



Advanced Electromagnetics:
21st Century Electromagnetics

Introduction to Engineered Materials

Lecture Outline

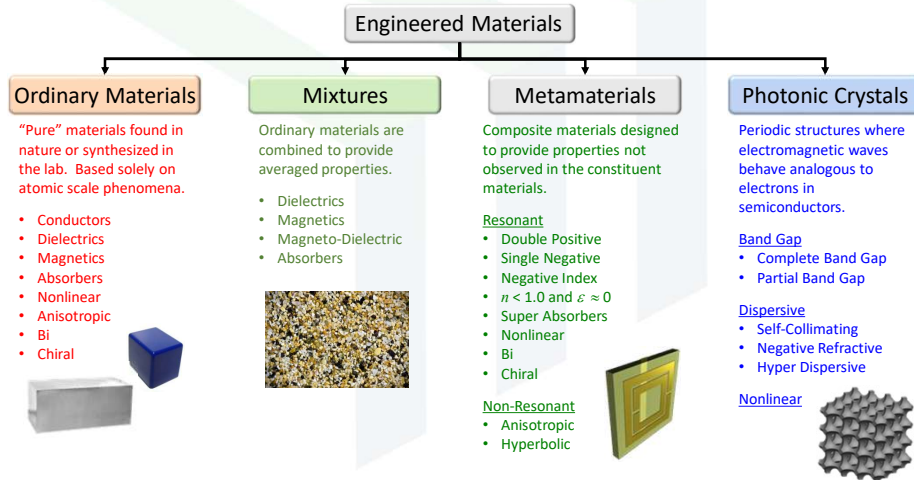
- Classifications of Engineered Materials
- Covered in this Lecture
 - Ordinary materials
 - Nonreciprocal and asymmetric materials and structures
 - Mixtures

Classification

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Classifications of Engineered Materials for Electromagnetics

Engineered materials are materials that are purposely tailored to exhibit useful and enabling electromagnetic properties.



Classification by Size and Frequency

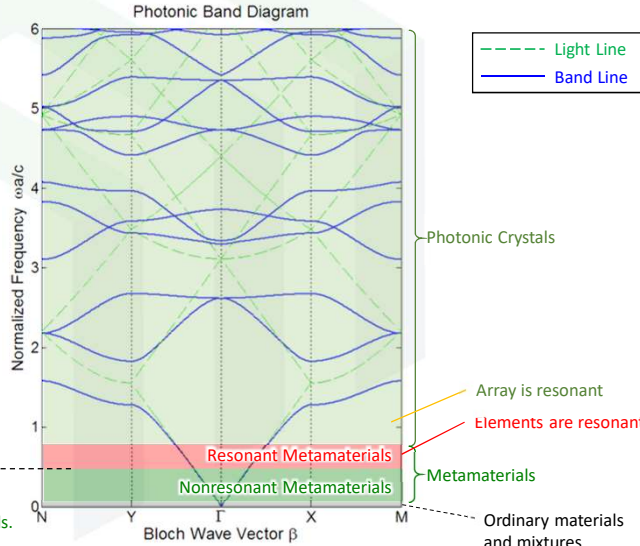
Photonic Crystals
 $a \sim \lambda/2$

Resonant Metamaterials
 $a \sim \lambda/10$

Nonresonant Metamaterials
 $a \ll \lambda/4$

Mixtures
 $a \ll \lambda/20$

Ordinary materials
 $a \sim$ atomic scale



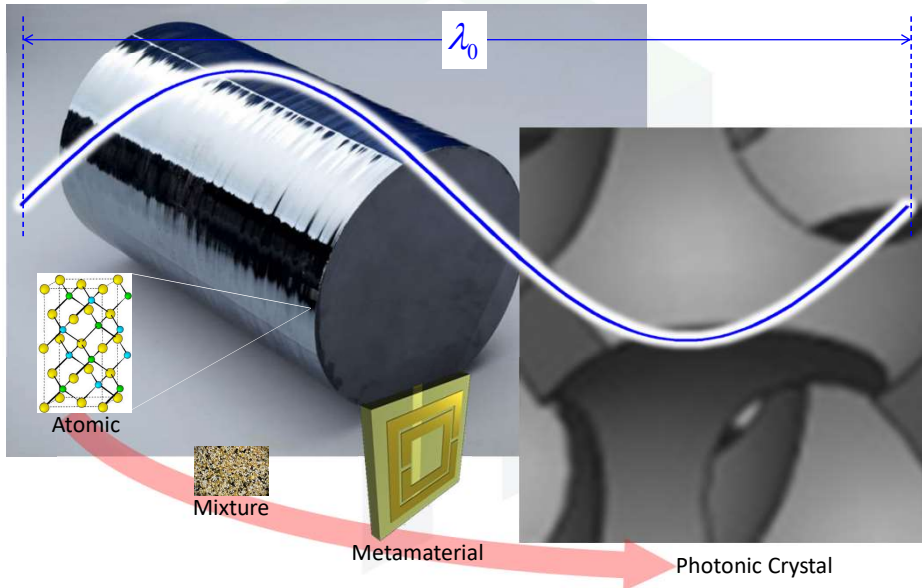
New properties emerge from resonance and interference.

Effective properties derive from "averages" of the constituent materials.



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Visualizing the Size Comparison



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Ordinary Materials

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Conductors

- Superconductors
 - Perfect conductivity
 - Requires cryogenic cooling
- Graphene
- Silver
 - Best conductivity
 - Does not oxidize into AgO until particles are less than 3 nm. AgO is conductive.
- Copper
 - 95% the conductivity of silver
 - Robust to oxidation, but CuO is an insulator.
- Gold
 - 94% conductivity of silver
 - Robust to oxidation
- Aluminum
 - Very inexpensive!
 - Oxidizes very easily and quickly. Al_2O_3 is a ceramic and insulator.
 - Like titanium, powders with particles less than 20 μm or so are explosive.
- Mercury
 - Okay conductivity
 - Liquid metal mirrors
 - Gravity switches

Dielectrics

- Teflon
 - Gold standard for dielectrics ($\epsilon_r = 2.1$)
- FR4
 - Common material in printed circuit boards (PCB)
- Water
 - Water is actually an insulator similar in conductivity to wood.
 - $\epsilon_r = 80$ at radio frequencies
- Titanium Dioxide (TiO_2)
 - Excellent thermal stability
 - $\epsilon_r = 100$
- Strontium Titanate (SrTiO_3)
 - Very high dielectric constant ($\epsilon_r > 300$)
 - Strong temperature dependence

