Advanced Electromagnetics:
21st Century Electromagnetics

Preliminary Topics in Electromagnetics

Lecture Outline

- Maxwell's equations
- Wave vectors and polarization
- Scattering at an interface
- Scattering from multiple interfaces
- Phase, group, and energy velocity
- *Bonus Topics*
  - *Lenses*
  - *Gaussian beams*

Maxwell's Equations

GOVERNING EQUATIONS FOR CLASSICAL ELECTROMAGNETICS

Pioneering 21st Century Electromagnetics and Photonics

http://emlab.utep.edu

	Integral Form	Differential Form	Name
Time-Domain	$Q_e(t)=\oint\!\!\!\oint_S \vec{D}(t)\bullet d\vec{s}=\iiint_V \rho_v(t)\,dv$	$\nabla\bullet\vec{D}(t)=\rho_v(t)$	Gauss' Law
	$\oint\!\!\!\oint_S \vec{B}(t)\bullet d\vec{s}=0$	$\nabla\bullet\vec{B}(t)=0$	No Magnetic Charge
	$V_{\text{emf}}(t)=\oint_L \vec{E}(t)\bullet d\vec{\ell}=-\iint_S\left[\frac{\partial\vec{B}(t)}{\partial t}\right]\bullet d\vec{s}$	$\nabla\times\vec{E}(t)=-\frac{\partial\vec{B}(t)}{\partial t}$	Faraday's Law
	$I(t)=\oint_L \vec{H}(t)\bullet d\vec{\ell}=\iint_S\left[\vec{J}(t)+\frac{\partial\vec{D}(t)}{\partial t}\right]\bullet d\vec{s}$	$\nabla\times\vec{H}(t)=\vec{J}(t)+\frac{\partial\vec{D}(t)}{\partial t}$	Ampere's Circuit Law
	$\oint\!\!\!\oint_S \vec{J}\bullet d\vec{s}=-\frac{\partial Q_e}{\partial t}$	$\nabla\bullet\vec{J}=-\frac{\partial\rho_v}{\partial t}$	Continuity of Current
	$\vec{D}(t)=[\varepsilon(t)]*\vec{E}(t)$ $\vec{B}(t)=[\mu(t)]*\vec{H}(t)$	Electric Response Magnetic Response	Constitutive Relations
Frequency-Domain	$Q_e=\oint\!\!\!\oint_S \vec{D}\bullet d\vec{s}=\iiint_V \rho_v\,dv$	$\nabla\bullet\vec{D}=\rho_v$	Gauss' Law
	$\oint\!\!\!\oint_S \vec{B}\bullet d\vec{s}=0$	$\nabla\bullet\vec{B}=0$	No Magnetic Charge
	$V_{\text{emf}}=\oint_L \vec{E}\bullet d\vec{\ell}=-\iint_S\left[j\omega\vec{B}\right]\bullet d\vec{s}$	$\nabla\times\vec{E}=-j\omega\vec{B}$	Faraday's Law
	$I=\oint_L \vec{H}\bullet d\vec{\ell}=\iint_S\left[\vec{J}+j\omega\vec{D}\right]\bullet d\vec{s}$	$\nabla\times\vec{H}=\vec{J}+j\omega\vec{D}$	Ampere's Circuit Law
	$\oint\!\!\!\oint_S \vec{J}\bullet d\vec{s}=-j\omega Q_e$	$\nabla\bullet\vec{J}=-j\omega\rho_v$	Continuity of Current
	$\vec{D}=[\varepsilon]\vec{E}$ $\vec{B}=[\mu]\vec{H}$	Electric Response Magnetic Response	Constitutive Relations

Parameter Definitions

Electric Field Intensity, E (V/m)
Electric Flux Density, D (C/m^2)
Magnetic Field Intensity, H (A/m)
Magnetic Flux Density, B (Wb/m^2)
Electric Current Density, J (A/m^2)
Volume Charge Density, ρ_v (C/m^3)
Permittivity, ε (F/m)
Permeability, μ (H/m)
Electrical Conductivity, σ (1/Ω·m)

Constants

Permittivity: $[\varepsilon]=\varepsilon_0[\varepsilon_r]$
$\varepsilon_0=8.8541878176\times10^{-12}$ (F/m)

Permeability: $[\mu]=\mu_0[\mu_r]$
$\mu_0\approx4\pi\times10^{-7}$ (H/m)
$\mu_0=1.2566370614\times10^{-6}$ (H/m)

Impedance: $\eta_0\approx120\pi$ (Ω)
$\eta_0=376.73031346177$ (Ω)

Speed of Light: $c_0=299,792,458$ (m/s)

Lorentz Force Law

$$\vec{F}=q\vec{E}+q\left(\vec{v}\times\vec{B}\right)$$

Sign Convention

e^{-jkz} For propagation in the $+z$ direction.

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All Together Now…

Divergence Equations

$$\nabla \bullet \vec{B} = 0$$

$$\nabla \bullet \vec{D} = \rho_v$$

Curl Equations

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

What produces fields

Constitutive Relations

$$\vec{D}(t) = \left[\varepsilon(t)\right] * \vec{E}(t)$$

$$\vec{B}(t) = \left[\mu(t)\right] * \vec{H}(t)$$

$*$ means convolution

$=$ means tensor

How fields interact with materials

Physical Boundary Conditions

μ_1 and ε_1

μ_2 and ε_2

Tangential components of $\vec{E}$ and $\vec{H}$ are continuous across an interface.

$E_{1,\mathrm{T}}$ $E_{2,\mathrm{T}}$

$H_{1,\mathrm{T}}$ $H_{2,\mathrm{T}}$

Fields normal to the interface are discontinuous across an interface.

$\varepsilon_1 E_{1,\mathrm{N}}$ $\varepsilon_2 E_{2,\mathrm{N}}$

$\mu_1 H_{1,\mathrm{N}}$ $\mu_2 H_{2,\mathrm{N}}$

These are more complicated boundary conditions, physically and analytically.

Note: Normal components of $\vec{D}$ and $\vec{B}$ are continuous across the interface.

Tangential component of the wave vector is continuous across an interface.

$k_{1,\mathrm{T}}$ $k_{2,\mathrm{T}}$

Wave Vectors and Polarization

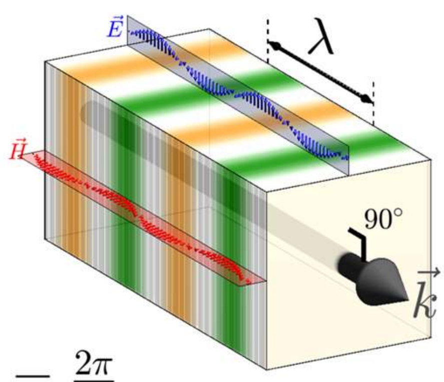
Wave Vector $\vec{k}$

The wave vector $\vec{k}$ conveys two pieces of information: (1) Magnitude conveys the wavelength λ inside the medium, and (2) direction conveys the direction of the wave and is perpendicular to the wave fronts.

$$\vec{k} = k_x \hat{a}_x + k_y \hat{a}_y + k_z \hat{a}_z$$

$$\left|\vec{k}\right| = \frac{2\pi}{\lambda} = \frac{2\pi n}{\lambda_0}$$

If the frequency of the wave is known and constant, and it usually is, the magnitude of $\vec{k}$ conveys the *refractive index n* of the material the wave is in.



$$|\vec{k}| = \frac{2\pi}{\lambda}$$

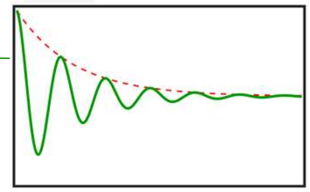
The Complex Wave Number $\tilde{k}$

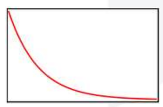
A wave travelling in the $+z$ direction can be written in terms of the complex wave number $\tilde{k}$ as

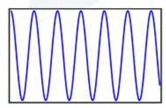
$$\vec{E}(z)=\vec{P}e^{-j\tilde{k}z} \qquad \tilde{k}=k'-jk''$$

Substituting $\tilde{k}=k'+jk''$ into the wave solution gives

$$\vec{E}(z)=\vec{P}e^{-j(k'-jk'')z} \quad =\overbrace{\vec{P}\underbrace{e^{-k''z}}_{\text{attenuation}}\underbrace{e^{-jk'z}}_{\text{oscillation}}}^{\text{attenuation \& oscillation}}$$







Attenuation Coefficient α and Phase Constant β

A wave travelling the $+z$ direction can also be written in terms of an attenuation coefficient α and a phase constant β and as

$$\vec{E}(z)=\vec{E}_0\, e^{-k''z}e^{-jk'z}$$

$$\vec{E}(z)=\vec{E}_0\, \underbrace{e^{-\alpha z}}_{\text{attenuation}}\ \underbrace{e^{-j\beta z}}_{\text{oscillation}}$$

$$\vec{E}(z)=\vec{E}_0\, e^{-\gamma' z}e^{-j\gamma'' z}$$

$$\vec{E}(z)=\vec{E}_0\, \underbrace{e^{-\alpha z}}_{\text{attenuation}}\ \underbrace{e^{-j\beta z}}_{\text{oscillation}}$$

This provides the physical meaning of the real and imaginary parts of the complex wave number $\tilde{k}$ and propagation constant $\tilde{\gamma}$.

