frac{\tilde{H}_x^{i,j,k} - \tilde{H}_x^{i,j,k-1}}{\Delta z} - \frac{\tilde{H}_z^{i,j,k} - \tilde{H}_z^{i,j,k-1}}{\Delta x} = \frac{\epsilon_{yy}^{i,j,k}}{c_0} \frac{E_y^{i,j,k} - E_y^{i,j,k-1}}{\Delta t}$$

$$\frac{\tilde{H}_y^{i,j,k} - \tilde{H}_y^{i,j,k-1}}{\Delta x} - \frac{\tilde{H}_x^{i,j,k} - \tilde{H}_x^{i,j,k-1}}{\Delta y} = \frac{\epsilon_{zz}^{i,j,k}}{c_0} \frac{E_z^{i,j,k} - E_z^{i,j,k-1}}{\Delta t}$$



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Numerical Dispersion Relation

The numerical dispersion relation for a 3D Yee grid is

$$\left[\frac{1}{\Delta x} \sin\left(\frac{\tilde{k}_x \Delta x}{2}\right) \right]^2 + \left[\frac{1}{\Delta y} \sin\left(\frac{\tilde{k}_y \Delta y}{2}\right) \right]^2 + \left[\frac{1}{\Delta z} \sin\left(\frac{\tilde{k}_z \Delta z}{2}\right) \right]^2 = \left[\frac{1}{v \Delta t} \sin\left(\frac{\omega \Delta t}{2}\right) \right]^2$$

The numerical dispersion relation for a 2D Yee grid is

$$\left[\frac{1}{\Delta x} \sin\left(\frac{\tilde{k}_x \Delta x}{2}\right) \right]^2 + \left[\frac{1}{\Delta y} \sin\left(\frac{\tilde{k}_y \Delta y}{2}\right) \right]^2 = \left[\frac{1}{v \Delta t} \sin\left(\frac{\omega \Delta t}{2}\right) \right]^2$$

The numerical dispersion relation for a 1D Yee grid is

$$\left[\frac{1}{\Delta z} \sin\left(\frac{\tilde{k}_z \Delta z}{2}\right) \right]^2 = \left[\frac{1}{v \Delta t} \sin\left(\frac{\omega \Delta t}{2}\right) \right]^2$$

$\tilde{v} \equiv$ velocity of numerical wave

Limiting Case

As the time step Δt and the grid resolution Δx , Δy , and Δz approach zero, the numerical dispersion relation reduces exactly to the dispersion relation of a real material.

$$\left[\frac{1}{\Delta x} \sin\left(\frac{\tilde{k}_x \Delta x}{2}\right) \right]^2 + \left[\frac{1}{\Delta y} \sin\left(\frac{\tilde{k}_y \Delta y}{2}\right) \right]^2 + \left[\frac{1}{\Delta z} \sin\left(\frac{\tilde{k}_z \Delta z}{2}\right) \right]^2 = \left[\frac{1}{v \Delta t} \sin\left(\frac{\omega \Delta t}{2}\right) \right]^2$$

↓

$$\left[\frac{\tilde{k}_x}{2} \right]^2 + \left[\frac{\tilde{k}_y}{2} \right]^2 + \left[\frac{\tilde{k}_z}{2} \right]^2 = \left[\frac{\omega}{2v} \right]^2$$

↓

$$\tilde{k}_x^2 + \tilde{k}_y^2 + \tilde{k}_z^2 = \left(\frac{\omega}{v} \right)^2$$

Numerical Phase Velocity

The 3D numerical dispersion relation is solved for k numerically by iterating until a valid solution is found. For propagation along a single axis, the dispersion relation reduces to 1D and can be solved analytically for \tilde{k}_z .

$$\left[\frac{1}{\Delta z} \sin\left(\frac{\tilde{k}_z \Delta z}{2}\right) \right]^2 = \left[\frac{1}{v\Delta t} \sin\left(\frac{\omega\Delta t}{2}\right) \right]^2 \rightarrow \tilde{k}_z = \frac{2}{\Delta z} \sin^{-1}\left[\frac{\Delta z}{v\Delta t} \sin\left(\frac{\omega\Delta t}{2}\right) \right]$$

The speed that a wave will travel through a grid is defined as

$$\tilde{v}_p = \frac{\omega}{\tilde{k}}$$

This becomes

$$\tilde{v}_p = \frac{\omega\Delta z/2}{\sin^{-1}\left[\frac{\Delta z}{v\Delta t} \sin\left(\frac{\omega\Delta t}{2}\right) \right]}$$

Note, waves in a grid propagate at a slightly different velocity than a physical wave due to the additional interaction with the grid.

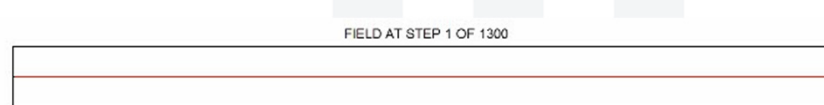
Typically they travel slower.

Animation of Grid Dispersion

Simulation with near-zero grid dispersion...



Simulation with strong grid dispersion...



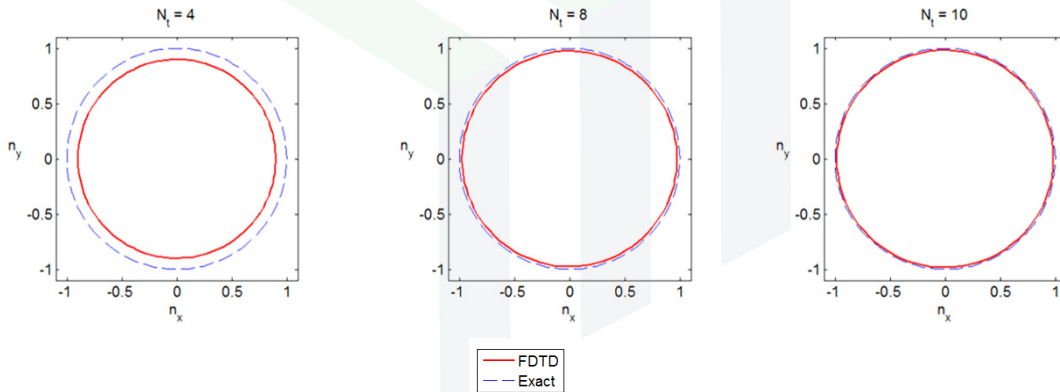
Notes:

- Grid dispersion error accumulates the farther a wave propagates.
- Grid dispersion will be more severe on larger grids.
- This is yet another consideration for grid resolution → size of the grid

Impact of the Time Step

The time step imposes an isotropic dispersion.

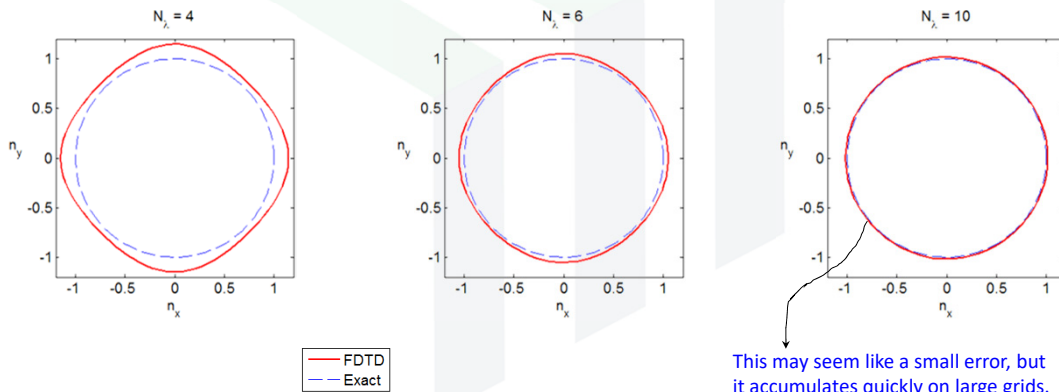
$$\Delta t = \frac{1}{N_t f_0} \quad N_t \equiv \text{number of time steps per wave period}$$



Impact of the Grid Resolution

The grid imposes an anisotropic dispersion (spatial dispersion).

$$\Delta = \frac{\lambda_0}{N_\lambda} \quad N_\lambda \equiv \text{number of grid points per wavelength}$$



Methods for Mitigating Grid Dispersion

- Increase grid resolution (i.e. decrease Δz)
- Adjust numerical phase velocity
- Use a hexagonal grid
- Use higher-order accurate derivatives
- Use discrete Fourier transforms to calculate derivatives
- Adopt more advanced algorithms like M24.

Adjusting Numerical Phase Velocity (1 of 3)

The numerical phase velocity of a wave in 1D-FDTD is

$$\tilde{v}_p = \frac{\omega\Delta z/2}{\sin^{-1}\left[\left(\frac{n}{c_0}\right)\frac{\Delta z}{\Delta t}\sin\left(\frac{\omega\Delta t}{2}\right)\right]} \neq \frac{c_0}{n}$$

The phase velocity of a wave in FDTD is different than a physical wave due to numerical artifacts of the Yee grid.

We can force $v_p = c_0/n$ by introducing a correction factor f .

$$\tilde{v}_p = \frac{c_0}{n} = \frac{\omega\Delta z/2}{\sin^{-1}\left[\left(\frac{fn}{c_0}\right)\frac{\Delta z}{\Delta t}\sin\left(\frac{\omega\Delta t}{2}\right)\right]}$$

The correction factor f multiplies the refractive index so as to adjust the phase velocity in the most straightforward manner possible.

Adjusting Numerical Phase Velocity (2 of 3)

We derive an equation for the correct factor by solving this equation for f

$$f = \frac{c_0 \Delta t \sin(k_0 n \Delta z / 2)}{n \Delta z \sin(k_0 c_0 \Delta t / 2)}$$

In practice, the grid is not homogeneous and the refractive index is set to some average value.

$$f = \frac{c_0 \Delta t \sin(k_0 n_{\text{avg}} \Delta z / 2)}{n_{\text{avg}} \Delta z \sin(k_0 c_0 \Delta t / 2)}$$

f will typically be just slightly less than 1.0.