Magnetics

- Diamagnetism
 - Creates a magnetic field in opposition to an externally applied magnetic field.
 - Negative susceptibility.
 - Repelled by magnetic fields.
 - Very weak phenomenon.
 - Does not retain magnetism after external field is removed.
 - Water
- Paramagnetism
 - Positive susceptibility
 - Attracted to magnetic fields
 - Very weak phenomenon
 - Does not retain magnetism after external field is removed.
- Ferromagnetism
 - Similar to paramagnetism, but retains magnetism after applied magnetic field is removed.
 - Strong enough to be felt by hand
 - Iron, nickel, cobalt, lodestone
- Antiferromagnetism
 - Transition metal compounds, hematite, chromium, FeMn, and NiO

Absorbers

- Salt water
- Graphite
- Think “black”

Anisotropic

- Usually very weak at radio frequencies
- More common at optical frequencies
- Uniaxial
- Biaxial

$$\vec{D} = [\epsilon] \vec{E}$$

$$\vec{B} = [\mu] \vec{H}$$

TWO CLASSES OF ANISOTROPIC MATERIALS

1. Ordinary Anisotropic Media

Symmetric tensors

Natural modes are linearly polarized

$$\epsilon_{ij} = \epsilon_{ji} \quad \text{and} \quad \mu_{ij} = \mu_{ji}$$

2. Gyrotropic Media

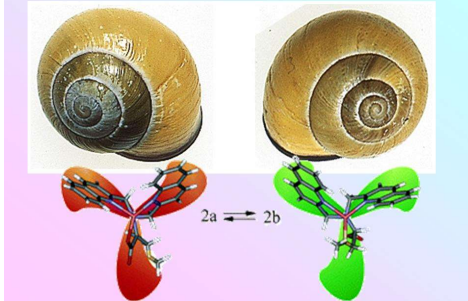
Antisymmetric tensors

Natural modes are circularly polarized

$$\epsilon_{ij} = -\epsilon_{ji} \quad \text{or} \quad \mu_{ij} = -\mu_{ji} \quad i \neq j$$

Gyrotropic (Chiral)

Chirality - Organic Chemistry



Fundamental eigen-states are circularly polarized beams.

Optical activity
Faraday rotation

Chiral Materials

- Sugar
- Quartz

Gyrotropic behavior can be produced by applying a magnetic field to a plasma, to a ferrite, or to some dielectric crystals.

$$[\varepsilon] = \begin{bmatrix} \varepsilon_a & -j\varepsilon_b & 0 \\ j\varepsilon_b & \varepsilon_a & 0 \\ 0 & 0 & \varepsilon_c \end{bmatrix} \quad \text{gyroelectric}$$

$$[\mu] = \begin{bmatrix} \mu_a & -j\mu_b & 0 \\ j\mu_b & \mu_a & 0 \\ 0 & 0 & \mu_c \end{bmatrix} \quad \text{gyromagnetic}$$

Bi-Isotropic and Bi-Anisotropic

In "bi" materials, the constitutive relations couple both electric and magnetic fields.

Bi-Isotropic

$$\vec{D} = \varepsilon \vec{E} + \xi \vec{H}$$

$$\vec{B} = \mu \vec{H} + \zeta \vec{E}$$

ε \equiv permittivity

μ \equiv permeability

ξ, ζ \equiv magnetoelectric coupling coefficients

Classification of Bi-Isotropic Media

$$\vec{D} = \varepsilon \vec{E} + (\chi - j\kappa) \sqrt{\mu\varepsilon} \vec{H} \quad \vec{B} = \mu \vec{H} + (\chi + j\kappa) \sqrt{\mu\varepsilon} \vec{E}$$

	Nonchiral ($\kappa = 0$)	Chiral ($\kappa \neq 0$)
Reciprocal ($\chi = 0$)	Simple isotropic medium	Pasteur medium
Nonreciprocal ($\chi \neq 0$)	Tellegen medium	General bi-isotropic medium

χ \equiv reciprocity parameter

κ \equiv chirality parameter

Bi-Anisotropic

$$\vec{D} = [\varepsilon] \vec{E} + [\xi] \vec{H}$$

$$\vec{B} = [\mu] \vec{H} + [\zeta] \vec{E}$$

Nonreciprocal & Asymmetric Materials and Structures

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Related, But Easily Confused Phenomena

- Reciprocity
- Asymmetric Devices
- Time-Reversal Symmetry
- Phase Conjugation

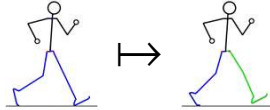
Time-Reversal Symmetry

Imagine if time could be reversed.

$$t \mapsto -t$$

Mathematically this is a simple transformation.

Would this just look like a movie playing backwards?



Answer: only if the physics in the movie has time-reversal symmetry. Most things do.

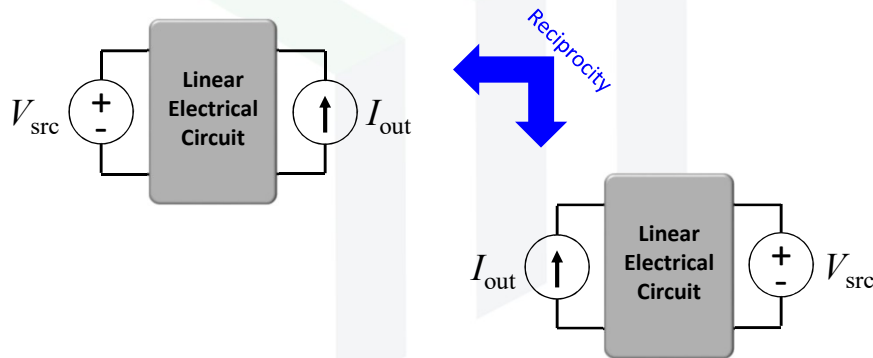
If a physical system has time-reversal symmetry, the laws retain the same mathematical form when time is reversed.