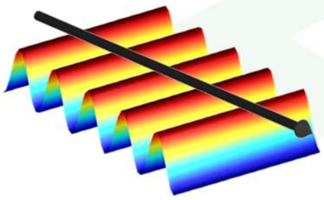
$$\tilde{k}=\beta-j\alpha \qquad \tilde{\gamma}=\alpha+j\beta$$

$$\alpha=-\mathrm{Im}\left[\tilde{k}\right] \qquad \alpha=\mathrm{Re}\left[\tilde{\gamma}\right]$$

$$\beta=\mathrm{Re}\left[\tilde{k}\right] \qquad \beta=\mathrm{Im}\left[\tilde{\gamma}\right]$$

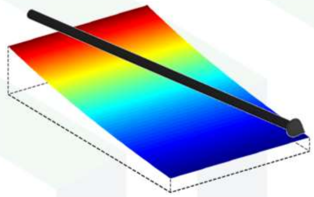
1D Waves with Complex Wave Number $\tilde{k}$

Purely Real k



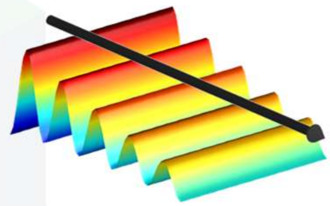
- Uniform amplitude
- Oscillations move power
- Considered to be a propagating wave

Purely Imaginary k



- Decaying amplitude
- No oscillations, no flow of power
- Considered to be evanescent

Complex k



- Decaying amplitude
- Oscillations move power
- Considered to be a propagating wave (not evanescent)

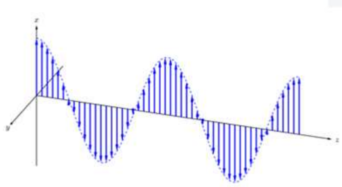
This implies that these are the only 2.5 configurations that electromagnetic fields can take on.

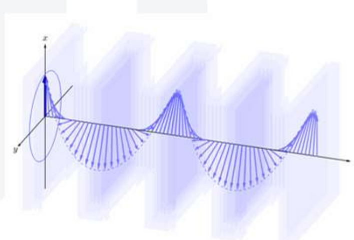
What is Polarization?

Polarization is that property of a electromagnetic wave which describes the time-varying direction and relative magnitude of the electric field vector.

$$\vec{E}(\vec{r}) = \boxed{\vec{P}} e^{-j\vec{k}\bullet\vec{r}}$$

Polarization Vector

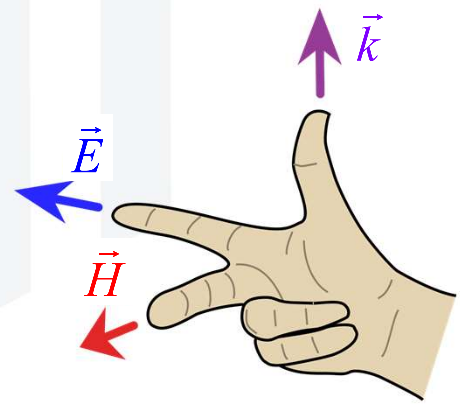




Plane Waves in LHI Media are TEM

In linear, homogeneous, and isotropic (LHI) media, the electric field $\vec{E}$, magnetic field $\vec{H}$, and direction of the wave $\vec{k}$ are all perpendicular to each other. These are called transverse electromagnetic (TEM) waves.

$$\vec{E} \perp \vec{k} \perp \vec{H}$$



Further $\vec{E}$, $\vec{H}$, and $\vec{k}$ follow a right-hand rule.

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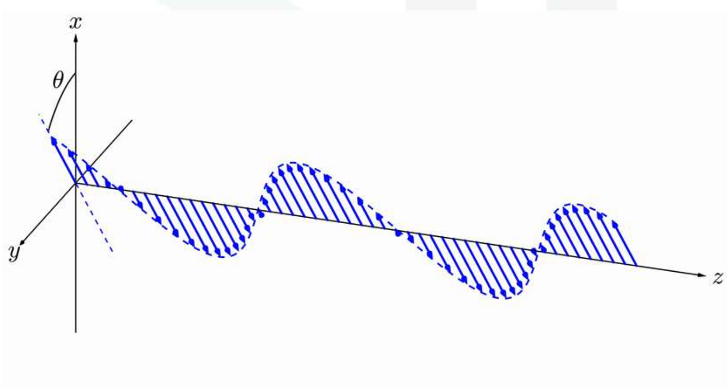
Linear Polarization (LP)

An electromagnetic wave has linear polarization if the electric field oscillation is confined to a single plane.

$$\vec{E} = E_0 \left(\cos\theta \hat{a}_x + \sin\theta \hat{a}_y \right) e^{-jkz}$$



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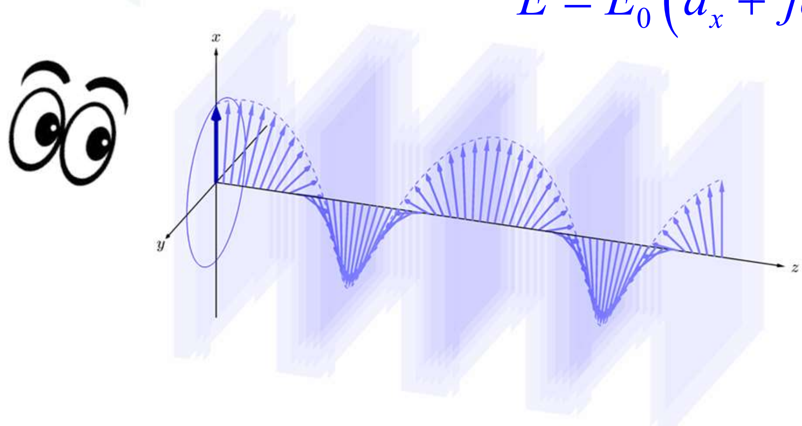
Right Circular Polarization (RCP)

An electromagnetic wave has right circular polarization if the electric field rotates clockwise when viewed from behind.

$$\vec{E} = E_0\left(\hat{a}_x + j\hat{a}_y\right)e^{-jkz}$$



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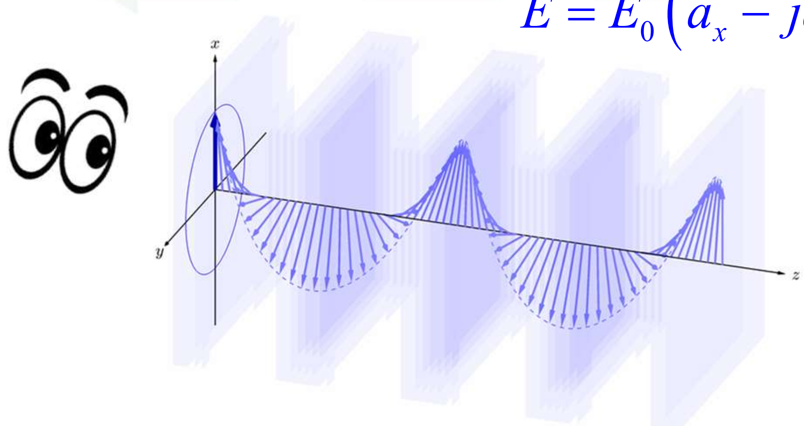
Left Circular Polarization (LCP)

An electromagnetic wave has left circular polarization if the electric field rotates counterclockwise when viewed from behind.

$$\vec{E} = E_0\left(\hat{a}_x - j\hat{a}_y\right)e^{-jkz}$$



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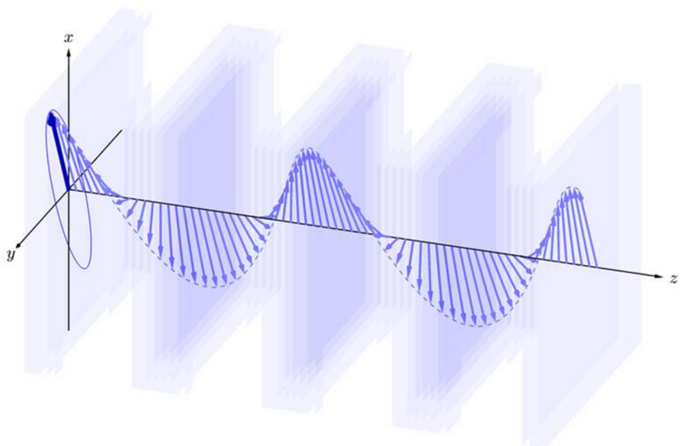
Elliptical Polarization (EP)

An electromagnetic wave has elliptical polarization if the electric field rotates with time to form an ellipse.