Adjusting Numerical Phase Velocity (3 of 3)

Grid dispersion can be compensated by adjusting the refractive index across the grid according to the correction factor f . This is accomplished by modifying the permittivity and permeability functions as follows.

$$\begin{aligned} \tilde{\epsilon}_r &= f \epsilon_r \\ \tilde{\mu}_r &= f \mu_r \end{aligned}$$

We can only perfectly compensate for dispersion at one frequency, in one refractive index, and in one direction.

In practice, we compensate at the center frequency and for some average refractive index. We also choose the dominant direction that waves travel or we can just average the correction factor over all angles or compensate at an angle of 22.5°.

Example of Adjusting Numerical Phase Velocity

Suppose you are constructing a 1D FDTD simulation of a device in air operating at around 1.5 GHz. You have calculated your FDTD parameters to be

$$\Delta z = 3.0 \text{ cm}$$

$$\Delta t = 50 \text{ ps}$$

$$k_0 = \frac{2\pi f}{c_0} = 20.9585 \text{ m}^{-1}$$

Compensate for numerical dispersion.

$$f = \frac{c_0 \Delta t \sin(k_0 n \Delta z / 2)}{n \Delta z \sin(c_0 k_0 \Delta t / 2)} = \frac{(299792458 \frac{\text{m}}{\text{s}})(5 \times 10^{-11} \text{ s})}{(1.0)(0.03 \text{ m})} \frac{\sin[0.5(20.9585 \text{ m}^{-1})(1.0)(0.03 \text{ m})]}{\sin[0.5(299792458 \frac{\text{m}}{\text{s}})(20.9585 \text{ m}^{-1})(5 \times 10^{-11} \text{ s})]} = 0.9877$$

```
% COMPENSATE FOR NUMERICAL DISPERSION
```

```
f = c0*dt/(n*dz) * sin(k0*n*dz/2)/sin(c0*k0*dt/2);
```

```
UR = f*UR;
```

```
ER = f*ER;
```

Conclusions About Numerical Dispersion

- Waves on a grid propagate slower than a physical wave
- The time step imposes isotropic dispersion
- The grid imposes anisotropic dispersion
- Primary effect is to push the spectral response to lower frequencies (longer wavelengths)
- We can make the numerical dispersion arbitrarily small by increasing the grid resolution and decreasing the time step.
- We can compensate for numerical dispersion by adjusting the free space permittivity and permeability

Incorporating Loss

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Maxwell's Equations

An easy way to account for absorption loss in FDTD is through the material conductivity σ . Start with Maxwell's equations in the following form:

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \cdot \vec{D} = \rho_v$$

$$\vec{D}(t) = \epsilon_0 \epsilon_r \vec{E}(t)$$

$$\vec{B}(t) = \mu_0 \mu_r \vec{H}(t)$$

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Maxwell's Equations with Conductivity

The current density \vec{J} is related to conductivity σ and the electric field \vec{E} through:

$$\vec{J} = \sigma \vec{E} \quad \text{This is essentially Ohm's law for electromagnetics.}$$

Substituting this and the constitutive relations into Maxwell's curl equations yields

$$\nabla \times \vec{H} = \sigma \vec{E} + \epsilon_0 \epsilon_r \frac{\partial \vec{E}}{\partial t}$$

$$\nabla \times \vec{E} = -\mu_0 \mu_r \frac{\partial \vec{H}}{\partial t}$$

Normalized Maxwell's Equations

Just like before, the magnetic field is normalized according to:

$$\vec{\tilde{H}} = \sqrt{\frac{\mu_0}{\epsilon_0}} \vec{H}$$

Substituting this into Maxwell's curl equations leads to

$$\nabla \times \vec{\tilde{H}} = \eta_0 \sigma \vec{E} + \frac{\epsilon_r}{c_0} \frac{\partial \vec{E}}{\partial t}$$

$$\nabla \times \vec{E} = -\frac{\mu_r}{c_0} \frac{\partial \vec{\tilde{H}}}{\partial t}$$

Expansion of Curl-H Equation

Only one of the curl equations has changed so it is only necessary to derive new equations for the first curl equation.

$$\nabla \times \vec{H} = \eta_0 \sigma \vec{E} + \frac{\epsilon_r}{c_0} \frac{\partial \vec{E}}{\partial t}$$

This vector equation can be expanded into three scalar equations.

$$\begin{aligned} \frac{\partial \tilde{H}_z}{\partial y} - \frac{\partial \tilde{H}_y}{\partial z} &= \eta_0 \sigma_{xx} E_x + \frac{\epsilon_{xx}}{c_0} \frac{\partial E_x}{\partial t} \\ \frac{\partial \tilde{H}_x}{\partial z} - \frac{\partial \tilde{H}_z}{\partial x} &= \eta_0 \sigma_{yy} E_y + \frac{\epsilon_{yy}}{c_0} \frac{\partial E_y}{\partial t} \\ \frac{\partial \tilde{H}_y}{\partial x} - \frac{\partial \tilde{H}_x}{\partial y} &= \eta_0 \sigma_{zz} E_z + \frac{\epsilon_{zz}}{c_0} \frac{\partial E_z}{\partial t} \end{aligned}$$

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Finite-Difference Approximation (1 of 2)

Start with the first partial differential equation.

$$\frac{\partial \tilde{H}_z}{\partial y} - \frac{\partial \tilde{H}_y}{\partial z} = \eta_0 \sigma_{xx} E_x + \frac{\epsilon_{xx}}{c_0} \frac{\partial E_x}{\partial t}$$

Approximating this with finite-differences on a Yee grid leads to:

$$\frac{\tilde{H}_z \Big|_{t+\frac{\Delta t}{2}}^{i,j,k} - \tilde{H}_z \Big|_{t+\frac{\Delta t}{2}}^{i,j-1,k}}{\Delta y} - \frac{\tilde{H}_y \Big|_{t+\frac{\Delta t}{2}}^{i,j,k} - \tilde{H}_y \Big|_{t+\frac{\Delta t}{2}}^{i,j,k-1}}{\Delta z} = \eta_0 \left(\sigma_{xx} \Big|_{t+\frac{\Delta t}{2}}^{i,j,k} \right) E_x \Big|_{t+\frac{\Delta t}{2}}^{i,j,k} + \frac{\epsilon_{xx} \Big|_{t+\frac{\Delta t}{2}}^{i,j,k}}{c_0} \frac{E_x \Big|_{t+\Delta t}^{i,j,k} - E_x \Big|_t^{i,j,k}}{\Delta t}$$

What time step should be used here?

$$E_x \Big|_{t+\Delta t}^{i,j,k} \text{ or } E_x \Big|_t^{i,j,k} \text{ or ?}$$

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Finite-Difference Approximation (2 of 2)

REMEMBER: All terms in a finite-difference equation MUST exist at the same point in time and space.

Here, each term must exist at $t + \Delta t/2$, but we only have \vec{E} stored at t or $t + \Delta t$.

\vec{E} must be interpolated at $t + \Delta t/2$.

$$E_x \Big|_{t+\frac{\Delta t}{2}} \cong \frac{E_x \Big|_{t+\Delta t} + E_x \Big|_t}{2}$$

The finite-difference equation is then

$$\frac{\tilde{H}_z \Big|_{t+\frac{\Delta t}{2}}^{i,j,k} - \tilde{H}_z \Big|_{t+\frac{\Delta t}{2}}^{i,j-1,k}}{\Delta y} - \frac{\tilde{H}_y \Big|_{t+\frac{\Delta t}{2}}^{i,j,k} - \tilde{H}_y \Big|_{t+\frac{\Delta t}{2}}^{i,j,k-1}}{\Delta z} = \eta_0 \sigma_{xx} \Big|_{t+\frac{\Delta t}{2}}^{i,j,k} \frac{E_x \Big|_{t+\Delta t}^{i,j,k} + E_x \Big|_t^{i,j,k}}{2} + \frac{\epsilon_{xx} \Big|_{t+\frac{\Delta t}{2}}^{i,j,k}}{c_0} \frac{E_x \Big|_{t+\Delta t}^{i,j,k} - E_x \Big|_t^{i,j,k}}{\Delta t}$$

Derivation of the Update Equation (1 of 3)

Now there is a finite-difference equation with two occurrences of E_x at the future time step. The algebra is more involved, but the equation is still just solved for E_x at the future time step.

$$\frac{\tilde{H}_z \Big|_{t+\frac{\Delta t}{2}}^{i,j,k} - \tilde{H}_z \Big|_{t+\frac{\Delta t}{2}}^{i,j-1,k}}{\Delta y} - \frac{\tilde{H}_y \Big|_{t+\frac{\Delta t}{2}}^{i,j,k} - \tilde{H}_y \Big|_{t+\frac{\Delta t}{2}}^{i,j,k-1}}{\Delta z} = \eta_0 \sigma_{xx} \Big|_{t+\frac{\Delta t}{2}}^{i,j,k} \frac{E_x \Big|_{t+\Delta t}^{i,j,k} + E_x \Big|_t^{i,j,k}}{2} + \frac{\epsilon_{xx} \Big|_{t+\frac{\Delta t}{2}}^{i,j,k}}{c_0} \frac{E_x \Big|_{t+\Delta t}^{i,j,k} - E_x \Big|_t^{i,j,k}}{\Delta t}$$

Derivation of the Update Equation (2 of 3)

Expand equation and collect common terms on right side of equation.

$$\begin{aligned} \frac{\tilde{H}_z|_{t+\frac{\Delta t}{2}}^{i,j,k} - \tilde{H}_z|_{t+\frac{\Delta t}{2}}^{i,j-1,k}}{\Delta y} - \frac{\tilde{H}_y|_{t+\frac{\Delta t}{2}}^{i,j,k} - \tilde{H}_y|_{t+\frac{\Delta t}{2}}^{i,j,k-1}}{\Delta z} &= \eta_0 \sigma_{xx} \left|^{i,j,k} \frac{E_x|_{t+\Delta t}^{i,j,k} + E_x|_t^{i,j,k}}{2} + \frac{\varepsilon_{xx} \left|^{i,j,k} E_x|_{t+\Delta t}^{i,j,k} - E_x|_t^{i,j,k}}{c_0 \Delta t} \right. \right. \\ &= \frac{\eta_0 \sigma_{xx} \left|^{i,j,k}}{2} E_x|_{t+\Delta t}^{i,j,k} + \frac{\eta_0 \sigma_{xx} \left|^{i,j,k}}{2} E_x|_t^{i,j,k} + \frac{\varepsilon_{xx} \left|^{i,j,k}}{c_0 \Delta t} E_x|_{t+\Delta t}^{i,j,k} - \frac{\varepsilon_{xx} \left|^{i,j,k}}{c_0 \Delta t} E_x|_t^{i,j,k} \right. \right. \\ &= \left(\frac{\eta_0 \sigma_{xx} \left|^{i,j,k}}{2} + \frac{\varepsilon_{xx} \left|^{i,j,k}}{c_0 \Delta t} \right) E_x|_{t+\Delta t}^{i,j,k} + \left(\frac{\eta_0 \sigma_{xx} \left|^{i,j,k}}{2} - \frac{\varepsilon_{xx} \left|^{i,j,k}}{c_0 \Delta t} \right) E_x|_t^{i,j,k} \right. \right. \end{aligned}$$