Ways To Break Time-Reversal Symmetry

- Magnetic Materials
 - Most common
 - Anything magnetized is nonreciprocal.
 - Usually a weak response, but can be enhanced using photonic crystals and metamaterials.
 - Faraday rotation.
 - Rotation has "absolute" sense in space
 - rotation is opposite for waves travelling in opposite direction
 - Optical activity
 - Rotation due to chirality
- Nonlinear Materials
 - Material properties change with time
 - May require high power or sensitive components
- Acousto-Optical Materials
 - An electromagnetic wave travelling through an acoustic wave
- Composite right-hand/left-hand (CRLH) materials
 - Possess nonreciprocal phase

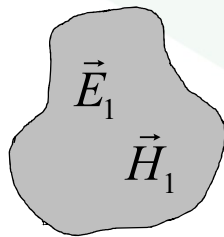
Lorentz Reciprocity Theorem for Electrical Circuits

Reciprocity in linear electrical circuits states that the positions of a voltage source and ammeter can be interchanged without affecting the reading.

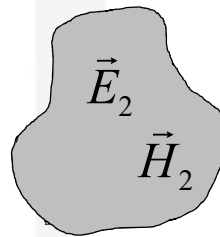


Derivation of Lorentz Reciprocity Theorem (1 of 2)

Consider two allowed states of electromagnetic fields. In order to be allowed states, the electromagnetic fields must satisfy Maxwell's equations. Assuming time-harmonic quantities, these may be



$$\begin{aligned}\nabla \times \vec{E}_1 &= -\vec{M}_1 - j\omega[\mu]\vec{H}_1 \\ \nabla \times \vec{H}_1 &= \vec{J}_1 + j\omega[\varepsilon]\vec{E}_1\end{aligned}$$



$$\begin{aligned}\nabla \times \vec{E}_2 &= -\vec{M}_2 - j\omega[\mu]\vec{H}_2 \\ \nabla \times \vec{H}_2 &= \vec{J}_2 + j\omega[\varepsilon]\vec{E}_2\end{aligned}$$

Derivation of Lorentz Reciprocity Theorem (2 of 2)

$$\vec{E}_1 \quad \nabla \times \vec{E}_1 = -\vec{M}_1 - j\omega[\mu]\vec{H}_1 \quad \text{Eq. (1a)}$$

$$\vec{H}_1 \quad \nabla \times \vec{H}_1 = \vec{J}_1 + j\omega[\varepsilon]\vec{E}_1 \quad \text{Eq. (1b)}$$

1. Dot multiply (1a) with \vec{H}_2 and (1b) with \vec{E}_2 .

$$\vec{H}_2 \cdot \nabla \times \vec{E}_1 = -\vec{H}_2 \cdot \vec{M}_1 - j\omega\vec{H}_2 \cdot [\mu]\vec{H}_1 \quad \text{Eq. (3a)}$$

$$\vec{E}_2 \cdot \nabla \times \vec{H}_1 = \vec{E}_2 \cdot \vec{J}_1 + j\omega\vec{E}_2 \cdot [\varepsilon]\vec{E}_1 \quad \text{Eq. (3b)}$$

2. Subtract (3a) from (3b).

$$\begin{aligned} \vec{E}_2 \cdot \nabla \times \vec{H}_1 - \vec{H}_2 \cdot \nabla \times \vec{E}_1 \\ = \vec{E}_2 \cdot \vec{J}_1 + \vec{H}_2 \cdot \vec{M}_1 + j\omega\vec{E}_2 \cdot [\varepsilon]\vec{E}_1 + j\omega\vec{H}_2 \cdot [\mu]\vec{H}_1 \end{aligned} \quad \text{Eq. (5)}$$

3. Use $\nabla \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot \nabla \times \vec{A} - \vec{A} \cdot \nabla \times \vec{B}$

$$\begin{aligned} \nabla \cdot (\vec{H}_1 \times \vec{E}_2) \\ = \vec{E}_2 \cdot \vec{J}_1 + \vec{H}_2 \cdot \vec{M}_1 + j\omega\vec{E}_2 \cdot [\varepsilon]\vec{E}_1 + j\omega\vec{H}_2 \cdot [\mu]\vec{H}_1 \end{aligned} \quad \text{Eq. (7)}$$

4. Subtract (7) from (8).

$$\begin{aligned} -\nabla \cdot (\vec{E}_1 \times \vec{H}_2 - \vec{E}_2 \times \vec{H}_1) = \vec{E}_1 \cdot \vec{J}_2 - \vec{E}_2 \cdot \vec{J}_1 + \vec{H}_1 \cdot \vec{M}_2 - \vec{H}_2 \cdot \vec{M}_1 \\ + j\omega(\vec{E}_1 \cdot [\varepsilon]\vec{E}_2 - \vec{E}_2 \cdot [\varepsilon]\vec{E}_1) + j\omega(\vec{H}_1 \cdot [\mu]\vec{H}_2 - \vec{H}_2 \cdot [\mu]\vec{H}_1) \end{aligned}$$

$$\vec{E}_2 \quad \nabla \times \vec{E}_2 = -\vec{M}_2 - j\omega[\mu]\vec{H}_2 \quad \text{Eq. (2a)}$$

$$\vec{H}_2 \quad \nabla \times \vec{H}_2 = \vec{J}_2 + j\omega[\varepsilon]\vec{E}_2 \quad \text{Eq. (2b)}$$

1. Dot multiply (2a) with \vec{H}_1 and (2b) with \vec{E}_1 .

$$\vec{H}_1 \cdot \nabla \times \vec{E}_2 = -\vec{H}_1 \cdot \vec{M}_2 - j\omega\vec{H}_1 \cdot [\mu]\vec{H}_2 \quad \text{Eq. (4a)}$$

$$\vec{E}_1 \cdot \nabla \times \vec{H}_2 = \vec{E}_1 \cdot \vec{J}_2 + j\omega\vec{E}_1 \cdot [\varepsilon]\vec{E}_2 \quad \text{Eq. (4b)}$$

2. Subtract (4a) from (4b).

$$\begin{aligned} \vec{E}_1 \cdot \nabla \times \vec{H}_2 - \vec{H}_1 \cdot \nabla \times \vec{E}_2 \\ = \vec{E}_1 \cdot \vec{J}_2 + \vec{H}_1 \cdot \vec{M}_2 + j\omega\vec{E}_1 \cdot [\varepsilon]\vec{E}_2 + j\omega\vec{H}_1 \cdot [\mu]\vec{H}_2 \end{aligned} \quad \text{Eq. (6)}$$

3. Use $\nabla \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot \nabla \times \vec{A} - \vec{A} \cdot \nabla \times \vec{B}$

$$\begin{aligned} \nabla \cdot (\vec{H}_2 \times \vec{E}_1) \\ = \vec{E}_1 \cdot \vec{J}_2 + \vec{H}_1 \cdot \vec{M}_2 + j\omega\vec{E}_1 \cdot [\varepsilon]\vec{E}_2 + j\omega\vec{H}_1 \cdot [\mu]\vec{H}_2 \end{aligned} \quad \text{Eq. (8)}$$

General Forms of the Lorentz Reciprocity Theorem

The following equations are the most general forms of the Lorentz reciprocity theorem.