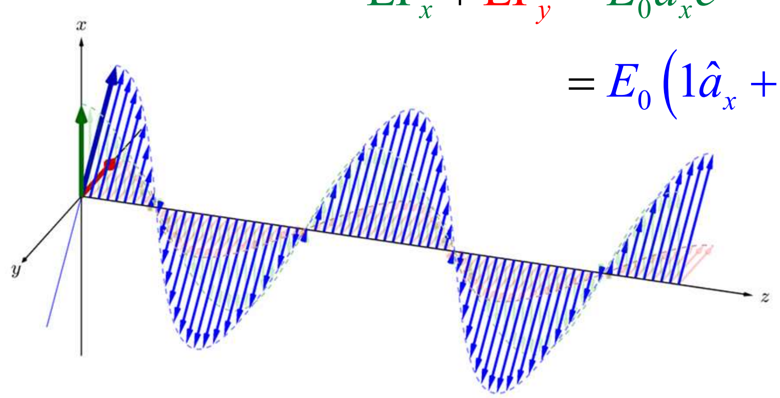


Continuum of LP+LP Polarizations

As the phase difference δ between the two LP waves changes, the resulting polarization oscillates between RCP, LCP, $LP_{45°}$, $LP_{-45°}$, and EP.

$$\mathrm{LP}_x + \mathrm{LP}_y = E_0 \hat{a}_x e^{-jkz} + E_0 e^{j\delta} \hat{a}_y e^{-jkz}$$

$$= E_0 \left(1\hat{a}_x + e^{j\delta} \hat{a}_y\right) e^{-jkz}$$

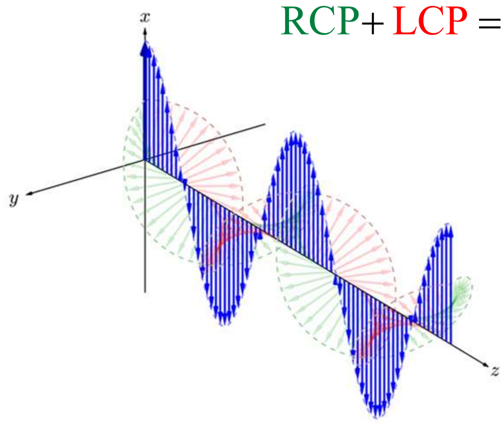


Continuum of CP+CP Polarizations

As the phase difference θ between the two CP waves changes, the resulting polarization is LP where the phase θ translates into a tilt angle.

$$\text{RCP}+\text{LCP} = E_0\left(1\hat{a}_x + j\hat{a}_y\right)e^{-jkz} + E_0\left(1\hat{a}_x - j\hat{a}_y\right)e^{j\theta}e^{-jkz}$$

$$= E_0\left[\cos\theta\hat{a}_x + \sin\theta\hat{a}_y\right]e^{-jkz}$$



Why is Polarization Important?

- Different polarizations can behave differently in a device
- Orthogonal polarizations will not interfere with each other
- Polarization becomes critical when analyzing devices on the scale of a wavelength
- Focusing properties are different
- Reflection/transmission can be different
- Frequency of resonators
- Cutoff conditions for filters, waveguides, etc.

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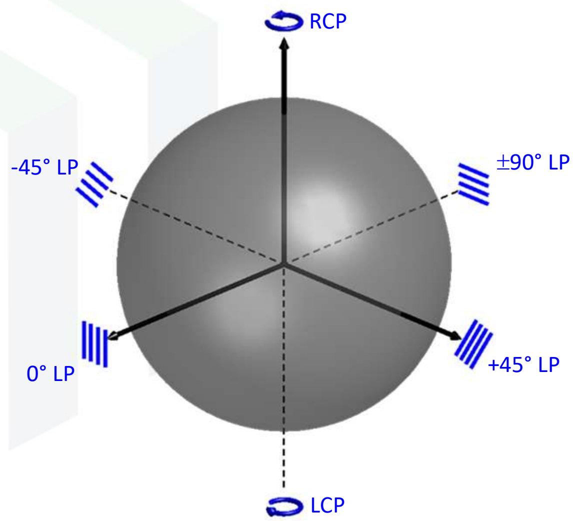
Poincaré Sphere

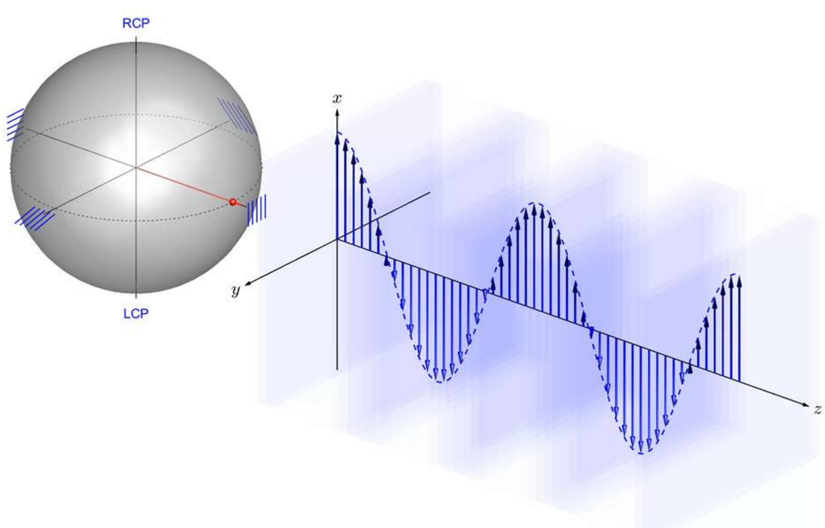
The polarization of a wave can be mapped to a unique point on the Poincaré sphere.

Points on opposite sides of the sphere are orthogonal.

See Balanis, Chap. 4.



Continuum of All Polarizations

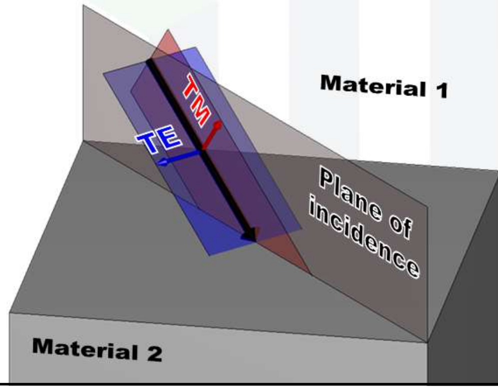


TE and TM

The labels "TE" and "TM" are used when the orientation of a linearly polarized wave is being described <u>relative to a device</u>.

TE/perpendicular/s – the electric field $\vec{E}$ is polarized perpendicular to the plane of incidence.

TM/parallel/p – the electric field $\vec{E}$ is polarized parallel to the plane of incidence.



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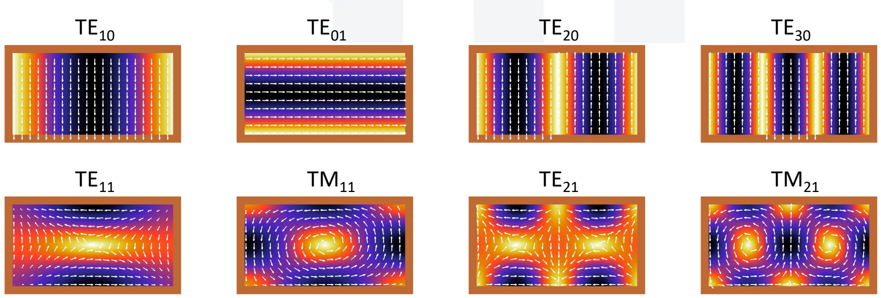
CAUTION: Another Place You Will See TE and TM Labels

The labels TE and TM are also used to describe modes in homogeneous metallic waveguides.

There really is no comparing TE and TM waveguide modes to TE and TM plane waves incident on a interface. It is a completely different concept.



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Scattering at an Interface

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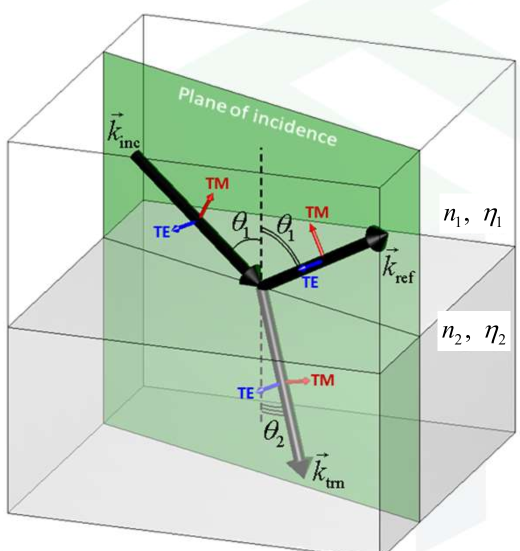
Reflection, Transmission and Refraction at an Interface