Solve for E_x at the future time step.

$$\begin{aligned} \left(\frac{\varepsilon_{xx} \left|^{i,j,k}}{c_0 \Delta t} + \frac{\eta_0 \sigma_{xx} \left|^{i,j,k}}{2} \right) E_x|_{t+\Delta t}^{i,j,k} &= \left(\frac{\varepsilon_{xx} \left|^{i,j,k}}{c_0 \Delta t} - \frac{\eta_0 \sigma_{xx} \left|^{i,j,k}}{2} \right) E_x|_t^{i,j,k} + \left(\frac{\tilde{H}_z|_{t+\frac{\Delta t}{2}}^{i,j,k} - \tilde{H}_z|_{t+\frac{\Delta t}{2}}^{i,j-1,k}}{\Delta y} - \frac{\tilde{H}_y|_{t+\frac{\Delta t}{2}}^{i,j,k} - \tilde{H}_y|_{t+\frac{\Delta t}{2}}^{i,j,k-1}}{\Delta z} \right) \right. \\ E_x|_{t+\Delta t}^{i,j,k} &= \left(\frac{\varepsilon_{xx} \left|^{i,j,k}}{c_0 \Delta t} - \frac{\eta_0 \sigma_{xx} \left|^{i,j,k}}{2} \right) E_x|_t^{i,j,k} + \left(\frac{\tilde{H}_z|_{t+\frac{\Delta t}{2}}^{i,j,k} - \tilde{H}_z|_{t+\frac{\Delta t}{2}}^{i,j-1,k}}{\Delta y} - \frac{\tilde{H}_y|_{t+\frac{\Delta t}{2}}^{i,j,k} - \tilde{H}_y|_{t+\frac{\Delta t}{2}}^{i,j,k-1}}{\Delta z} \right) \right. \\ &= \left(\frac{\varepsilon_{xx} \left|^{i,j,k}}{c_0 \Delta t} + \frac{\eta_0 \sigma_{xx} \left|^{i,j,k}}{2} \right) E_x|_t^{i,j,k} + \left(\frac{1}{\frac{\varepsilon_{xx} \left|^{i,j,k}}{c_0 \Delta t} + \frac{\eta_0 \sigma_{xx} \left|^{i,j,k}}{2}} \right) \left(\frac{\tilde{H}_z|_{t+\frac{\Delta t}{2}}^{i,j,k} - \tilde{H}_z|_{t+\frac{\Delta t}{2}}^{i,j-1,k}}{\Delta y} - \frac{\tilde{H}_y|_{t+\frac{\Delta t}{2}}^{i,j,k} - \tilde{H}_y|_{t+\frac{\Delta t}{2}}^{i,j,k-1}}{\Delta z} \right) \right. \end{aligned}$$

Derivation of the Update Equation (3 of 3)

Rearrange the update coefficients

$$E_x|_{t+\Delta t}^{i,j,k} = \left(\frac{\varepsilon_{xx} \left|^{i,j,k}}{c_0 \Delta t} - \frac{\eta_0 \sigma_{xx} \left|^{i,j,k}}{2} \right) E_x|_t^{i,j,k} + \left(\frac{1}{\frac{\varepsilon_{xx} \left|^{i,j,k}}{c_0 \Delta t} + \frac{\eta_0 \sigma_{xx} \left|^{i,j,k}}{2}} \right) \left(\frac{\tilde{H}_z|_{t+\frac{\Delta t}{2}}^{i,j,k} - \tilde{H}_z|_{t+\frac{\Delta t}{2}}^{i,j-1,k}}{\Delta y} - \frac{\tilde{H}_y|_{t+\frac{\Delta t}{2}}^{i,j,k} - \tilde{H}_y|_{t+\frac{\Delta t}{2}}^{i,j,k-1}}{\Delta z} \right) \right.$$

$$E_x|_{t+\Delta t}^{i,j,k} = (m_{Ex1} \left|^{i,j,k}) E_x|_t^{i,j,k} + (m_{Ex2} \left|^{i,j,k}) \left(\frac{\tilde{H}_z|_{t+\frac{\Delta t}{2}}^{i,j,k} - \tilde{H}_z|_{t+\frac{\Delta t}{2}}^{i,j-1,k}}{\Delta y} - \frac{\tilde{H}_y|_{t+\frac{\Delta t}{2}}^{i,j,k} - \tilde{H}_y|_{t+\frac{\Delta t}{2}}^{i,j,k-1}}{\Delta z} \right)$$

$$m_{Ex1} \left|^{i,j,k} = \frac{\frac{\varepsilon_{xx} \left|^{i,j,k}}{c_0 \Delta t} - \frac{\eta_0 \sigma_{xx} \left|^{i,j,k}}{2}}{\frac{\varepsilon_{xx} \left|^{i,j,k}}{c_0 \Delta t} + \frac{\eta_0 \sigma_{xx} \left|^{i,j,k}}{2}} = \frac{2\varepsilon_0 \varepsilon_{xx} \left|^{i,j,k} - \Delta t \sigma_{xx} \left|^{i,j,k}}{2\varepsilon_0 \varepsilon_{xx} \left|^{i,j,k} + \Delta t \sigma_{xx} \left|^{i,j,k}} \right.$$

The update coefficients have gotten big and ugly!

$$m_{Ex2} \left|^{i,j,k} = \frac{1}{\frac{\varepsilon_{xx} \left|^{i,j,k}}{c_0 \Delta t} + \frac{\eta_0 \sigma_{xx} \left|^{i,j,k}}{2}} = \frac{2\varepsilon_0 c_0 \Delta t}{2\varepsilon_0 \varepsilon_{xx} \left|^{i,j,k} + \Delta t \sigma_{xx} \left|^{i,j,k}} \right.$$

These will get even worse later. ☺

Confirmation We Are Probably Correct

A good test to see if there is a mistake is to set the conductivity to zero and see if the update equation reduces to the standard update equation.

$$E_x|_{t+\Delta t}^{i,j,k} = \left(m_{Ex1}|^{i,j,k} \right) E_x|_t^{i,j,k} + \left(m_{Ex2}|^{i,j,k} \right) \left(\frac{\tilde{H}_z|_{t+\frac{\Delta t}{2}}^{i,j,k} - \tilde{H}_z|_{t+\frac{\Delta t}{2}}^{i,j-1,k}}{\Delta y} - \frac{\tilde{H}_y|_{t+\frac{\Delta t}{2}}^{i,j,k} - \tilde{H}_y|_{t+\frac{\Delta t}{2}}^{i,j,k-1}}{\Delta z} \right)$$

$$m_{Ex1}|^{i,j,k} = \frac{2\varepsilon_0 \varepsilon_{xx}|^{i,j,k} - \cancel{\Delta t \sigma_{xx}|^{i,j,k}}}{2\varepsilon_0 \varepsilon_{xx}|^{i,j,k} + \cancel{\Delta t \sigma_{xx}|^{i,j,k}}} = 1$$

$$m_{Ex2}|^{i,j,k} = \frac{2\varepsilon_0 c_0 \Delta t}{2\varepsilon_0 \varepsilon_{xx}|^{i,j,k} + \cancel{\Delta t \sigma_{xx}|^{i,j,k}}} = \frac{c_0 \Delta t}{\varepsilon_{xx}|^{i,j,k}}$$



Frequency Dependent Materials

What are Frequency-Dependent Materials

The electromagnetic properties of all materials change with frequency due to the physics of the mechanisms producing electric and magnetic responses.

$$\epsilon_r(\omega) \Leftrightarrow \epsilon_r(t)$$

$$\mu_r(\omega) \Leftrightarrow \mu_r(t)$$

Frequency-Domain Maxwell's Equations

$$\nabla \times \vec{H}(\omega) = j\omega \vec{D}(\omega)$$

$$\nabla \times \vec{E}(\omega) = -j\omega \vec{B}(\omega)$$

$$\vec{D}(\omega) = \epsilon_0 \epsilon_r(\omega) \vec{E}(\omega)$$

$$\vec{B}(\omega) = \mu_0 \mu_r(\omega) \vec{H}(\omega)$$

Time-Domain Maxwell's Equations

$$\nabla \times \vec{H}(t) = \frac{\partial \vec{D}(t)}{\partial t}$$

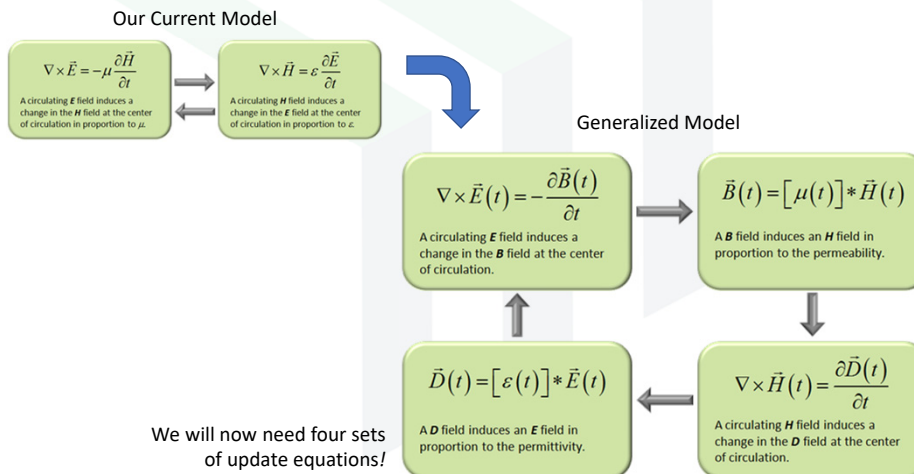
$$\nabla \times \vec{E}(t) = -\frac{\partial \vec{B}(t)}{\partial t}$$

$$\vec{D}(t) = \epsilon_0 \epsilon_r(t) * \vec{E}(t)$$

$$\vec{B}(t) = \mu_0 \mu_r(t) * \vec{H}(t)$$

Generalized Flow of Maxwell's Equations

As more complicated properties of materials are incorporated into an FDTD simulation, it makes more sense to generalize the flow of Maxwell's equations. This "compartmentalizes" the materials problem into a single and much simpler equation.