Differential Form

$$\begin{aligned} -\nabla \cdot (\vec{E}_1 \times \vec{H}_2 - \vec{E}_2 \times \vec{H}_1) = \vec{E}_1 \cdot \vec{J}_2 - \vec{E}_2 \cdot \vec{J}_1 + \vec{H}_1 \cdot \vec{M}_2 - \vec{H}_2 \cdot \vec{M}_1 \\ + j\omega(\vec{E}_1 \cdot [\varepsilon]\vec{E}_2 - \vec{E}_2 \cdot [\varepsilon]\vec{E}_1) \\ + j\omega(\vec{H}_1 \cdot [\mu]\vec{H}_2 - \vec{H}_2 \cdot [\mu]\vec{H}_1) \end{aligned}$$

Integral Form

$$\begin{aligned} -\iiint_S (\vec{E}_1 \times \vec{H}_2 - \vec{E}_2 \times \vec{H}_1) \cdot d\vec{s} = \iiint_V (\vec{E}_1 \cdot \vec{J}_2 - \vec{E}_2 \cdot \vec{J}_1 + \vec{H}_1 \cdot \vec{M}_2 - \vec{H}_2 \cdot \vec{M}_1) dv \\ + \iiint_V j\omega(\vec{E}_1 \cdot [\varepsilon]\vec{E}_2 - \vec{E}_2 \cdot [\varepsilon]\vec{E}_1) dv \\ + \iiint_V j\omega(\vec{H}_1 \cdot [\mu]\vec{H}_2 - \vec{H}_2 \cdot [\mu]\vec{H}_1) dv \end{aligned}$$

Special Case #1: *Source Free Media (i.e. dielectrics)*

In source free regions $\vec{J}_1 = \vec{J}_2 = \vec{M}_1 = \vec{M}_2 = 0$ and the reciprocity theorems reduce to

Differential Form

$$-\nabla \cdot (\vec{E}_1 \times \vec{H}_2 - \vec{E}_2 \times \vec{H}_1) = j\omega (\vec{E}_1 \cdot [\varepsilon] \vec{E}_2 - \vec{E}_2 \cdot [\varepsilon] \vec{E}_1) + j\omega (\vec{H}_1 \cdot [\mu] \vec{H}_2 - \vec{H}_2 \cdot [\mu] \vec{H}_1)$$

Integral Form

$$-\oint_S (\vec{E}_1 \times \vec{H}_2 - \vec{E}_2 \times \vec{H}_1) \cdot d\vec{s} = \iiint_V j\omega (\vec{E}_1 \cdot [\varepsilon] \vec{E}_2 - \vec{E}_2 \cdot [\varepsilon] \vec{E}_1) dv + \iiint_V j\omega (\vec{H}_1 \cdot [\mu] \vec{H}_2 - \vec{H}_2 \cdot [\mu] \vec{H}_1) dv$$

Special Case #2: *Media with Symmetric Tensors*

If $[\varepsilon]$ and $[\mu]$ are symmetric tensors or scalars, then

$$\vec{E}_1 \cdot [\varepsilon] \vec{E}_2 - \vec{E}_2 \cdot [\varepsilon] \vec{E}_1 = 0 \quad \text{and} \quad \vec{H}_1 \cdot [\mu] \vec{H}_2 - \vec{H}_2 \cdot [\mu] \vec{H}_1 = 0$$

Differential Form

$$-\nabla \cdot (\vec{E}_1 \times \vec{H}_2 - \vec{E}_2 \times \vec{H}_1) = \vec{E}_1 \cdot \vec{J}_2 - \vec{E}_2 \cdot \vec{J}_1 + \vec{H}_1 \cdot \vec{M}_2 - \vec{H}_2 \cdot \vec{M}_1$$

Integral Form

$$-\oint_S (\vec{E}_1 \times \vec{H}_2 - \vec{E}_2 \times \vec{H}_1) \cdot d\vec{s} = \iiint_V (\vec{E}_1 \cdot \vec{J}_2 - \vec{E}_2 \cdot \vec{J}_1 + \vec{H}_1 \cdot \vec{M}_2 - \vec{H}_2 \cdot \vec{M}_1) dv$$

Special Case #3: *Source Free + Symmetric Tensors*

This is perhaps the most common way the reciprocity theorem is written, however it is a very special case.

Differential Form

$$\nabla \cdot (\vec{E}_1 \times \vec{H}_2 - \vec{E}_2 \times \vec{H}_1) = 0$$

Integral Form

$$\iiint_S (\vec{E}_1 \times \vec{H}_2 - \vec{E}_2 \times \vec{H}_1) \cdot d\vec{s} = 0$$

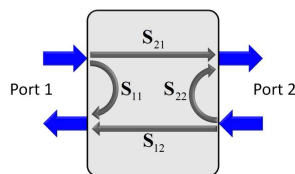
Nonreciprocal Materials and Devices

A material or device is “nonreciprocal” only if breaks the Lorentz reciprocity theorem.

This can be put in a mathematical framework through the scattering matrix $[S]$.

$$\begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}$$

s_{11} = reflection from Port 1
 s_{12} = transmission from Port 2 to Port 1
 s_{21} = transmission from Port 1 to Port 2
 s_{22} = reflection from Port 2

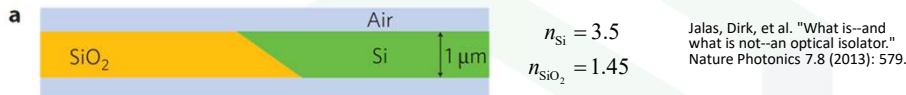


Reciprocal Systems: $S_{ij} = S_{ji}$ or $[S] = [S]^T$

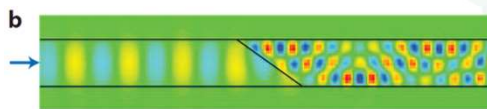
Nonreciprocal Systems: $S_{ij} \neq S_{ji}$ or $[S] \neq [S]^T$

Asymmetric Devices

Consider two waveguides butt-coupled with an inclined junction.