Angles

$$\theta_{\text{inc}} = \theta_{\text{ref}} = \theta_1$$

$$n_1 \sin\theta_1 = n_2 \sin\theta_2 \qquad \text{Snell's Law}$$

TE Polarization

$$r_{\text{TE}} = \frac{\eta_2 \cos\theta_1 - \eta_1 \cos\theta_2}{\eta_2 \cos\theta_1 + \eta_1 \cos\theta_2}$$

$$t_{\text{TE}} = \frac{2\eta_2 \cos\theta_1}{\eta_2 \cos\theta_1 + \eta_1 \cos\theta_2}$$

$$1 + r_{\text{TE}} = t_{\text{TE}}$$

TM Polarization

$$r_{\text{TM}} = \frac{\eta_2 \cos\theta_2 - \eta_1 \cos\theta_1}{\eta_1 \cos\theta_1 + \eta_2 \cos\theta_2}$$

$$t_{\text{TM}} = \frac{2\eta_2 \cos\theta_1}{\eta_1 \cos\theta_1 + \eta_2 \cos\theta_2}$$

$$1 + r_{\text{TM}} = \frac{\cos\theta_2}{\cos\theta_1} t_{\text{TM}}$$

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Simplifications at Normal Incidence

For normal incidence, the law of reflection reduces to

$$\theta_{\text{inc}} = \theta_{\text{ref}} = \theta_1 = \theta_2 = 0°$$

In isotropic materials, the different polarizations reflect and transmit the same.

$$r = r_{\text{TE}} = r_{\text{TM}} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} \qquad R = |r|^2$$

$$t = t_{\text{TE}} = t_{\text{TM}} = \frac{2\eta_2}{\eta_2 + \eta_1} \qquad T = \frac{n_2}{n_1}|t|^2$$

It really does not make sense to talk about TE and TM for normal incidence because there is no plane of incidence from which to define it. All polarizations scatter the same.

$$1 + r = t$$

This is NOT conservation of power because these are field amplitude quantities, not power quantities.

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The Critical Angle (Total Reflection)

Above the critical angle θ_c, reflection is 100%

$$|r_{\text{TE}}| = \left|\frac{\eta_2 \cos\theta_c - \eta_1 \cos\theta_2}{\eta_2 \cos\theta_c + \eta_1 \cos\theta_2}\right| = 1$$

$$|r_{\text{TM}}| = \left|\frac{\eta_2 \cos\theta_2 - \eta_1 \cos\theta_c}{\eta_1 \cos\theta_c + \eta_2 \cos\theta_2}\right| = 1$$

This will happen when $\cos(\theta_2)$ is imaginary. These conditions are derived from Snell's Law.

$$\cos\theta_2 = \sqrt{1 - \sin^2\theta_2} = \sqrt{1 - \frac{n_1^2}{n_2^2}\sin^2\theta_c}$$

$$n_1 \sin\theta_c = n_2 \sin\theta_2$$

$$\cos^2 90° = 0 = 1 - \frac{n_1^2}{n_2^2}\sin^2\theta_c$$

$$\frac{n_1^2}{n_2^2}\sin^2\theta_c = 1$$

$$\theta_c = \sin^{-1}\left(\frac{n_2}{n_1}\right)$$

$$\boxed{\theta_1 \geq \theta_c = \sin^{-1}\left(\frac{n_2}{n_1}\right)}$$

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Brewster's Angle (Total Transmission)

TE Polarization

$$r_{\text{TE}} = \frac{\eta_2 \cos\theta_B - \eta_1 \cos\theta_2}{\eta_2 \cos\theta_B + \eta_1 \cos\theta_2} = 0 \quad \Rightarrow \quad \sin\theta_B = \sqrt{\left(\frac{\varepsilon_2}{\varepsilon_1} - \frac{\mu_2}{\mu_1}\right) \Big/ \left(\frac{\mu_1}{\mu_2} - \frac{\mu_2}{\mu_1}\right)} \qquad \frac{\varepsilon_2}{\varepsilon_1} \le \frac{\mu_1}{\mu_2}$$

As long as $\mu_1=\mu_2$ then there is no Brewster's angle.

Generally, most materials have a very week magnetic response and there is no Brewster's angle for TE polarized waves.

TM Polarization

$$r_{\text{TM}} = \frac{\eta_2 \cos\theta_2 - \eta_1 \cos\theta_1}{\eta_1 \cos\theta_1 + \eta_2 \cos\theta_2} = 0 \quad \Rightarrow \quad \sin\theta_B = \sqrt{\left(\frac{\varepsilon_2}{\varepsilon_1} - \frac{\mu_2}{\mu_1}\right) \Big/ \left(\frac{\varepsilon_2}{\varepsilon_1} - \frac{\varepsilon_1}{\varepsilon_2}\right)} \qquad \frac{\mu_2}{\mu_1} \le \frac{\varepsilon_1}{\varepsilon_2}$$

If $\varepsilon_1=\varepsilon_2$ then there is no Brewster's angle.

For materials with no magnetic response, the Brewster's angle equation reduces to

$$\tan\theta_B = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}} = \frac{n_2}{n_1} \qquad \varepsilon_1 \ge \varepsilon_2$$

This is the most well known equation.

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Notes on a Single Interface

- It is a change in impedance that causes reflections
- Snell's Law quantifies the angle of transmission
- Angle of transmission and reflection does not depend on polarization.
- The Fresnell equations quantify the amount of reflection and transmission
- Amount of reflection and transmission depends on the polarization
- For incident angles greater than the critical angle, a wave will be completely reflected regardless of its polarization.
- When a wave is incident at the Brewster's angle, a particular polarization will be completely transmitted.

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Validity of Law of Reflection and Snell's Law

There are no plane wave sources. Beam sources are more realistic.

A "beam" can be decomposed into a plane wave spectrum.

The Fresnel equations predict that each of the component plane wave will reflect and transmit with a different amplitude depending on its angle and polarization.

$$\theta_{\text{ref}} \overset{???}{\neq} \theta_{\text{inc}}$$

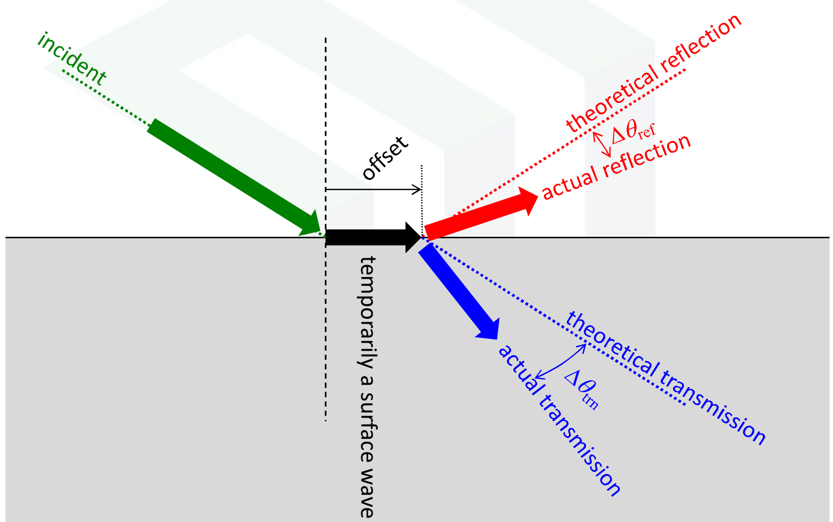
This means that the amplitude profile of a beam will be modified after reflection and transmission.

If the amplitude profile is modified, then the beam will propagate and diffract differently.

$$n_1 \sin\theta_1 \overset{???}{\neq} n_2 \sin\theta_2$$

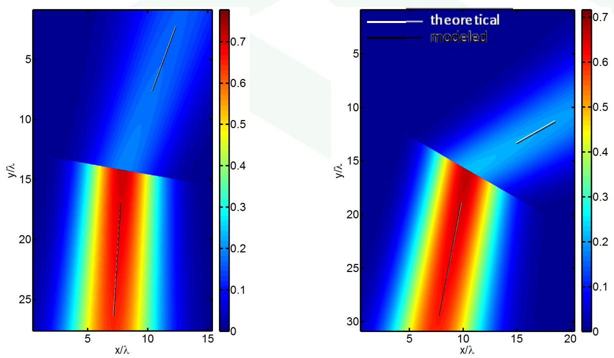
This ultimately means the reflected and transmitted beams will propagate at different angles than the law of reflection and Snell's law predict.

More Accurate Picture of Reflection and Transmission



Simulation Example

A Gaussian beam $5\lambda_0$ wide is incident from air onto glass with n = 1.52.