Lorentz Model for Dielectrics

EQUILIBRIUM STATE

POLARIZED STATE

Equation of Motion

$$m \frac{\partial^2 \vec{r}}{\partial t^2} + m\Gamma \frac{\partial \vec{r}}{\partial t} + m\omega_0^2 \vec{r} = -q\vec{E}$$

acceleration force $m \Rightarrow m_e$ mass of an electron
 frictional force Γ damping rate (loss/sec)
 restoring force $\omega_0 = \sqrt{\frac{k}{m}}$ natural frequency
 electric force

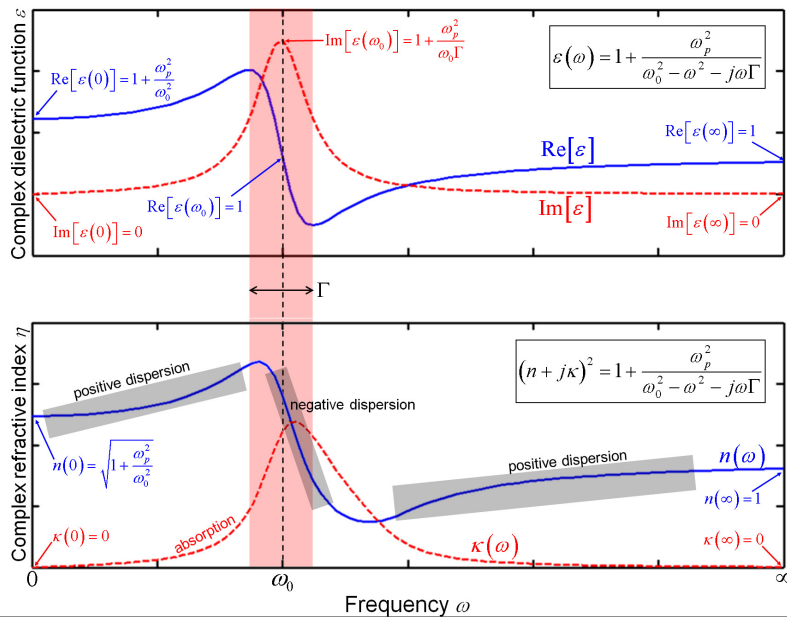
We can use the equation of motion to model the motion of electrons bound to an atom.

Lorentz Model

$$\epsilon_r(\omega) = 1 + \frac{\omega_p^2}{\omega_0^2 - \omega^2 + j\omega\Gamma}$$

$$\omega_p^2 = \frac{Nq^2}{\epsilon_0 m_e}$$

Typical Lorentz Response



Drude Model for Metals

The electrons in metals are not bound to a nucleus so there is no restoring force and hence no resonant frequency. The Lorentz model reduces to the Drude model when $\omega_0 = 0$.

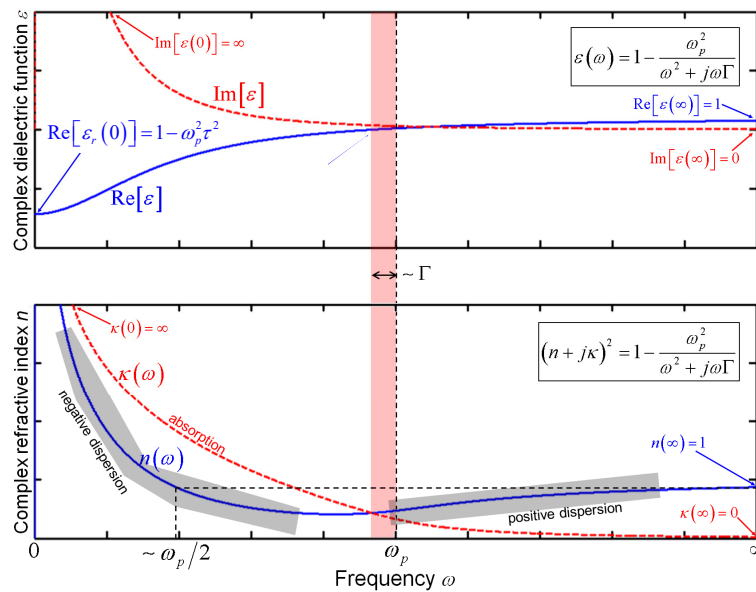
Drude Model

$$\epsilon_r(\omega) = 1 + \frac{\omega_p^2}{\cancel{\omega_0^2} - \omega^2 - j\omega\Gamma} = 1 - \frac{\omega_p^2}{\omega^2 + j\omega\Gamma}$$

$$\omega_p^2 = \frac{Nq^2}{\epsilon_0 m_e}$$

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Typical Drude Response



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Real Materials Have Multiple Resonances

At a macroscopic level, all resonance mechanisms can be characterized using the Lorentz model. This allows any number of resonances to be accounted for through a simple summation.

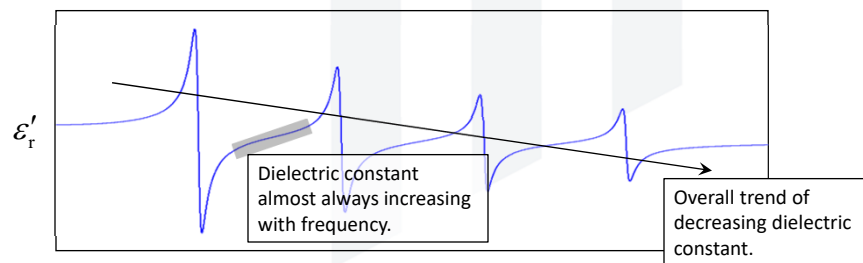
$$\epsilon_r(\omega) = 1 + \omega_p^2 \sum_{m=1}^M \frac{f_m}{\omega_{0,m}^2 - \omega^2 + j\omega\Gamma_m}$$

$M \equiv$ Number of resonators

$f_m \equiv$ Oscillator strength of the m^{th} resonator

$\omega_{0,m} \equiv$ Natural frequency of the m^{th} resonator

$\Gamma_m \equiv$ Damping rate of the m^{th} resonator



The Dielectric Constitutive Relation

\vec{D} and \vec{E} are related through the constitutive relation

$$\vec{D}(\omega) = \epsilon_0 \epsilon_r(\omega) \vec{E}(\omega)$$

This can also be written in terms of the material polarization \vec{P} .

$$\vec{D}(\omega) = \epsilon_0 \epsilon_\infty \vec{E}(\omega) + \vec{P}(\omega)$$

The material polarization \vec{P} can be written in terms of a sum of the generalized Lorentz-Drude model.

$$\vec{P}(\omega) = \omega_p^2 \vec{E}(\omega) \sum_{m=1}^M \frac{f_m}{\omega_{0,m}^2 - \omega^2 + j\omega\Gamma_m}$$

In practice, as few oscillators M as possible are used that still gives accurate material properties in the frequency range of interest.

Conversion to the Time-Domain

The polarization due a single Lorentz oscillator is

$$\vec{P}_m(\omega) = \frac{f_m \omega_p^2}{\omega_{0,m}^2 - \omega^2 + j\omega\Gamma_m} \vec{E}(\omega)$$

Multiplying both sides by the denominator on the right leads to

$$\begin{aligned} \omega_{0,m}^2 \vec{P}_m(\omega) - \omega^2 \vec{P}_m(\omega) + j\omega\Gamma_m \vec{P}_m(\omega) &= f_m \omega_p^2 \vec{E}(\omega) \\ \omega_{0,m}^2 \vec{P}_m(\omega) + (j\omega)^2 \vec{P}_m(\omega) + j\omega\Gamma_m \vec{P}_m(\omega) &= f_m \omega_p^2 \vec{E}(\omega) \end{aligned}$$

This equation is then converted to the time domain

$$\begin{aligned} \omega_{0,m}^2 \vec{P}_m(t) + \frac{\partial^2}{\partial t^2} \vec{P}_m(t) + \frac{\partial}{\partial t} \Gamma_m \vec{P}_m(t) &= f_m \omega_p^2 \vec{E}(t) \\ \frac{\partial^2}{\partial t^2} \vec{P}_m(t) + \frac{\partial}{\partial t} \Gamma_m \vec{P}_m(t) + \omega_{0,m}^2 \vec{P}_m(t) &= f_m \omega_p^2 \vec{E}(t) \end{aligned}$$

Governing Equations

It is best to not work with a second-order time derivatives so a polarization current \vec{J} (called displacement current in EM texts) is defined.

$$\vec{J}_m(t) = \frac{\partial}{\partial t} \vec{P}_m(t)$$

Now there is a system of two equations (governing equations) that only contain first-order time derivatives.

$$\begin{aligned} \frac{\partial}{\partial t} \vec{J}_m(t) + \Gamma_m \vec{J}_m(t) + \omega_{0,m}^2 \vec{P}_m(t) &= f_m \omega_p^2 \vec{E}(t) \\ \vec{J}_m(t) &= \frac{\partial}{\partial t} \vec{P}_m(t) \end{aligned}$$

Update Equations for \vec{P}

Update equations can now be derived for \vec{J} and \vec{P} that will need to be updated at every point in the grid where that material exists.