Excite the left waveguide with its fundamental mode. Refraction excites higher order modes and transmission is observed.



Now excite the right waveguide with its fundamental mode. Transmission is blocked due to total-internal reflection.



Applications of Nonreciprocal Materials and Devices

- Isolators
 - "Diodes" for electromagnetic waves
- Circulators
- Controlling polarization
 - Polarization rotation
- Resonators
 - Can be made to resonant independent of size of resonator
- Leaky-Wave Antennas
 - Enhanced "leakage" at terminal.

Nonreciprocal Materials and Devices

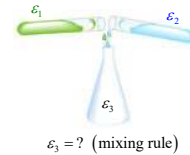
- Radio and Microwave Frequencies
 - Ferrites
 - Diodes and transistors
- Optics
 - Dielectrics based on yttrium iron garnet (YIG)
 - Sugar molecules
- Metamaterials
 - Chiral unit cells
 - Unit cells containing nonreciprocal materials

Mixtures

What are Mixing Rules?

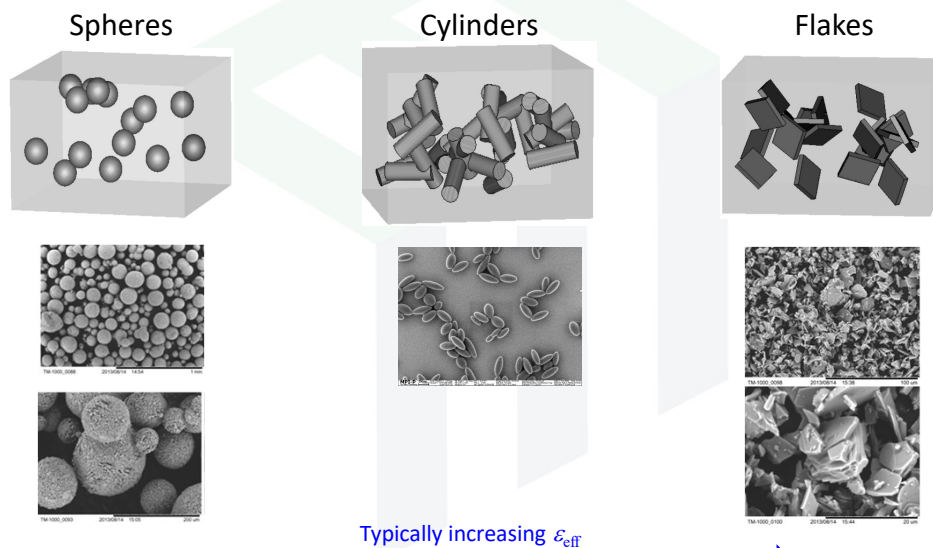
Sometimes it is desired to mix multiple materials together to realize some overall material property. The effective dielectric constant of the mixture ϵ_{eff} depends on many things:

- Shape of the particles
- Size of the particles
- Electromagnetic properties of the particles.
- Statistics on the particle distribution
- Volume fill fraction of the constituent materials.



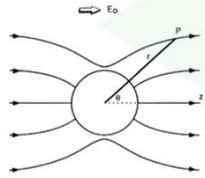
Typically materials are mixed without good knowledge of the statistics of the particle distribution. Therefore, these mixing rules, therefore, rarely match measured results exactly.

Particle Shapes



Influence of Particle Shape (1 of 2)

Particles distort the local field. We can think of each particle as generating its own field that combines with the applied field.



$$V = S_x \frac{x}{r^3} - E_0 x = S_y \frac{y}{r^3} - E_0 y = S_z \frac{z}{r^3} - E_0 z \quad \leftarrow \text{Potential away from particle}$$

$$S_x = \frac{E_0 v}{4\pi n_x} \quad S_y = \frac{E_0 v}{4\pi n_y} \quad S_z = \frac{E_0 v}{4\pi n_z} \quad \leftarrow \text{Strength of potential along an axis.}$$

$$v = \frac{4}{3}\pi R^3 \quad n_i \equiv \text{depolarizing factors due to shape of particle}$$

Sphere

$$n_x = n_y = n_z = \frac{1}{3}$$

$$\frac{S_z}{S_x} = \frac{n_x}{n_z} = 1$$

Prolate Ellipsoid (Rods)

$$n_x = n_y = \frac{1-n_z}{2}$$

$$n_z = \frac{1-e^2}{2e^3} \left(\ln \frac{1+e}{1-e} - 2e \right)$$

$$e = \sqrt{1-(b/a)^2}$$

$$\frac{S_z}{S_x} = \frac{n_x}{n_z}$$

Oblate Ellipsoid (Flake)

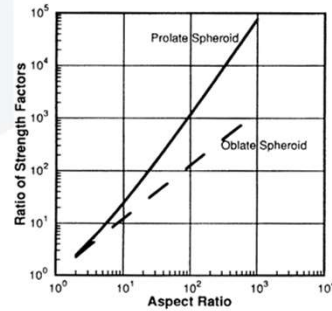
$$n_x = n_y = \frac{1-n_z}{2}$$

$$n_z = \frac{1-e^2}{2e^3} (e - \tan^{-1} e)$$

$$e = \sqrt{(b/a)^2 - 1}$$

$$\frac{S_z}{S_x} = \frac{n_x}{n_z}$$

H. S. Gokturk, et al., "Effects of Particle Shape and Size Distributions on the Electrical and Magnetic Properties of Nickel/Polyethylene Composites," J. Appl. Polym. Sci., Vol. 50, 1891-1901 (1993).

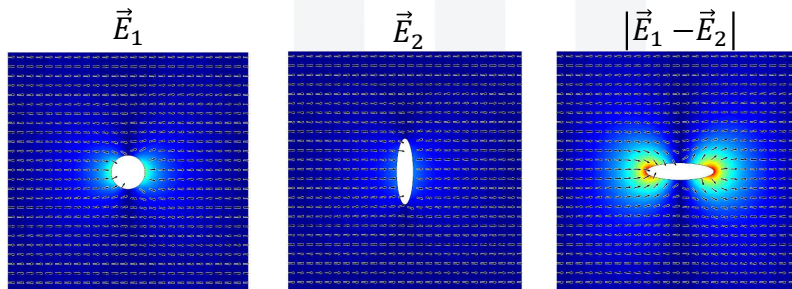


Influence of Particle Shape (2 of 2)

It may appear at first glance that the prolate spheroid (rods) produce a stronger electromagnetic response than the oblate (flakes).