$\theta_{inc} = 10°$ $\Delta\theta_{ref} \cong 0.32°$ $\Delta\theta_{trn} \cong 0.05°$

$\theta_{inc} = 30°$ $\Delta\theta_{ref} \cong 0.45°$ $\Delta\theta_{trn} \cong 0.11°$

$\theta_{inc} = 60°$ $\Delta\theta_{ref} \cong ?$ $\Delta\theta_{trn} \cong ?$

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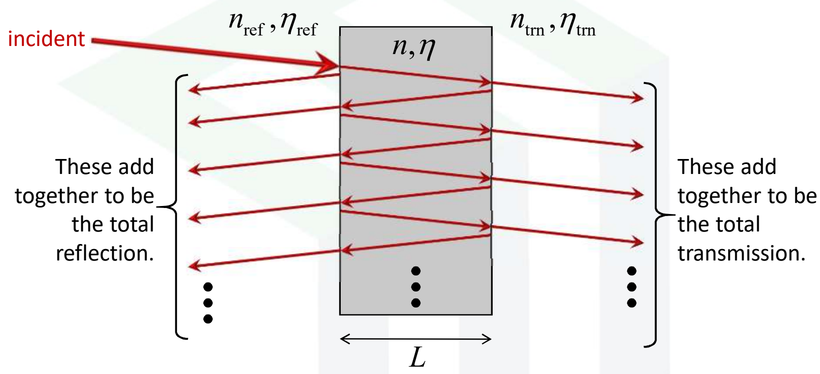
Scattering from Multiple Interfaces

Reflection and Transmission from a Slab



$$r = \frac{r_1 + r_2 e^{-j2k_0 nL}}{1 + r_1 r_2 e^{-j2k_0 nL}}$$

$$r_1 = \frac{\eta - \eta_{\text{ref}}}{\eta + \eta_{\text{ref}}} \qquad r_2 = \frac{\eta_{\text{trn}} - \eta}{\eta_{\text{trn}} + \eta}$$

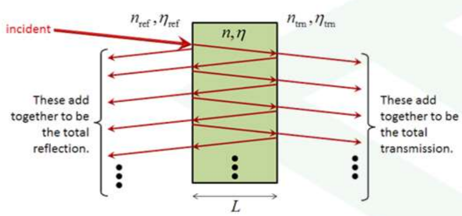
For small reflections,

$$r \simeq r_1 + r_2 e^{-j2k_0 nL}$$

EMPossible

Slide 35

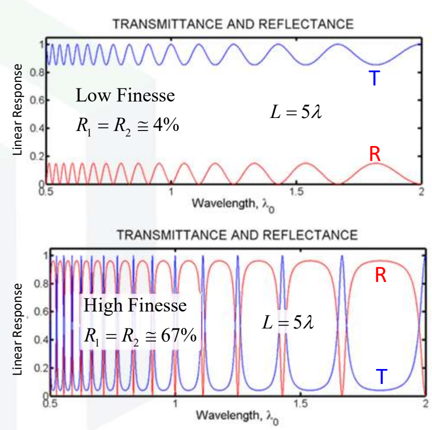
The Fabry-Perot Cavity



$$R_{\text{FP}} = \frac{R_1 + R_2 (1-A)^2 + 2\sqrt{R_1 R_2}(1-A)\cos\phi}{1 + R_1 R_2 + 2\sqrt{R_1 R_2}\cos\phi}$$

$$T_{\text{FP}} = \frac{T_1 T_2}{1 + R_1 R_2 + 2\sqrt{R_1 R_2}\cos\phi}$$

$R_1 \equiv$ reflectance at first interface
$R_2 \equiv$ reflectance at second interface
$T_1 = 1 - R_1 \equiv$ transmittance at first interface
$T_2 = 1 - R_2 \equiv$ transmittance at second interface
$A \equiv$ power loss through cavity
$\phi \equiv$ round trip phase shift in the cavity



For small reflections,

$$R_{\text{FP}} \simeq 2R(1 + \cos\phi)$$
$$T_{\text{FP}} \simeq 1 - 2R(1 + \cos\phi)$$

EMPossible

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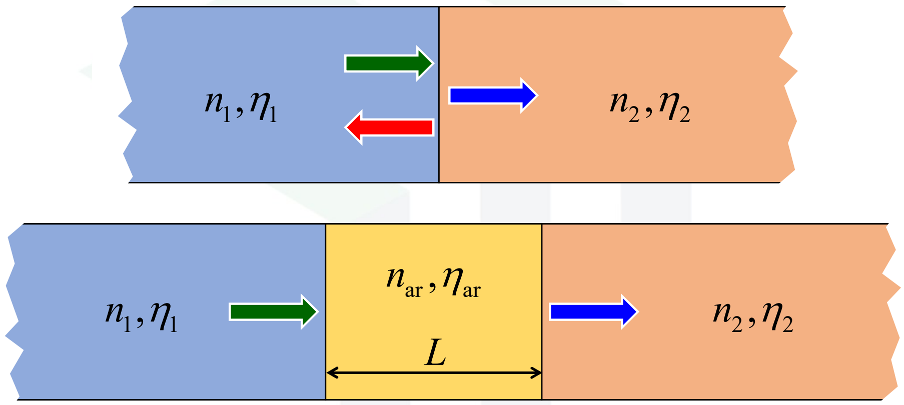
Anti-Reflection Coatings



General Case

$$\eta_{\text{ar}} = \sqrt{\eta_1 \eta_2} \qquad L = \frac{\lambda_0}{4n_{\text{ar}}}$$

$$n_{\text{ar}} = \sqrt{n_1 n_2} \qquad L = \frac{\lambda_0}{4n_{\text{ar}}}$$

No magnetic response

EMPossible

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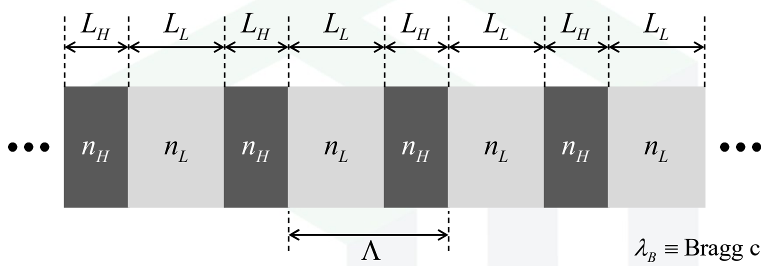
Bragg Gratings

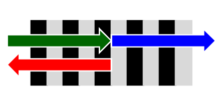


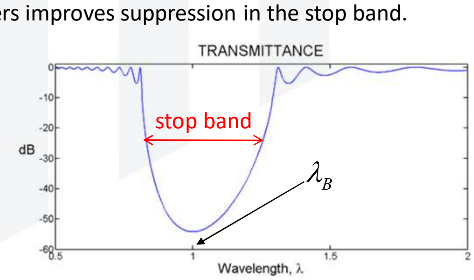
$$L_L = \frac{\lambda_B}{4n_L}$$

$$L_H = \frac{\lambda_B}{4n_H}$$

$\lambda_B \equiv$ Bragg center wavelength

A Bragg grating is typically composed of alternating layers of high and low refractive index. Each layer is $\lambda/4$ thick. Higher index contrast provides wider stop band. More layers improves suppression in the stop band.





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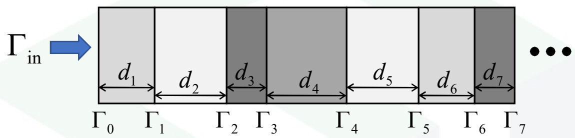
Multilayer Filters



For small reflection coefficients Γ, the overall reflection coefficient can be written as

$$\Gamma_{\text{in}} \cong \Gamma_0 + \Gamma_1 e^{-j2\theta} + \Gamma_2 e^{-j4\theta} + \cdots + \Gamma_M e^{-j2M\theta} \qquad \theta \cong \beta_1 d_1 \cong \cdots \cong \beta_M d_M$$

where Γ_i is the reflection coefficient at the i^{th} interface.

$$\Gamma_i = \frac{Z_{i+1} - Z_i}{Z_{i+1} + Z_i} = \frac{\eta_{i+1} - \eta_i}{\eta_{i+1} + \eta_i}$$

Any desired filter response can be designed by appropriate selection of Γ's and incorporating enough segments. The design process is essentially the same as for designing digital filters.

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Phase, Group, and Energy Velocity

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Phase Velocity v_p

The phase velocity v_p of a wave is the speed at which the phase of a single frequency wave propagates through space.

It is defined in terms of the angular frequency ω (number of oscillations per unit time) and the wave number k (number of oscillations per unit distance).

$$v_\mathrm{p} = \frac{\omega}{k}$$

Derivation of Phase Velocity

Start with the expression for a wave travelling in the x direction.

$$E(z;t) = \sin\underbrace{(kx - \omega t)}$$

Wave moves at a speed that keeps the argument of the sine function constant.

Set the argument equal to zero and rearrange the terms.

$$kx - \omega t = 0$$

$$kx = \omega t$$

$$\frac{x}{t} = \frac{\omega}{k}$$

$$v_p = \frac{x}{t} = \frac{\omega}{k}$$

This has units of distance/time, which is velocity.
This was derived from the phase ϕ of the wave $\sin(\phi)$.

Phase Refractive Index n_{p}

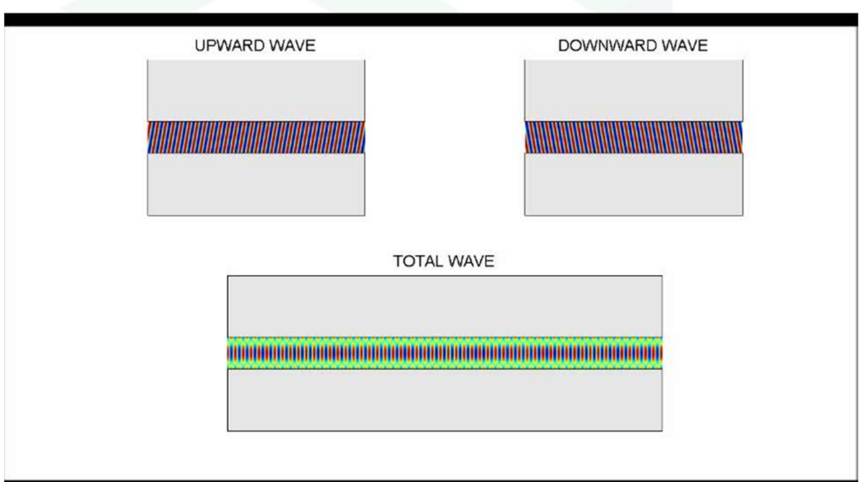
A medium can be characterized by it phase refractive index n_{p}. This is the factor describing how much slower phase is propagating than the speed of light in vacuum c_0.

$$v_{\mathrm{p}} = \frac{c_0}{n_{\mathrm{p}}}$$

EMPossible 43

Phase Velocity Can Approach Infinity in a Waveguide (1 of 2)

A low-order mode can be thought of as the sum of two plane waves at small angles relative to the longitudinal direction of the waveguide. Phase propagates slowly.