Observe that \vec{J} and \vec{P} are staggered in time just like was done for \vec{E} and \vec{H} . It is \vec{P} that is directly combined with \vec{E} , so let \vec{P} exist at the integer time steps and \vec{J} exist at the half-integer time steps.

$$\vec{J}_m(t) = \frac{\partial}{\partial t} \vec{P}_m(t) \rightarrow \begin{aligned} P_{x,m} \Big|_{t+\Delta t}^{i,j,k} &= P_{x,m} \Big|_t^{i,j,k} + \Delta t J_{x,m} \Big|_{t+\frac{\Delta t}{2}}^{i,j,k} \\ P_{y,m} \Big|_{t+\Delta t}^{i,j,k} &= P_{y,m} \Big|_t^{i,j,k} + \Delta t J_{y,m} \Big|_{t+\frac{\Delta t}{2}}^{i,j,k} \\ P_{z,m} \Big|_{t+\Delta t}^{i,j,k} &= P_{z,m} \Big|_t^{i,j,k} + \Delta t J_{z,m} \Big|_{t+\frac{\Delta t}{2}}^{i,j,k} \end{aligned}$$

Update Equations for \vec{J}

Start with the governing equation.

$$\frac{\partial}{\partial t} \vec{J}_m(t) + \Gamma_m \vec{J}_m(t) + \omega_{0,m}^2 \vec{P}_m(t) = f_m \omega_p^2 \vec{E}(t)$$

Approximate with finite-differences

$$\begin{aligned} \frac{J_{x,m} \Big|_{t+\frac{\Delta t}{2}}^{i,j,k} - J_{x,m} \Big|_{t-\frac{\Delta t}{2}}^{i,j,k}}{\Delta t} + \left(\Gamma_m \Big|^{i,j,k} \right) \frac{J_{x,m} \Big|_{t+\frac{\Delta t}{2}}^{i,j,k} + J_{x,m} \Big|_{t-\frac{\Delta t}{2}}^{i,j,k}}{2} + \left(\omega_{0,m}^2 \Big|^{i,j,k} \right) P_{x,m} \Big|_t^{i,j,k} \\ = \left(f_m \Big|^{i,j,k} \right) \left(\omega_p^2 \Big|^{i,j,k} \right) E_{x,m} \Big|_t^{i,j,k} \end{aligned}$$

Solve for the future value of \vec{J} .

$$J_{x,m} \Big|_{t+\frac{\Delta t}{2}}^{i,j,k} = \left[\frac{2 - \left(\Gamma_m \Big|^{i,j,k} \right) \Delta t}{2 + \left(\Gamma_m \Big|^{i,j,k} \right) \Delta t} \right] J_{x,m} \Big|_{t-\frac{\Delta t}{2}}^{i,j,k} - \left[\frac{2 \left(\omega_{0,m}^2 \Big|^{i,j,k} \right) \Delta t}{2 + \left(\Gamma_m \Big|^{i,j,k} \right) \Delta t} \right] P_{x,m} \Big|_t^{i,j,k} + \left[\frac{2 \left(f_m \Big|^{i,j,k} \right) \Delta t}{2 + \left(\Gamma_m \Big|^{i,j,k} \right) \Delta t} \right] \left(\omega_p^2 \Big|^{i,j,k} \right) E_{x,m} \Big|_t^{i,j,k}$$

Update Equations for \vec{E}

The constitutive relation for multiple resonances was

$$\vec{D}(t) = \epsilon_0 \epsilon_\infty \vec{E}(t) + \sum_{m=1}^M \vec{P}_m(t)$$

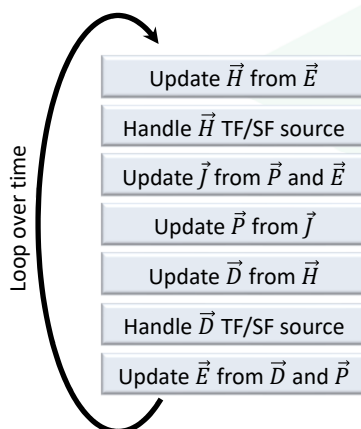
Approximating this numerically leads to

$$D_{x,m}|_t^{i,j,k} = \epsilon_0 \left(\epsilon_\infty |^{i,j,k} \right) E_x|_t^{i,j,k} + \sum_{m=1}^M P_{x,m}|_t^{i,j,k}$$

Solving this for E_x at the future time step gives the update equation.

$$E_x|_t^{i,j,k} = \left[\frac{1}{\epsilon_0 \left(\epsilon_\infty |^{i,j,k} \right)} \right] \left(D_{x,m}|_t^{i,j,k} - \sum_{m=1}^M P_{x,m}|_t^{i,j,k} \right)$$

Revised Block Diagram of Main FDTD Loop for Dispersive Dielectric Materials



$$J_{x,m}|_{t+\frac{\Delta t}{2}}^{i,j,k} = m_{Jx1}|^{i,j,k} J_{x,m}|_{t-\frac{\Delta t}{2}}^{i,j,k} - m_{Jx2}|^{i,j,k} P_{x,m}|_t^{i,j,k} + m_{Jx3}|^{i,j,k} E_{x,m}|_t^{i,j,k}$$

$$P_{x,m}|_{t+\Delta t}^{i,j,k} = P_{x,m}|_t^{i,j,k} + \Delta t J_{x,m}|_{t+\frac{\Delta t}{2}}^{i,j,k}$$

$$E_x|_t^{i,j,k} = \left[\frac{1}{\epsilon_0 \left(\epsilon_\infty |^{i,j,k} \right)} \right] \left(D_{x,m}|_t^{i,j,k} - \sum_{m=1}^M P_{x,m}|_t^{i,j,k} \right)$$

Note: We could do the same with $B=\mu H$ to handle dispersive magnetic materials.

1D Anisotropic FDTD

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Starting Point

Start with Maxwell's equations in the following form

$$\begin{aligned}\nabla \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} & \nabla \cdot \vec{D} &= \rho_v \\ \nabla \times \vec{H} &= \vec{J} + \frac{\partial \vec{D}}{\partial t} & \nabla \cdot \vec{B} &= 0\end{aligned}$$

and the constitutive relations for anisotropic media.

$$\begin{aligned}\vec{D} &= [\varepsilon] \vec{E} \\ \vec{B} &= [\mu] \vec{H}\end{aligned}$$

Assuming no sources (i.e. $\rho_v = 0$ and $\vec{J} = 0$), the equations above become

$$\begin{aligned}\nabla \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} & \nabla \cdot \vec{D} &= 0 & \vec{D} &= [\varepsilon] \vec{E} \\ \nabla \times \vec{H} &= \frac{\partial \vec{D}}{\partial t} & \nabla \cdot \vec{B} &= 0 & \vec{B} &= [\mu] \vec{H}\end{aligned}$$

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Normalize the Fields

Normalize the fields according to

$$\vec{\tilde{E}} = \frac{1}{\eta_0} \vec{E} \quad \vec{\tilde{D}} = c_0 \vec{D} \quad \vec{\tilde{B}} = \frac{1}{\mu_0} \vec{B}$$

The governing equations become

$$\begin{aligned} \nabla \times \vec{\tilde{E}} &= -\frac{1}{c_0} \frac{\partial \vec{\tilde{B}}}{\partial t} & \nabla \cdot \vec{\tilde{D}} &= 0 & \vec{\tilde{D}} &= [\epsilon_r] \vec{\tilde{E}} \\ \nabla \times \vec{\tilde{H}} &= \frac{1}{c_0} \frac{\partial \vec{\tilde{D}}}{\partial t} & \nabla \cdot \vec{\tilde{B}} &= 0 & \vec{\tilde{B}} &= [\mu_r] \vec{\tilde{H}} \end{aligned}$$

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Expansion into Cartesian Coordinates

Expand Maxwell's curl equations to get

$$\begin{aligned} \nabla \times \vec{\tilde{E}} = -\frac{1}{c_0} \frac{\partial \vec{\tilde{B}}}{\partial t} &\rightarrow \begin{aligned} \frac{\partial \tilde{E}_z}{\partial y} - \frac{\partial \tilde{E}_y}{\partial z} &= -\frac{1}{c_0} \frac{\partial \tilde{B}_x}{\partial t} \\ \frac{\partial \tilde{E}_x}{\partial z} - \frac{\partial \tilde{E}_z}{\partial x} &= -\frac{1}{c_0} \frac{\partial \tilde{B}_y}{\partial t} \\ \frac{\partial \tilde{E}_y}{\partial x} - \frac{\partial \tilde{E}_x}{\partial y} &= -\frac{1}{c_0} \frac{\partial \tilde{B}_z}{\partial t} \end{aligned} \\ \nabla \times \vec{\tilde{H}} = \frac{1}{c_0} \frac{\partial \vec{\tilde{D}}}{\partial t} &\rightarrow \begin{aligned} \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} &= \frac{1}{c_0} \frac{\partial \tilde{D}_x}{\partial t} \\ \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} &= \frac{1}{c_0} \frac{\partial \tilde{D}_y}{\partial t} \\ \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} &= \frac{1}{c_0} \frac{\partial \tilde{D}_z}{\partial t} \end{aligned} \end{aligned}$$

and the constitutive relations expand to

$$\begin{aligned} \tilde{D}_x &= \epsilon_{xx} \tilde{E}_x + \epsilon_{xy} \tilde{E}_y + \epsilon_{xz} \tilde{E}_z & \tilde{B}_x &= \mu_{xx} H_x + \mu_{xy} H_y + \mu_{xz} H_z \\ \tilde{D}_y &= \epsilon_{yx} \tilde{E}_x + \epsilon_{yy} \tilde{E}_y + \epsilon_{yz} \tilde{E}_z & \tilde{B}_y &= \mu_{yx} H_x + \mu_{yy} H_y + \mu_{yz} H_z \\ \tilde{D}_z &= \epsilon_{zx} \tilde{E}_x + \epsilon_{zy} \tilde{E}_y + \epsilon_{zz} \tilde{E}_z & \tilde{B}_z &= \mu_{zx} H_x + \mu_{zy} H_y + \mu_{zz} H_z \end{aligned}$$

These equations are not easily solved for $\vec{\tilde{E}}$ and $\vec{\tilde{H}}$!!

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Constitutive Relations

The constitutive relations must be solved for $\vec{\tilde{E}}$ and $\vec{\tilde{H}}$.