Given equal volume, the oblate spheroid has significantly greater strength factor ratios. For $a/b = 10$, oblate is 4.5 times stronger than the prolate.

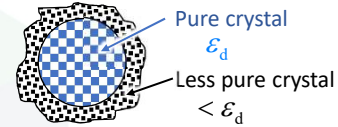
Conclusion – Asymmetric particles with large aspect ratios will produce a stronger electromagnetic response because they produce more “warping” of the field.



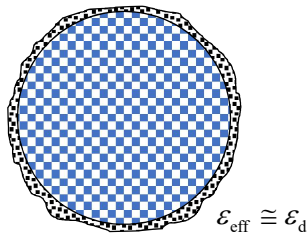
Color conveys amount of field distortion

Influence of Particle Size

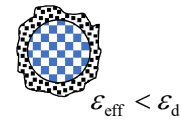
The crystalline structure on the outside of the particles is less pure and typically exhibits a lower dielectric constant than the purer crystal in the interior.



Large Particles: The outside region of large particles is less significant so large particles exhibit higher dielectric constant overall.

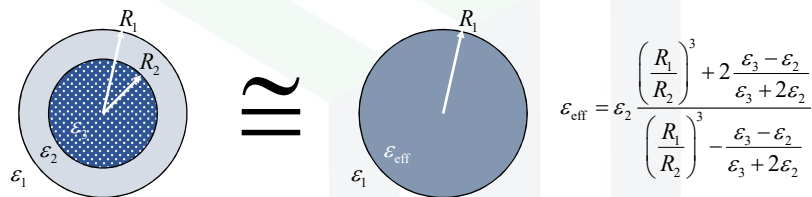


Small Particles: The outside region of small particles is more significant so they exhibit a lower overall dielectric constant.



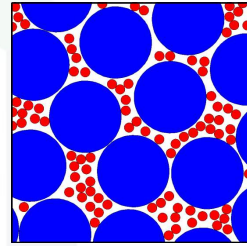
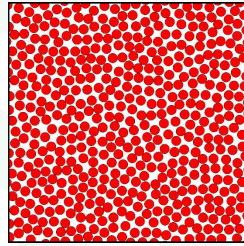
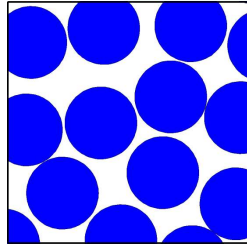
Model for Spherical Shell Particles

The induced electrostatic potential outside a shell particle is indistinguishable from an equivalent spherical particle of the same size with homogeneous permittivity ϵ_{eff} .



This is very often used to approximate biological cells and particles as homogeneous particles. The effective properties can then be incorporated into electromagnetic models or into higher level mixing rules.

Combinations of Different Particle Sizes



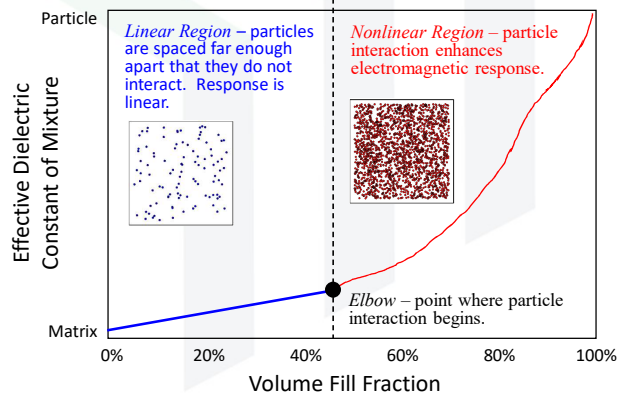
These mixtures have the same fill fraction because they are just scaled versions of each other.

Mixtures with mixed particle sizes give a higher fill fraction.

Small particles should be less than 1/10 of the size of the large particles.

Influence of Particle Spacing

The position of the elbow can be moved to lower volume fill fraction with more asymmetric particles.



Weiner Bounds

For mixtures, there exists limits on the range of possible effective permittivity values. The *Weiner bounds* give the maximum and minimum values.

Two Component Systems

$$\frac{1}{\epsilon_{\min}} = f \frac{1}{\epsilon_{r1}} + (1-f) \frac{1}{\epsilon_{r2}}$$

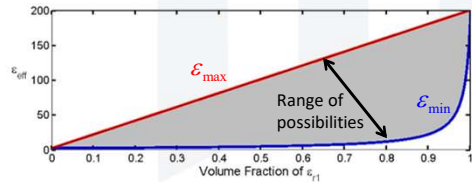
$$\epsilon_{\max} = f \epsilon_{r1} + (1-f) \epsilon_{r2}$$

Multiple Component Systems

$$\frac{1}{\epsilon_{\min}} = \sum_{m=1}^M \frac{f_m}{\epsilon_{r,m}} \quad 1 = \sum_{m=1}^M f_m$$

$$\epsilon_{\max} = \sum_{m=1}^M f_m \epsilon_{r,m}$$

Note: these are the equations we used for first-order effective medium theory.



Particles in a Uniform Matrix

Spherical Particles in a Matrix

$$\frac{\epsilon_{\text{eff}}}{\epsilon_m} = \frac{\epsilon_d (1 + 2f_d) + 2\epsilon_m (1 - f_d)}{\epsilon_d (1 - f_d) + \epsilon_m (2 + f_d)}$$

Low fill fraction f_d

$$\frac{\epsilon_{\text{eff}} - \epsilon_m}{3\epsilon_{\text{eff}}} = \frac{\epsilon_d - \epsilon_m}{\epsilon_d + 2\epsilon_{\text{eff}}} f_d$$

High fill fraction f_d

Cylindrical Particles in a Matrix

$$\epsilon_{\text{eff}} - \epsilon_m = \frac{(\epsilon_d - \epsilon_m)(\epsilon_d + 5\epsilon_m)}{3(\epsilon_d + \epsilon_m)} f_d$$