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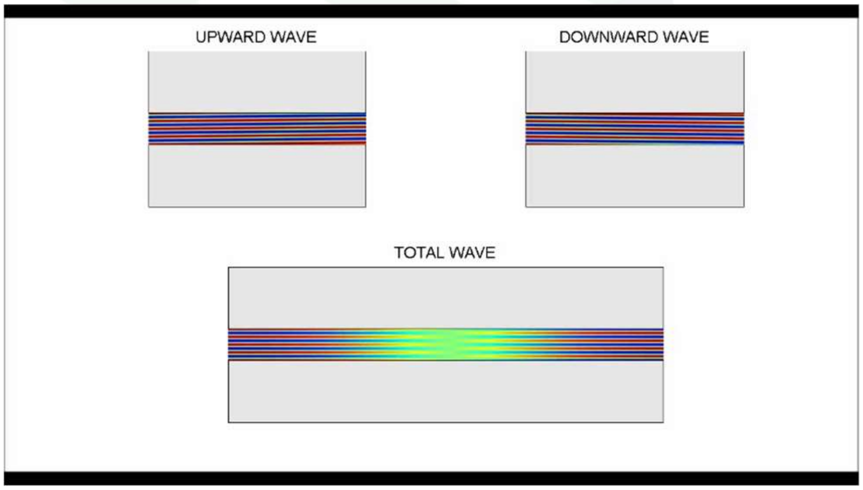
Phase Velocity Can Approach Infinity in a Waveguide (2 of 2)

A high-order mode can be thought of as the sum of two plane waves at large angles relative to the longitudinal direction of the waveguide. Phase propagates quickly.



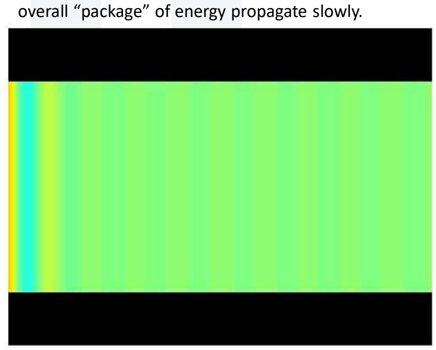
Group Velocity v_{g}

The group velocity v_{g} is the speed and direction in which the envelope of the wave's amplitude propagates.

It is defined as

$$v_{\mathrm{g}} = \frac{\partial \omega}{\partial k}$$

Here, the wave appears to be very fast, but the overall "package" of energy propagate slowly.



Group Refractive Index n_{g}

A medium can be characterized by its group refractive index n_{g}. This is the factor describing how much slower the envelope of a wave is propagating than the speed of light in vacuum c_0.

$$v_{\mathrm{g}} = \frac{c_0}{n_{\mathrm{g}}}$$

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Phase Vs. Group Velocity

By their definitions, the phase velocity • applies only to a wave at a single frequency.

The group velocity • applies to a packet of waves covering some spectrum.

The phase and group velocities are often the same, but they can be different.

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Derivation of Group Velocity (1 of 7)

By definition, group velocity applies to a wave composed of more than one frequency. A wave composed of two frequencies can be written as

$$E(z;t) = \underbrace{\sin(k_1 x - \omega_1 t)}_{\text{Wave 1}} + \underbrace{\sin(k_2 x - \omega_2 t)}_{\text{Wave 2}}$$

Derivation of Group Velocity (2 of 7)

Using the following identity from trigonometry,

$$\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

the sum of two waves becomes

$$\begin{aligned} E(z;t) &= \sin(k_1 x - \omega_1 t) + \sin(k_2 x - \omega_2 t) \\ &= 2\sin\left[\frac{(k_2 x - \omega_2 t) + (k_1 x - \omega_1 t)}{2}\right]\cos\left[\frac{(k_2 x - \omega_2 t) - (k_1 x - \omega_1 t)}{2}\right] \\ &= 2\sin\left[\left(\frac{k_2 + k_1}{2}\right)x - \left(\frac{\omega_2 + \omega_1}{2}\right)t\right]\cos\left[\left(\frac{k_2 - k_1}{2}\right)x - \left(\frac{\omega_2 - \omega_1}{2}\right)t\right] \end{aligned}$$

Derivation of Group Velocity (3 of 7)

The last equation can be written in terms of the center frequency ω_c and the bandwidth $\Delta\omega$.

$$E(z;t) = 2\sin(k_c x - \omega_c t)\cos(\Delta k \cdot x - \Delta\omega \cdot t)$$

$$k_c = \frac{k_2 + k_1}{2} \qquad \Delta k = \frac{k_2 - k_1}{2}$$

$$\omega_c = \frac{\omega_2 + \omega_1}{2} \qquad \Delta\omega = \frac{\omega_2 - \omega_1}{2}$$

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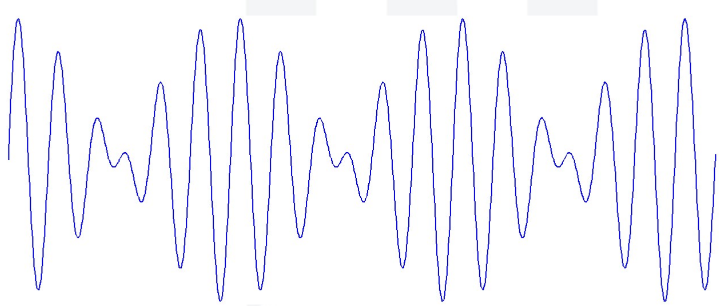
Derivation of Group Velocity (4 of 7)

Interpret this as a sine wave at frequency ω_c and wave number k_c that is modulated by a cosine function.

$$E(z;t) = 2\sin(k_c x - \omega_c t)\cos(\Delta k \cdot x - \Delta\omega \cdot t)$$



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Derivation of Group Velocity (5 of 7)

The $\cos(\Delta k \cdot x - \Delta\omega \cdot t)$ term was identified as the envelope.

How quickly does this move?

It moves at a speed that keeps the argument of the cosine function constant.

$$\Delta k \cdot x - \Delta\omega \cdot t = \text{constant}$$

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Derivation of Group Velocity (6 of 7)

To derive a velocity term (dx/dt), we differentiate this equation.

$$\Delta k \cdot x - \Delta\omega \cdot t = 0$$

$$\Delta k \cdot dx - \Delta\omega \cdot dt = 0$$

The equation can now be manipulated to derive a quantity with units of velocity.

$$\Delta k \cdot dx - \Delta\omega \cdot dt = 0$$

$$\Delta k \cdot dx = \Delta\omega \cdot dt$$

$$\frac{dx}{dt} = \frac{\Delta\omega}{\Delta k} \longrightarrow v \equiv \frac{dx}{dt} = \frac{\Delta\omega}{\Delta k}$$

EMPossible 54

Derivation of Group Velocity (7 of 7)

Take the limit as the deltas become very small.

$$v_{\mathrm{g}} = \lim_{\substack{\Delta\omega\to 0\\ \Delta k\to 0}} \frac{\Delta\omega}{\Delta k} = \frac{d\omega}{dk}$$

The equivalent equation in more than one dimension is

$$\vec{v}_{\mathrm{g}} = \nabla_{\vec{k}}\,\omega\left(\vec{k}\right) \qquad \omega\left(\vec{k}\right) \equiv \text{dispersion relation}$$

EMPossible

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Ordinary Materials

In an ordinary material, the dispersion relation is

$$\left(\frac{\omega}{c_0}\right)^2 n^2 = k^2 \qquad \text{or} \qquad \omega\left(k\right) = \frac{c_0}{n}k$$

The phase velocity is

$$v_{\mathrm{p}} = \frac{c_0}{n}$$

The group velocity is

$$v_{\mathrm{g}} = \frac{d\omega}{dk} = \frac{c_0}{n}$$

Phase velocity and group velocity are equal.

When are they not equal?