This is most easily accomplished using the *impermissivity* $[\psi]$ and *impermeability* $[\xi]$ tensors.

$$\vec{\tilde{D}} = [\epsilon_r] \vec{\tilde{E}} \rightarrow \vec{\tilde{E}} = [\psi] \vec{\tilde{D}} \quad \vec{\tilde{B}} = [\mu_r] \vec{\tilde{H}} \rightarrow \vec{\tilde{H}} = [\xi] \vec{\tilde{B}}$$

$$[\psi] = [\epsilon_r]^{-1} \quad [\xi] = [\mu_r]^{-1}$$

These new equations expand to

$$\begin{aligned} \tilde{E}_x &= \psi_{xx} \tilde{D}_x + \psi_{xy} \tilde{D}_y + \psi_{xz} \tilde{D}_z & H_x &= \xi_{xx} \tilde{B}_x + \xi_{xy} \tilde{B}_y + \xi_{xz} \tilde{B}_z \\ \tilde{E}_y &= \psi_{yx} \tilde{D}_x + \psi_{yy} \tilde{D}_y + \psi_{yz} \tilde{D}_z & H_y &= \xi_{yx} \tilde{B}_x + \xi_{yy} \tilde{B}_y + \xi_{yz} \tilde{B}_z \\ \tilde{E}_z &= \psi_{zx} \tilde{D}_x + \psi_{zy} \tilde{D}_y + \psi_{zz} \tilde{D}_z & H_z &= \xi_{zx} \tilde{B}_x + \xi_{zy} \tilde{B}_y + \xi_{zz} \tilde{B}_z \end{aligned}$$

Calculating the Impermittivity Tensor $[\psi]$

The impermittivity tensor $[\psi]$ cannot be calculated by just inverting the individual tensor elements of the permittivity. That is

$$\psi_{ij} \neq \frac{1}{\epsilon_{ij}}$$

Instead, the entire impermittivity tensor $[\psi]$ is calculated by inverting the entire permittivity tensor $[\epsilon]$.

$$\begin{bmatrix} \psi_{xx} & \psi_{xy} & \psi_{xz} \\ \psi_{yx} & \psi_{yy} & \psi_{yz} \\ \psi_{zx} & \psi_{zy} & \psi_{zz} \end{bmatrix} = \begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{yx} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz} \end{bmatrix}^{-1}$$

```
% CALCULATE IMPERMITTIVITY TENSOR ON 2X GRID
Z = zeros(Nx2,Ny2,Nz2);
Y2xx = Z; Y2xy = Z; Y2xz = Z;
Y2yx = Z; Y2yy = Z; Y2yz = Z;
Y2zx = Z; Y2zy = Z; Y2zz = Z;
for m = 1 : Nx2*Ny2*Nz2
    er = [ ER2xx(m) ER2xy(m) ER2xz(m) ; ...
           ER2yx(m) ER2yy(m) ER2yz(m) ; ...
           ER2zx(m) ER2zy(m) ER2zz(m) ];
    y = inv(er);
    Y2xx(m) = y(1,1); Y2xy(m) = y(1,2); Y2xz(m) = y(1,3);
    Y2yx(m) = y(2,1); Y2yy(m) = y(2,2); Y2yz(m) = y(2,3);
    Y2zx(m) = y(3,1); Y2zy(m) = y(3,2); Y2zz(m) = y(3,3);
end
```

Finite-Difference Approximation of Curl Equations

$$\begin{aligned}
 \frac{\partial \tilde{E}_z}{\partial y} - \frac{\partial \tilde{E}_y}{\partial z} &= \frac{1}{c_0} \frac{\partial \tilde{B}_x}{\partial t} & \frac{\tilde{E}_z^{i,j+1,k}|_l - \tilde{E}_z^{i,j,k}|_l}{\Delta y} - \frac{\tilde{E}_y^{i,j,k+1}|_l - \tilde{E}_y^{i,j,k}|_l}{\Delta z} &= \frac{1}{c_0} \frac{\tilde{B}_x^{i,j,k}|_{l+\frac{\Delta t}{2}} - \tilde{B}_x^{i,j,k}|_{l-\frac{\Delta t}{2}}}{\Delta t} \\
 \frac{\partial \tilde{E}_x}{\partial z} - \frac{\partial \tilde{E}_z}{\partial x} &= \frac{1}{c_0} \frac{\partial \tilde{B}_y}{\partial t} & \frac{\tilde{E}_x^{i,j,k+1}|_l - \tilde{E}_x^{i,j,k}|_l}{\Delta z} - \frac{\tilde{E}_z^{i+1,j,k}|_l - \tilde{E}_z^{i,j,k}|_l}{\Delta x} &= \frac{1}{c_0} \frac{\tilde{B}_y^{i,j,k}|_{l+\frac{\Delta t}{2}} - \tilde{B}_y^{i,j,k}|_{l-\frac{\Delta t}{2}}}{\Delta t} \\
 \frac{\partial \tilde{E}_y}{\partial x} - \frac{\partial \tilde{E}_x}{\partial y} &= \frac{1}{c_0} \frac{\partial \tilde{B}_z}{\partial t} & \frac{\tilde{E}_y^{i+1,j,k}|_l - \tilde{E}_y^{i,j,k}|_l}{\Delta x} - \frac{\tilde{E}_x^{i,j+1,k}|_l - \tilde{E}_x^{i,j,k}|_l}{\Delta y} &= \frac{1}{c_0} \frac{\tilde{B}_z^{i,j,k}|_{l+\frac{\Delta t}{2}} - \tilde{B}_z^{i,j,k}|_{l-\frac{\Delta t}{2}}}{\Delta t}
 \end{aligned}$$

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$$\begin{aligned}
 \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} &= \frac{1}{c_0} \frac{\partial \tilde{D}_x}{\partial t} & \frac{H_z^{i,j,k}|_{l+\frac{\Delta t}{2}} - H_z^{i,j-1,k}|_{l+\frac{\Delta t}{2}}}{\Delta y} - \frac{H_y^{i,j,k}|_{l+\frac{\Delta t}{2}} - H_y^{i,j,k-1}|_{l+\frac{\Delta t}{2}}}{\Delta z} &= \frac{1}{c_0} \frac{\tilde{D}_x^{i,j,k}|_{l+\Delta t} - \tilde{D}_x^{i,j,k}|_l}{\Delta t} \\
 \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} &= \frac{1}{c_0} \frac{\partial \tilde{D}_y}{\partial t} & \frac{H_x^{i,j,k}|_{l+\frac{\Delta t}{2}} - H_x^{i,j,k-1}|_{l+\frac{\Delta t}{2}}}{\Delta z} - \frac{H_z^{i,j,k}|_{l+\frac{\Delta t}{2}} - H_z^{i-1,j,k}|_{l+\frac{\Delta t}{2}}}{\Delta x} &= \frac{1}{c_0} \frac{\tilde{D}_y^{i,j,k}|_{l+\Delta t} - \tilde{D}_y^{i,j,k}|_l}{\Delta t} \\
 \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} &= \frac{1}{c_0} \frac{\partial \tilde{D}_z}{\partial t} & \frac{H_y^{i,j,k}|_{l+\frac{\Delta t}{2}} - H_y^{i-1,j,k}|_{l+\frac{\Delta t}{2}}}{\Delta x} - \frac{H_x^{i,j,k}|_{l+\frac{\Delta t}{2}} - H_x^{i,j-1,k}|_{l+\frac{\Delta t}{2}}}{\Delta y} &= \frac{1}{c_0} \frac{\tilde{D}_z^{i,j,k}|_{l+\Delta t} - \tilde{D}_z^{i,j,k}|_l}{\Delta t}
 \end{aligned}$$

Finite-Difference Approximation of Constitutive Relations (1 of 2)

$$\begin{aligned}
 \tilde{E}_x^{i,j,k} &= \frac{\psi_{xx}^{i,j,k} \tilde{D}_x^{i,j,k} + \psi_{xy}^{i+1,j-1,k} \tilde{D}_y^{i+1,j-1,k} + \psi_{xy}^{i,j-1,k} \tilde{D}_y^{i,j-1,k} + \psi_{xy}^{i+1,j,k} \tilde{D}_y^{i+1,j,k} + \psi_{xy}^{i,j,k} \tilde{D}_y^{i,j,k}}{4} \\
 &+ \frac{\psi_{xz}^{i+1,j,k} \tilde{D}_z^{i+1,j,k} + \psi_{xz}^{i+1,j,k-1} \tilde{D}_z^{i+1,j,k-1} + \psi_{xz}^{i,j,k} \tilde{D}_z^{i,j,k} + \psi_{xz}^{i,j,k-1} \tilde{D}_z^{i,j,k-1}}{4} \\
 \tilde{E}_y^{i,j,k} &= \frac{\psi_{yx}^{i,j,k} \tilde{D}_x^{i,j,k} + \psi_{yx}^{i-1,j,k} \tilde{D}_x^{i-1,j,k} + \psi_{yx}^{i,j+1,k} \tilde{D}_x^{i,j+1,k} + \psi_{yx}^{i-1,j+1,k} \tilde{D}_x^{i-1,j+1,k}}{4} \\
 &+ \frac{\psi_{yy}^{i,j,k} \tilde{D}_y^{i,j,k} + \psi_{yz}^{i,j,k} \tilde{D}_z^{i,j,k} + \psi_{yz}^{i,j,k-1} \tilde{D}_z^{i,j,k-1} + \psi_{yz}^{i,j+1,k} \tilde{D}_z^{i,j+1,k} + \psi_{yz}^{i,j+1,k-1} \tilde{D}_z^{i,j+1,k-1}}{4} \\
 \tilde{E}_z^{i,j,k} &= \frac{\psi_{zx}^{i,j,k} \tilde{D}_x^{i,j,k} + \psi_{zx}^{i,j,k+1} \tilde{D}_x^{i,j,k+1} + \psi_{zx}^{i-1,j,k} \tilde{D}_x^{i-1,j,k} + \psi_{zx}^{i-1,j,k+1} \tilde{D}_x^{i-1,j,k+1}}{4} \\
 &+ \frac{\psi_{zy}^{i,j,k} \tilde{D}_y^{i,j,k} + \psi_{zy}^{i,j,k+1} \tilde{D}_y^{i,j,k+1} + \psi_{zy}^{i,j-1,k} \tilde{D}_y^{i,j-1,k} + \psi_{zy}^{i,j-1,k+1} \tilde{D}_y^{i,j-1,k+1}}{4} \\
 &+ \psi_{zz}^{i,j,k} \tilde{D}_z^{i,j,k}
 \end{aligned}$$