Low fill fraction f_d

$$1 - f_d = \frac{(\epsilon_d - \epsilon_{\text{eff}})(2\epsilon_m + \epsilon_d + 3\epsilon_{\text{eff}})}{(\epsilon_d - \epsilon_m)(\epsilon_d + 5\epsilon_{\text{eff}})}$$

High fill fraction f_d

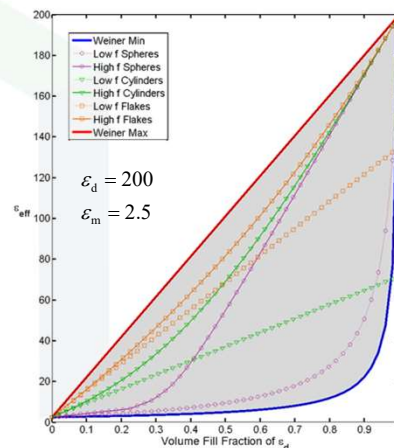
Flake Particles in a Matrix

$$\epsilon_{\text{eff}} - \epsilon_m = \frac{(\epsilon_d - \epsilon_m)(\epsilon_m + 2\epsilon_d)}{3\epsilon_d} f_d$$

Low fill fraction f_d

$$\frac{\epsilon_{\text{eff}}}{\epsilon_d} = \frac{3\epsilon_m + 2f_d(\epsilon_d - \epsilon_m)}{3\epsilon_d - f_d(\epsilon_d - \epsilon_m)}$$

High fill fraction f_d



Jyh Sheen, Zuo-Wen Hong, Weihsing Lui, Wei-Lung Mao, Chin-An Chen, "Study of dielectric constants of binary composites at microwave frequency by mixture laws derived from three basic particle shapes," *European Polymer Journal*, Vol. 45, pp. 1316-1321, 2009.

Exponential Mixing Rules

Standard Exponential Model

$$(\epsilon_{\text{eff}})^k = f_d (\epsilon_d)^k + (1 - f_d) (\epsilon_m)^k$$

$$k = \begin{cases} -1 & \text{Weiner min} \\ 1 & \text{Weiner max} \\ 1/2 & \text{refractive model} \\ 1/3 & \text{random model} \end{cases}$$

Refractive model maintains consistent electrical length nL .
Random model is most realistic.

$$k = \begin{cases} f_d - 0.35 & \text{Wakino model} \\ 1.65 f_d + 0.265 & \text{Stolzle model} \end{cases}$$

Random mixture of two materials.

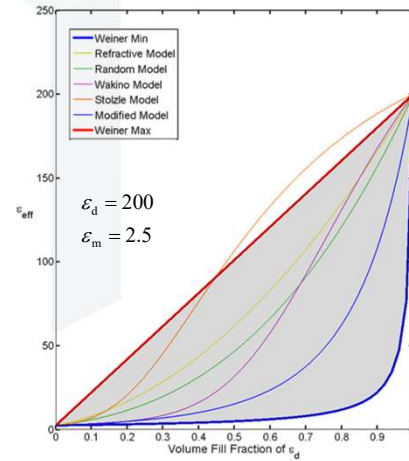
Modified Model with Frequency Dependence

$$(\epsilon_{\text{eff}})^k = z f_d (\epsilon_d)^k + (1 - z f_d) (\epsilon_m)^k$$

$$k = A f_d + B \quad A = 11.0 f^{0.0208} - 14.748$$

$$z = 3.23 f^{0.0416} - 6.481 \quad B = -0.22 f^{0.0537} + 1.0582$$

Jyh Sheen, Zuo-Wen Hong, "Microwave Measurements of Dielectric Constants by Exponential and Logarithmic Mixing Equations," *PIER*, Vol. 100, pp. 13-26, 2010.



Logarithmic Mixing Rules

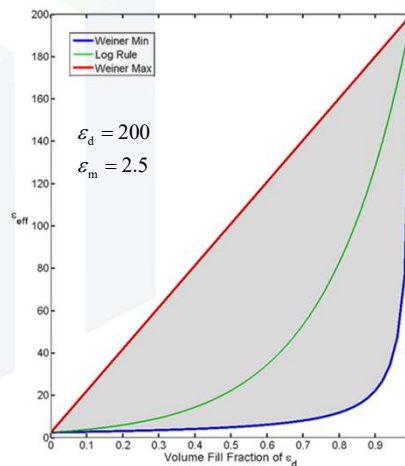
Standard Logarithmic Model

$$\log \epsilon_{\text{eff}} = f_d \log \epsilon_d + (1 - f_d) \log \epsilon_m$$

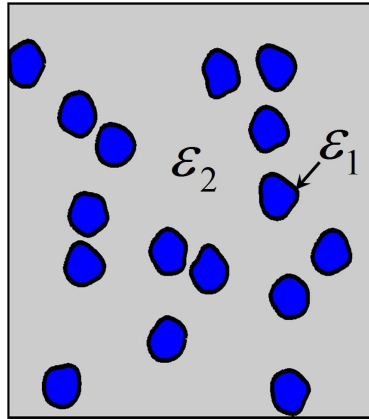
This is sometimes called the Lichtenecker mixing rule as it was published in 1926.

Lichtenecker, K. "Dielectric constant of natural and synthetic mixtures." *Phys. Z* 27 (1926): 115.

Jyh Sheen, Zuo-Wen Hong, "Microwave Measurements of Dielectric Constants by Exponential and Logarithmic Mixing Equations," *PIER*, Vol. 100, pp. 13-26, 2010.



Maxwell-Garnett Effective Media



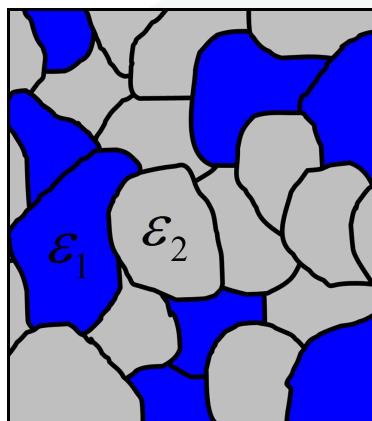
Maxwell-Garnett effective materials are composed of nanoparticles of material ϵ_{r1} sparsely dispersed into a host material of ϵ_{r2} .

$$\frac{\epsilon_{MG} - \epsilon_{r2}}{\epsilon_{MG} + 2\epsilon_{r2}} = f \frac{\epsilon_{r1} - \epsilon_{r2}}{\epsilon_{r1} + 2\epsilon_{r2}}$$

f is the volume fill factor of material ϵ_{r1} .