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Dispersive Materials (1 of 2)

In a dispersive material, the refractive index n can be different at different frequencies. The dispersion relation is differentiated as follows.

$$\omega n = c_0 k$$

$$\omega dn + n d\omega = c_0 dk$$

Rearrange the terms to arrive at

$$\frac{c_0}{n} = \frac{\omega}{n}\frac{dn}{dk} + \frac{d\omega}{dk}$$

This is group velocity, v_{g}

This is phase velocity, v_{p}

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Dispersive Materials (2 of 2)

The previous work leads to an expression relating group and phase velocity.

$$v_p = \frac{\omega}{n}\frac{dn}{dk} + v_g$$

Solving this for group velocity yields

$$v_g = v_p - \frac{\omega}{n}\frac{dn}{dk} = v_p\left(1 - \frac{k}{n}\frac{dn}{dk}\right) \qquad k = \frac{\omega}{v_p}$$

From this, it is the dn/dk term that is responsible for $v_{\text{g}} \neq v_{\text{p}}$.

Any time the refractive index n is not constant, the medium is said to have dispersion and the group velocity will deviate from the phase velocity.

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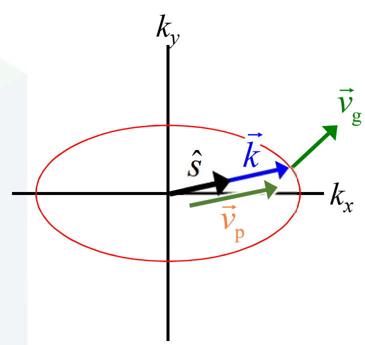
Phase and Group Refractive Indices

From the previous equations the phase refractive index n_{p} and group refractive index n_{g} can be written as

$$n_{\mathrm{p}} = \frac{kc_0}{\omega}$$

$$\begin{aligned} n_{\mathrm{g}} &= n_{\mathrm{p}} + \omega \frac{dn_p}{d\omega} \\ &= n_{\mathrm{p}} - \lambda_0 \frac{dn_{\mathrm{p}}}{d\lambda_0} \end{aligned}$$

Summary of Phase, Group and Energy Velocity



Phase Velocity $\vec{v}_{\mathrm{p}}$

Phase velocity describes the speed and direction of the phase of a wave.

$$\vec{v}_{\mathrm{p}} = \frac{\omega}{|\vec{k}|}\hat{s} \qquad n_{\mathrm{p}} = \frac{c_0}{v_{\mathrm{p}}}$$

Group Velocity $\vec{v}_{\mathrm{g}}$

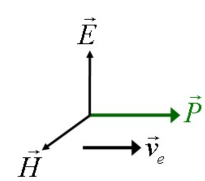
Group velocity describes the speed and direction of the envelope of a pulse.

$$\vec{v}_{\mathrm{g}} = \nabla_{\vec{k}}\omega\left(\vec{k}\right) \qquad n_{\mathrm{g}} = \frac{c_0}{v_{\mathrm{g}}} \qquad \vec{v}_{\mathrm{g}} = \vec{v}_{\mathrm{p}} \quad \text{if no dispersion}$$

Energy Velocity $\vec{v}_{\mathrm{e}}$

Energy velocity describes the speed and direction of the energy.

$$\vec{v}_{\mathrm{e}} = \frac{\vec{P}}{U} \qquad n_{\mathrm{e}} = \frac{c_0}{v_{\mathrm{e}}} \qquad \vec{v}_{\mathrm{e}} = \vec{v}_{\mathrm{g}} \quad \text{in linear materials}$$



Lenses

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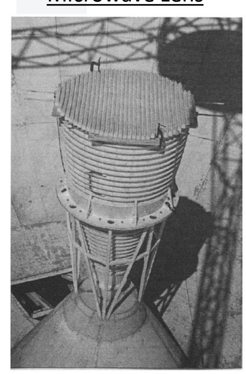
Lenses

Lenses are structures that focus electromagnetic waves.
Lenses are also used to collimate a beam or diverge a beam.

Optical Lens

Microwave Lens



W. Chalodhorn, D. R. Deboer, "Use of Microwave Lenses in Phase Retrieval Microwave Holography of Reflector Antennas," IEEE Trans. Ant. Prop., vol. 50, no. 9, pp. 1274-1284, 2002.

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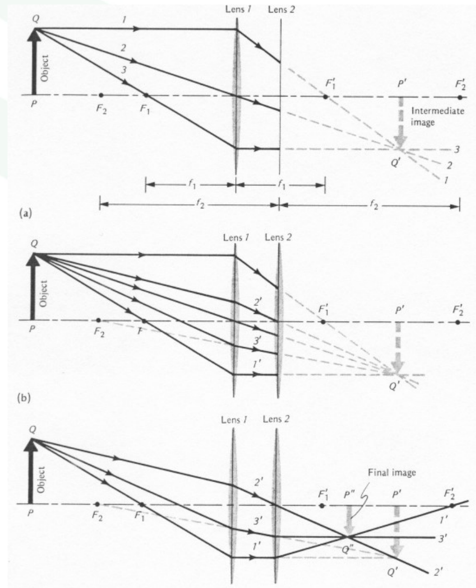
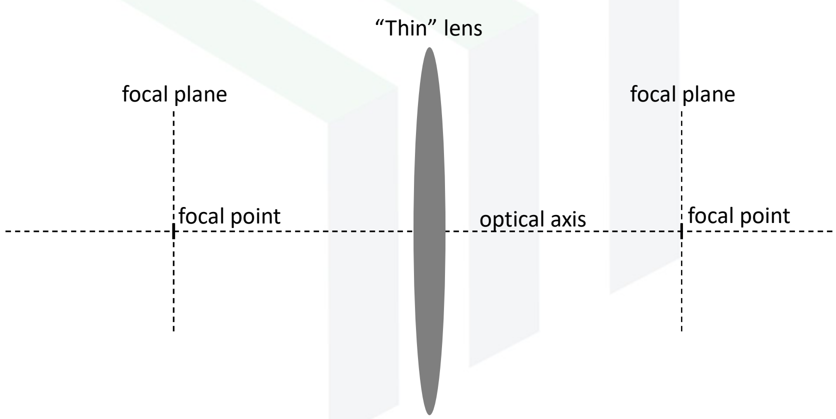
Ray Tracing

Ray tracing is a graphical technique to determine the direction of a beam that passes through the lens.

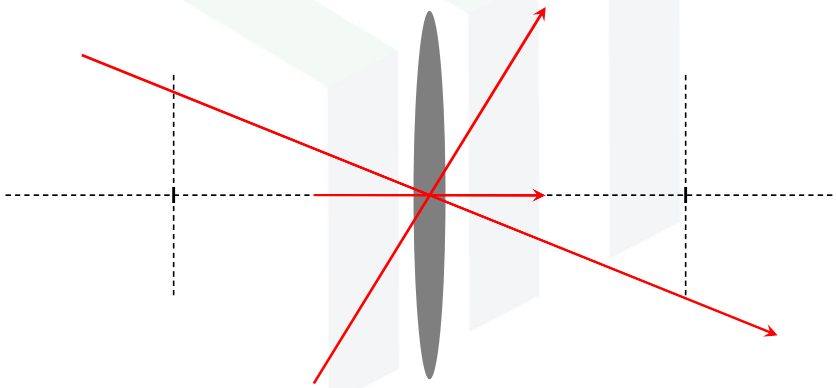


Ray Tracing Definitions



Ray Tracing: *Rule #1*

The direction of a ray passing through the center of the lens remains unchanged.



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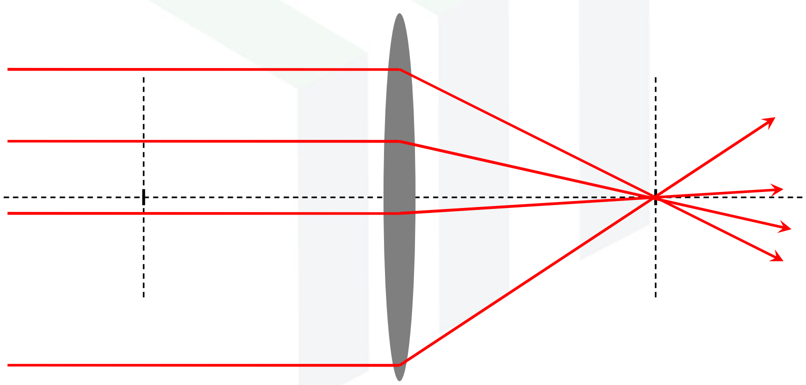
Ray Tracing: *Rule #2*

A ray parallel to the optical axis will pass through the focal point.