Finite-Difference Approximation of Constitutive Relations (2 of 2)

$$\begin{aligned}
 H_x^{i,j,k} &= \frac{\zeta_{xx}^{i,j,k} \tilde{B}_x^{i,j,k}}{4} + \frac{\zeta_{xy}^{i,j,k} \tilde{B}_y^{i,j,k} + \zeta_{xy}^{i-1,j,k} \tilde{B}_y^{i-1,j,k} + \zeta_{xy}^{i,j+1,k} \tilde{B}_y^{i,j+1,k} + \zeta_{xy}^{i-1,j+1,k} \tilde{B}_y^{i-1,j+1,k}}{4} \\
 &\quad + \frac{\zeta_{xz}^{i,j,k} \tilde{B}_z^{i,j,k} + \zeta_{xz}^{i-1,j,k} \tilde{B}_z^{i-1,j,k} + \zeta_{xz}^{i,j,k+1} \tilde{B}_z^{i,j,k+1} + \zeta_{xz}^{i-1,j,k+1} \tilde{B}_z^{i-1,j,k+1}}{4} \\
 H_y^{i,j,k} &= \frac{\zeta_{yx}^{i,j,k} \tilde{B}_x^{i,j,k} + \zeta_{yx}^{i+1,j,k} \tilde{B}_x^{i+1,j,k} + \zeta_{yx}^{i,j-1,k} \tilde{B}_x^{i,j-1,k} + \zeta_{yx}^{i+1,j-1,k} \tilde{B}_x^{i+1,j-1,k}}{4} \\
 &\quad + \frac{\zeta_{yy}^{i,j,k} \tilde{B}_y^{i,j,k}}{4} + \frac{\zeta_{yz}^{i,j,k} \tilde{B}_z^{i,j,k} + \zeta_{yz}^{i,j-1,k} \tilde{B}_z^{i,j-1,k} + \zeta_{yz}^{i,j,k+1} \tilde{B}_z^{i,j,k+1} + \zeta_{yz}^{i,j-1,k+1} \tilde{B}_z^{i,j-1,k+1}}{4} \\
 H_z^{i,j,k} &= \frac{\zeta_{zx}^{i,j,k} \tilde{B}_x^{i,j,k} + \zeta_{zx}^{i+1,j,k} \tilde{B}_x^{i+1,j,k} + \zeta_{zx}^{i,j,k-1} \tilde{B}_x^{i,j,k-1} + \zeta_{zx}^{i+1,j,k-1} \tilde{B}_x^{i+1,j,k-1}}{4} \\
 &\quad + \frac{\zeta_{zy}^{i,j,k} \tilde{B}_y^{i,j,k} + \zeta_{zy}^{i,j+1,k} \tilde{B}_y^{i,j+1,k} + \zeta_{zy}^{i,j,k-1} \tilde{B}_y^{i,j,k-1} + \zeta_{zy}^{i,j+1,k-1} \tilde{B}_y^{i,j+1,k-1}}{4} \\
 &\quad + \frac{\zeta_{zz}^{i,j,k} \tilde{B}_z^{i,j,k}}{4}
 \end{aligned}$$

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Reduction to 1D (1 of 2)

For 1D problems,

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial y} = 0$$

$$\begin{aligned}
 -\frac{\tilde{E}_y^{k+1}|_l - \tilde{E}_y^k|_l}{\Delta z} &= -\frac{1}{c_0} \frac{\tilde{B}_x^k|_{l+\frac{\Delta z}{2}} - \tilde{B}_x^k|_{l-\frac{\Delta z}{2}}}{\Delta t} \\
 \frac{\tilde{E}_x^{k+1}|_l - \tilde{E}_x^k|_l}{\Delta z} &= -\frac{1}{c_0} \frac{\tilde{B}_y^k|_{l+\frac{\Delta z}{2}} - \tilde{B}_y^k|_{l-\frac{\Delta z}{2}}}{\Delta t} \\
 \tilde{B}_z &= 0 \\
 -\frac{H_y^k|_{l+\frac{\Delta z}{2}} - H_y^{k-1}|_{l+\frac{\Delta z}{2}}}{\Delta z} &= \frac{1}{c_0} \frac{\tilde{D}_x^k|_{l+\Delta t} - \tilde{D}_x^k|_l}{\Delta t} \\
 \frac{H_x^k|_{l+\frac{\Delta z}{2}} - H_x^{k-1}|_{l+\frac{\Delta z}{2}}}{\Delta z} &= \frac{1}{c_0} \frac{\tilde{D}_y^k|_{l+\Delta t} - \tilde{D}_y^k|_l}{\Delta t} \\
 \tilde{D}_z &= 0
 \end{aligned}$$

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Reduction to 1D (2 of 2)

$$\begin{aligned}\tilde{E}_x^k &= \psi_{xx}^k \tilde{D}_x^k + \psi_{xy}^k \tilde{D}_y^k \\ \tilde{E}_y^k &= \psi_{yx}^k \tilde{D}_x^k + \psi_{yy}^k \tilde{D}_y^k \\ \tilde{E}_z^k &= \frac{\psi_{zx}^k \tilde{D}_x^k + \psi_{zx}^{k+1} \tilde{D}_x^{k+1}}{2} + \frac{\psi_{zy}^k \tilde{D}_y^k + \psi_{zy}^{k+1} \tilde{D}_y^{k+1}}{2} \\ \\ H_x^k &= \zeta_{xx}^k \tilde{B}_x^k + \zeta_{xy}^k \tilde{B}_y^k \\ H_y^k &= \zeta_{yx}^k \tilde{B}_x^k + \zeta_{yy}^k \tilde{B}_y^k \\ H_z^k &= \frac{\zeta_{zx}^k \tilde{B}_x^k + \zeta_{zx}^{k-1} \tilde{B}_x^{k-1}}{2} + \frac{\zeta_{zy}^k \tilde{B}_y^k + \zeta_{zy}^{k-1} \tilde{B}_y^{k-1}}{2}\end{aligned}$$

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Update Equations (1 of 2)

$$\begin{aligned}\tilde{B}_x^k \Big|_{t+\frac{\Delta t}{2}} &= \tilde{B}_x^k \Big|_{t-\frac{\Delta t}{2}} + m \left(\tilde{E}_y^{k+1} \Big|_t - \tilde{E}_y^k \Big|_t \right) \\ \tilde{B}_y^k \Big|_{t+\frac{\Delta t}{2}} &= \tilde{B}_y^k \Big|_{t-\frac{\Delta t}{2}} - m \left(\tilde{E}_x^{k+1} \Big|_t - \tilde{E}_x^k \Big|_t \right) \\ \\ \tilde{D}_x^k \Big|_{t+\Delta t} &= \tilde{D}_x^k \Big|_t - m \left(H_y^k \Big|_{t+\frac{\Delta t}{2}} - H_y^{k-1} \Big|_{t+\frac{\Delta t}{2}} \right) \\ \tilde{D}_y^k \Big|_{t+\Delta t} &= \tilde{D}_y^k \Big|_t + m \left(H_x^k \Big|_{t+\frac{\Delta t}{2}} - H_x^{k-1} \Big|_{t+\frac{\Delta t}{2}} \right) \\ \\ m &= \frac{\Delta t \cdot c_0}{\Delta z}\end{aligned}$$

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Update Equations (2 of 2)

$$\tilde{E}_x^k = \psi_{xx}^k \tilde{D}_x^k + \psi_{xy}^k \tilde{D}_y^k$$

$$\tilde{E}_y^k = \psi_{yx}^k \tilde{D}_x^k + \psi_{yy}^k \tilde{D}_y^k$$

$$\tilde{E}_z^k = \frac{\psi_{zx}^k \tilde{D}_x^k + \psi_{zx}^{k+1} \tilde{D}_x^{k+1}}{2} + \frac{\psi_{zy}^k \tilde{D}_y^k + \psi_{zy}^{k+1} \tilde{D}_y^{k+1}}{2}$$

$$H_x^k = \zeta_{xx}^k \tilde{B}_x^k + \zeta_{xy}^k \tilde{B}_y^k$$

$$H_y^k = \zeta_{yx}^k \tilde{B}_x^k + \zeta_{yy}^k \tilde{B}_y^k$$

$$H_z^k = \frac{\zeta_{zx}^k \tilde{B}_x^k + \zeta_{zx}^{k-1} \tilde{B}_x^{k-1}}{2} + \frac{\zeta_{zy}^k \tilde{B}_y^k + \zeta_{zy}^{k-1} \tilde{B}_y^{k-1}}{2}$$

TF/SF Source (1 of 2)

The following update equations span the TF/SF interface. In the SF region, these are the \vec{B} field update equations.

$$\tilde{B}_x^k \Big|_{t+\frac{\Delta t}{2}} = \tilde{B}_x^k \Big|_{t-\frac{\Delta t}{2}} + m \left(\tilde{E}_y^{k+1} \Big|_t - \tilde{E}_y^k \Big|_t \right) \quad \tilde{B}_y^k \Big|_{t+\frac{\Delta t}{2}} = \tilde{B}_y^k \Big|_{t-\frac{\Delta t}{2}} - m \left(\tilde{E}_x^{k+1} \Big|_t - \tilde{E}_x^k \Big|_t \right)$$

In the TF region, these are the \vec{D} field update equations.

$$\tilde{D}_x^k \Big|_{t+\Delta t} = \tilde{D}_x^k \Big|_t - m \left(H_y^k \Big|_{t+\frac{\Delta t}{2}} - H_y^{k-1} \Big|_{t+\frac{\Delta t}{2}} \right) \quad \tilde{D}_y^k \Big|_{t+\Delta t} = \tilde{D}_y^k \Big|_t + m \left(H_x^k \Big|_{t+\frac{\Delta t}{2}} - H_x^{k-1} \Big|_{t+\frac{\Delta t}{2}} \right)$$

TF/SF Source (2 of 2)

The \vec{B} field update equations exist in the SF region but contain TF terms. It is necessary to subtract the source from the TF terms.