G. A. Niklasson, C. G. Granqvist, O. Hunderi, "Effective medium models for the optical properties of inhomogeneous materials," Appl. Opt. 20, 26-30 (1981).

Bruggeman Effective Media



Bruggeman effective media can be viewed as a very dense case of Maxwell-Garnett media.

$$f \frac{\epsilon_{r1} - \epsilon_B}{\epsilon_{r1} + 2\epsilon_B} + (1-f) \frac{\epsilon_{r2} - \epsilon_B}{\epsilon_{r2} + 2\epsilon_B} = 0$$

The parameter f is the volume fill factor of ϵ_{r1} , but is perhaps better interpreted as the probability that any block of material will contain ϵ_{r1} .

G. A. Niklasson, C. G. Granqvist, O. Hunderi, "Effective medium models for the optical properties of inhomogeneous materials," Appl. Opt. 20, 26-30 (1981).

Solutions to Maxwell-Garnett and Bruggeman

Maxwell-Garnett

$$\epsilon_{\text{MG}} = \epsilon_{r2} \frac{1+2A}{1-A}$$

$$A = f \frac{\epsilon_{r1} - \epsilon_{r2}}{\epsilon_{r1} + 2\epsilon_{r2}}$$

Bruggeman

$$\epsilon_{\text{B}} = \frac{-B + \sqrt{B^2 - 4AC}}{2A}$$

$$A = 2$$

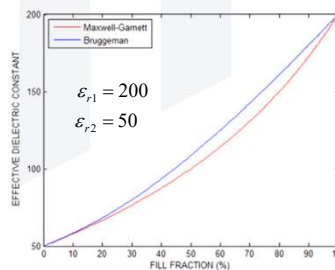
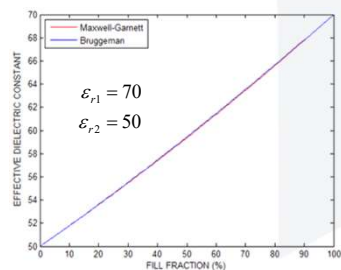
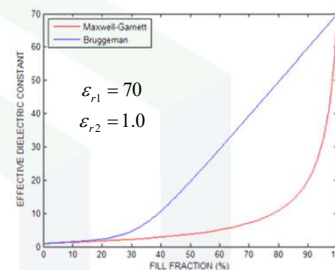
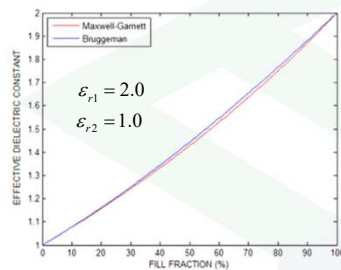
$$C = -\epsilon_{r1}\epsilon_{r2}$$

$$B = (\epsilon_{r1} - 2\epsilon_{r2}) + 3f(\epsilon_{r2} - \epsilon_{r1})$$

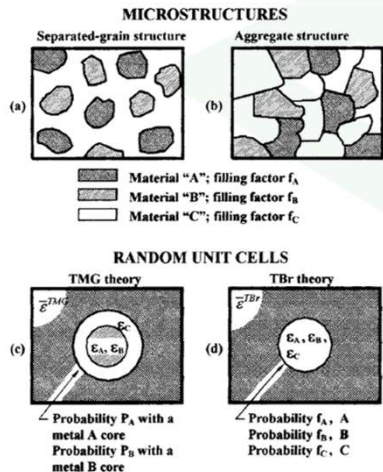
This model can be used to determine the dielectric constant of ϵ_{r1} given a measurement of the effective dielectric constant ϵ_{B} .

$$\epsilon_{r1} = \frac{f + 2A}{f - A} \quad A = (1-f) \frac{\epsilon_{\text{B}} - \epsilon_{r2}}{2\epsilon_{\text{B}} + \epsilon_{r2}}$$

Maxwell-Garnett Vs. Bruggeman



Three-Component Models



Three-Component Maxwell-Garnett

$$P_A \frac{(\epsilon_C - \bar{\epsilon}^{TMG})(\epsilon_A + 2\epsilon_C) + f_{AB}(2\epsilon_C + \bar{\epsilon}^{TMG})(\epsilon_A - \epsilon_C)}{(\epsilon_C + 2\bar{\epsilon}^{TMG})(\epsilon_A + 2\epsilon_C) + f_{AB}(2\epsilon_C - 2\bar{\epsilon}^{TMG})(\epsilon_A - \epsilon_C)} + P_B \frac{(\epsilon_C - \bar{\epsilon}^{TMG})(\epsilon_B + 2\epsilon_C) + f_{AB}(2\epsilon_C + \bar{\epsilon}^{TMG})(\epsilon_B - \epsilon_C)}{(\epsilon_C + 2\bar{\epsilon}^{TMG})(\epsilon_B + 2\epsilon_C) + f_{AB}(2\epsilon_C - 2\bar{\epsilon}^{TMG})(\epsilon_B - \epsilon_C)} = 0, \quad (8)$$

Three-Component Bruggeman

$$f_A \frac{\epsilon_A - \bar{\epsilon}^{TBr}}{\epsilon_A + 2\bar{\epsilon}^{TBr}} + f_B \frac{\epsilon_B - \bar{\epsilon}^{TBr}}{\epsilon_B + 2\bar{\epsilon}^{TBr}} + f_C \frac{\epsilon_C - \bar{\epsilon}^{TBr}}{\epsilon_C + 2\bar{\epsilon}^{TBr}} = 0.$$

R. Luo, "Effective medium theories for the optical properties of three-component composite materials," Appl. Opt. 36, 8153-8158 (1997).

Arbitrary Number of Constituents

$$\frac{\epsilon_{\text{eff}} - \epsilon_{\text{host}}}{\epsilon_{\text{eff}} + 2\epsilon_{\text{host}}} = \sum_{i=1}^N f_i \frac{\epsilon_i - \epsilon_{\text{host}}}{\epsilon_i + 2\epsilon_{\text{host}}} \quad \text{Lorentz-Lorenz effective medium approximation.}$$

$$\sum_{i=1}^N f_i = 1$$

The limits of ϵ_{eff} are set by the Wiener bounds.

Structure and anisotropy are not considered by this equation.