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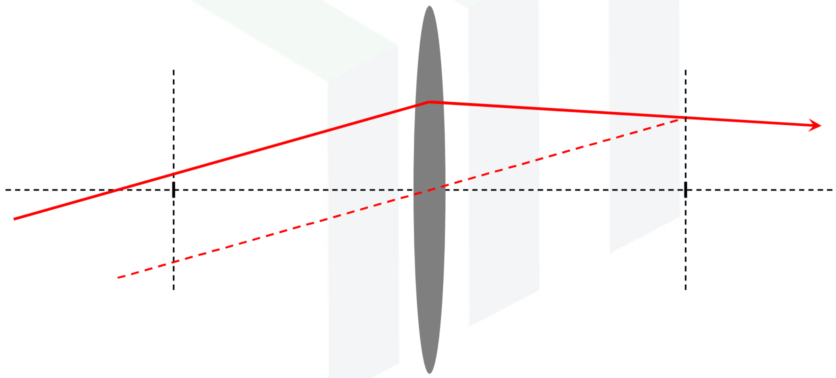
Ray Tracing: *Rule #3*

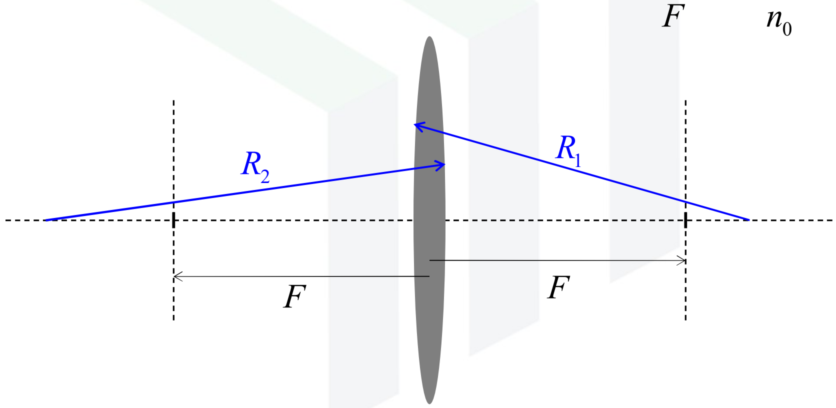
An arbitrary ray will pass through the focal plane at the same point as a parallel ray passing through the center of the lens.



Lens's Makers Formula

The focal length of a thin lens is approximately

$$\frac{1}{F} = \frac{n_{\text{lens}} - n_0}{n_0}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

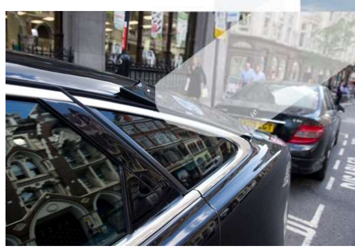


Death Rays From a Skyscraper

The curved glass on a skyscraper in London acts like a lens. In late August / early September, the sun is at just the right angle to focus light down onto the street.

Here, it melted part of an expensive Jaguar.

20 Fenchurch Street, London.



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Gaussian Beams

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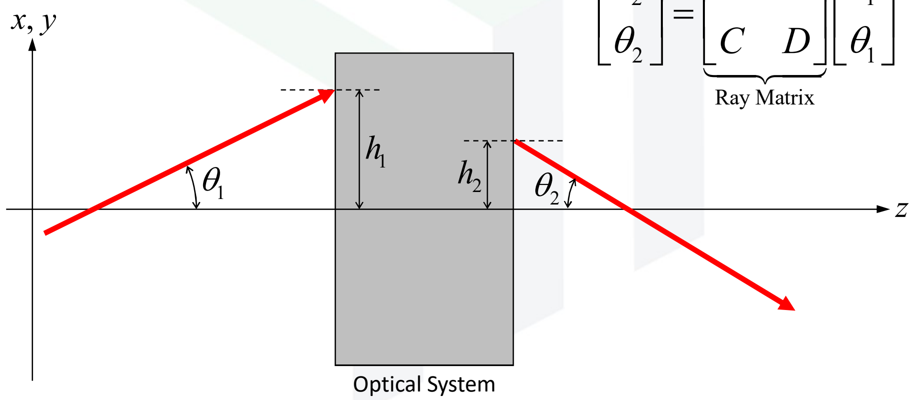
Ray Matrix

A ray matrix relates the height and slope of the input and output rays of an optical system.

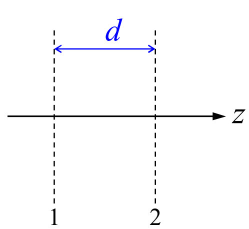
$$\begin{bmatrix} h_2 \\ \theta_2 \end{bmatrix} = \underbrace{\begin{bmatrix} A & B \\ C & D \end{bmatrix}}_{\text{Ray Matrix}} \begin{bmatrix} h_1 \\ \theta_1 \end{bmatrix}$$



Common Ray Matrices

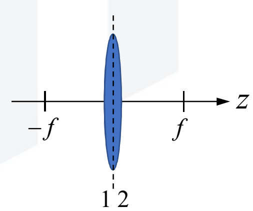
Unperturbed Distance

$$\begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}$$



Thin Lens

$$\begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix}$$



Combining Ray Matrices

Ray matrices are combined using standard matrix multiplication.

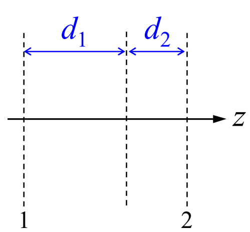
$$\begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \rightarrow \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} = \begin{bmatrix} A_3 & B_3 \\ C_3 & D_3 \end{bmatrix}$$

$$\begin{bmatrix} A_3 & B_3 \\ C_3 & D_3 \end{bmatrix} = \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix}$$

Note the order of these matrices.

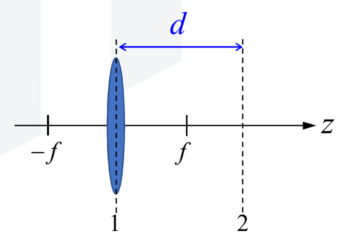
Distance-Distance

$$\begin{bmatrix} 1 & d_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & d_1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & (d_1 + d_2) \\ 0 & 1 \end{bmatrix}$$



Lens-Distance

$$\begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix} = \begin{bmatrix} (1 - d/f) & d \\ -1/f & 1 \end{bmatrix}$$



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The Gaussian Beam Equation

$$E(x,y,z) = \underbrace{E_0}_{\text{amplitude}} \cdot \underbrace{\frac{w_0}{w(z)} \exp\left[-\frac{r^2}{w^2(z)}\right]}_{\text{amplitude profile}} \cdot \underbrace{\exp\left\{-j\left[kz - \tan^{-1}\left(\frac{z}{z_0}\right)\right]\right\}}_{\text{longitudinal phase}} \cdot \underbrace{\exp\left[-j\frac{kr^2}{2R(z)}\right]}_{\text{radial phase}}$$

$$w(z) = w_0\sqrt{1 + \left(\frac{z}{z_0}\right)^2}$$

Beam width (radius)

$$R(z) = z\left[1 + \left(\frac{z_0}{z}\right)^2\right]$$

Radius of curvature

$$z_0 = \frac{\pi n w_0^2}{\lambda_0}$$

Division between near-field and far-field.
Beam diverges linearly for $z > z_0$.
$w(z_0) = z_0\sqrt{2}$

$w_0 \equiv$ minimum spot size
$r \equiv$ distance from z-axis
$r = x^2 + y^2$

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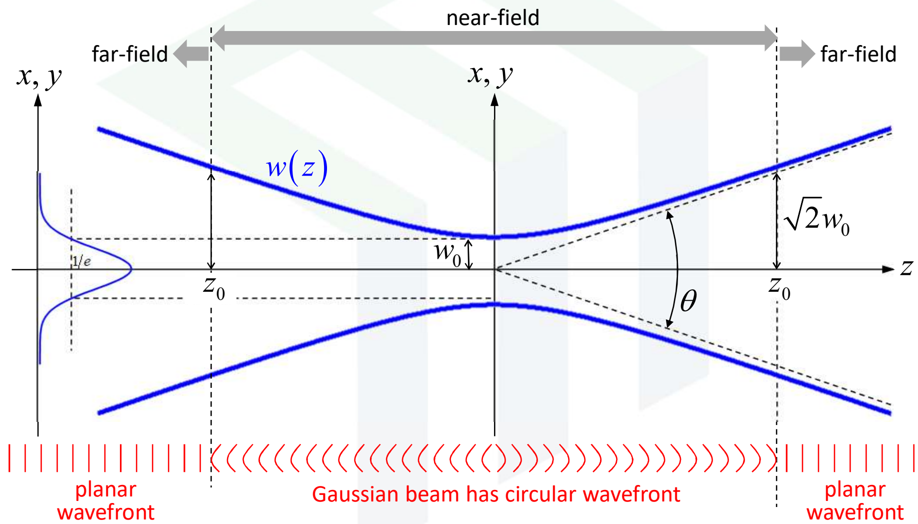
Geometry of the Gaussian Beam



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ABCD Law

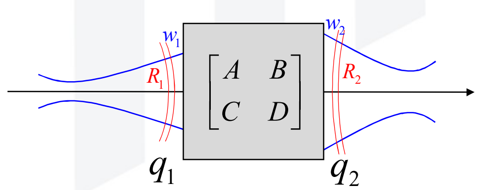
The ABCD law allows the radius of curvature R and beam width w to be carried through a system described by a ray matrix.

The ABCD law is

$$\boxed{q_2(z) = \frac{Aq_1(z)+B}{Cq_1(z)+D}} \quad \text{or} \quad \boxed{\frac{1}{q_2(z)} = \frac{C + D/q_1(z)}{A + B/q_1(z)}}$$

Here the beam parameters $R(z)$ and $w(z)$ are combined into the complex beam parameter $q(z)$.

$$\frac{1}{q(z)} = \frac{1}{R(z)} - j\frac{\lambda_0}{\pi n w^2(z)}$$



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