$$\begin{aligned}\tilde{B}_x^{k_{\text{src}}-1} \Big|_{t+\frac{\Delta t}{2}} &= \tilde{B}_x^{k_{\text{src}}-1} \Big|_{t-\frac{\Delta t}{2}} + m \left[\left(\tilde{E}_y^{k_{\text{src}}} \Big|_t - \tilde{E}_{y,\text{src}}^{k_{\text{src}}} \Big|_t \right) - \tilde{E}_y^{k_{\text{src}}-1} \Big|_t \right] \\ &= \tilde{B}_x^{k_{\text{src}}-1} \Big|_{t-\frac{\Delta t}{2}} + m \left[\tilde{E}_y^{k_{\text{src}}} \Big|_t - \tilde{E}_y^{k_{\text{src}}-1} \Big|_t \right] - m \tilde{E}_{y,\text{src}}^{k_{\text{src}}} \Big|_t\end{aligned}$$

$$\begin{aligned}\tilde{B}_y^{k_{\text{src}}-1} \Big|_{t+\frac{\Delta t}{2}} &= \tilde{B}_y^{k_{\text{src}}-1} \Big|_{t-\frac{\Delta t}{2}} - m \left[\left(\tilde{E}_x^{k_{\text{src}}} \Big|_t - \tilde{E}_{x,\text{src}}^{k_{\text{src}}} \Big|_t \right) - \tilde{E}_x^{k_{\text{src}}-1} \Big|_t \right] \\ &= \tilde{B}_y^{k_{\text{src}}-1} \Big|_{t-\frac{\Delta t}{2}} - m \left[\tilde{E}_x^{k_{\text{src}}} \Big|_t - \tilde{E}_x^{k_{\text{src}}-1} \Big|_t \right] + m \tilde{E}_{x,\text{src}}^{k_{\text{src}}} \Big|_t\end{aligned}$$

The \vec{D} field update equations exist in the TF region but contain SF terms. We must add the source to the SF terms.

$$\begin{aligned}\tilde{D}_x^{k_{\text{src}}} \Big|_{t+\Delta t} &= \tilde{D}_x^{k_{\text{src}}} \Big|_t - m \left[H_y^{k_{\text{src}}} \Big|_{t+\frac{\Delta t}{2}} - \left(H_y^{k_{\text{src}}-1} \Big|_{t+\frac{\Delta t}{2}} + \tilde{H}_y^{k_{\text{src}}-1} \Big|_{t+\frac{\Delta t}{2}} \right) \right] \\ &= \tilde{D}_x^{k_{\text{src}}} \Big|_t - m \left[H_y^{k_{\text{src}}} \Big|_{t+\frac{\Delta t}{2}} - H_y^{k_{\text{src}}-1} \Big|_{t+\frac{\Delta t}{2}} \right] + m \tilde{H}_y^{k_{\text{src}}-1} \Big|_{t+\frac{\Delta t}{2}}\end{aligned}$$

$$\begin{aligned}\tilde{D}_y^{k_{\text{src}}} \Big|_{t+\Delta t} &= \tilde{D}_y^{k_{\text{src}}} \Big|_t + m \left[H_x^{k_{\text{src}}} \Big|_{t+\frac{\Delta t}{2}} - \left(H_x^{k_{\text{src}}-1} \Big|_{t+\frac{\Delta t}{2}} + \tilde{H}_x^{k_{\text{src}}-1} \Big|_{t+\frac{\Delta t}{2}} \right) \right] \\ &= \tilde{D}_y^{k_{\text{src}}} \Big|_t + m \left[H_x^{k_{\text{src}}} \Big|_{t+\frac{\Delta t}{2}} - H_x^{k_{\text{src}}-1} \Big|_{t+\frac{\Delta t}{2}} \right] - m \tilde{H}_x^{k_{\text{src}}-1} \Big|_{t+\frac{\Delta t}{2}}\end{aligned}$$

The Source Terms

$$\tilde{E}_x^{\text{src}} \Big|_t^{k_{\text{src}}} = P_x \cdot g(t)$$

$$\tilde{E}_y^{\text{src}} \Big|_t^{k_{\text{src}}} = P_y \cdot g(t)$$

$$H_x^{\text{src}} \Big|_{t+\frac{\Delta t}{2}}^{k_{\text{src}}-1} = -P_y \sqrt{\frac{\epsilon_{r,\text{inc}}}{\mu_{r,\text{inc}}}} \cdot g(t + \delta_x) \quad \delta_x = \frac{n_{\text{inc}} \Delta z}{2c_0} + \frac{\Delta t}{2}$$

$$H_y^{\text{src}} \Big|_{t+\frac{\Delta t}{2}}^{k_{\text{src}}-1} = P_x \sqrt{\frac{\epsilon_{r,\text{inc}}}{\mu_{r,\text{inc}}}} \cdot g(t + \delta_y) \quad \delta_y = \frac{n_{\text{inc}} \Delta z}{2c_0} - \frac{\Delta t}{2}$$

Sequence of Update Equations

Step 1 -- Update \vec{B}

$$\tilde{B}_x^k|_{t+\frac{\Delta t}{2}} = \tilde{B}_x^k|_{t-\frac{\Delta t}{2}} + m(\tilde{E}_y^{k+1}|_t - \tilde{E}_y^k|_t) \quad \tilde{B}_y^k|_{t+\frac{\Delta t}{2}} = \tilde{B}_y^k|_{t-\frac{\Delta t}{2}} - m(\tilde{E}_x^{k+1}|_t - \tilde{E}_x^k|_t)$$

Step 2 -- Update \vec{H}

$$H_x^{i,j,k} = \zeta_{xx}^k \tilde{B}_x^k + \zeta_{xy}^k \tilde{B}_y^k \quad H_y^{i,j,k} = \zeta_{yx}^k \tilde{B}_x^k + \zeta_{yy}^k \tilde{B}_y^k \quad H_z^{i,j,k} = \frac{\zeta_{zx}^k \tilde{B}_x^k + \zeta_{zx}^{k-1} \tilde{B}_x^{k-1}}{2} + \frac{\zeta_{zy}^k \tilde{B}_y^k + \zeta_{zy}^{k-1} \tilde{B}_y^{k-1}}{2}$$

Step 3 -- Update \vec{D}


$$\tilde{D}_x^k|_{t+\Delta t} = \tilde{D}_x^k|_t - m(H_y^k|_{t+\frac{\Delta t}{2}} - H_y^{k-1}|_{t+\frac{\Delta t}{2}}) \quad \tilde{D}_y^k|_{t+\Delta t} = \tilde{D}_y^k|_t + m(H_x^k|_{t+\frac{\Delta t}{2}} - H_x^{k-1}|_{t+\frac{\Delta t}{2}})$$

Step 4 -- Update \vec{E}

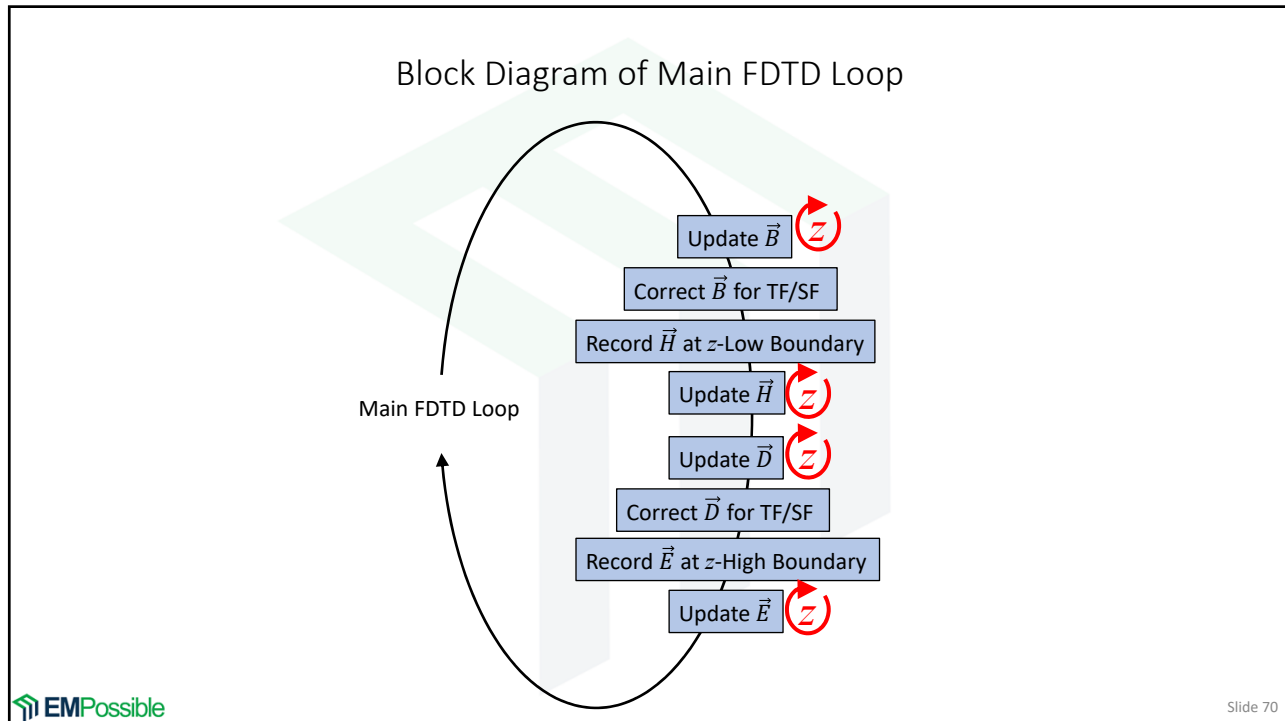
$$\tilde{E}_x^k = \psi_{xx}^k \tilde{D}_x^k + \psi_{xy}^k \tilde{D}_y^k$$

$$\tilde{E}_y^k = \psi_{yx}^k \tilde{D}_x^k + \psi_{yy}^k \tilde{D}_y^k$$

$$\tilde{E}_z^k = \frac{\psi_{zx}^k \tilde{D}_x^k + \psi_{zx}^{k+1} \tilde{D}_x^{k+1}}{2} + \frac{\psi_{zy}^k \tilde{D}_y^k + \psi_{zy}^{k+1} \tilde{D}_y^{k+1}}{2}$$

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