



Computational Science:
Introduction to Finite-Difference Time-Domain

Scattering Analysis

Lecture Outline

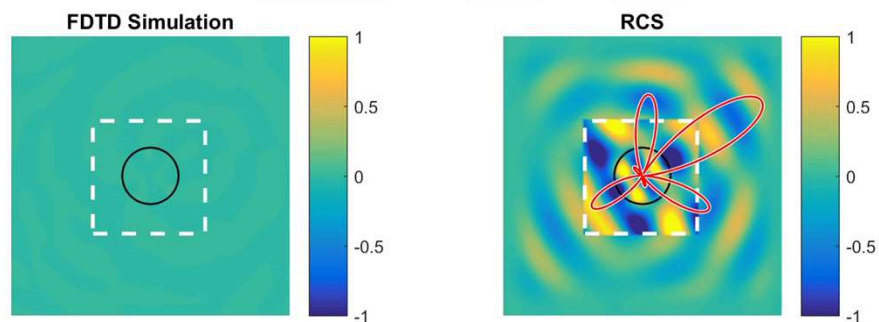
- Introduction
- FDTD for Scattering Simulations
- Formulation of NF2FF Transformation
- Implementation
- NF2FF in Two Dimensions
- Simulation Examples

Introduction

Slide 3

Animation of Scattering Analysis

- Finite object
- TF/SF on all surrounding sides
- Interested primarily in the scattered-field
- Usually the goal is to quantify scattering in the far field.

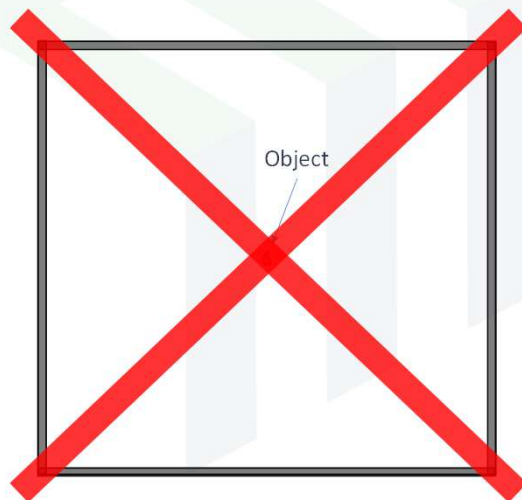


Slide 4

What is NF2FF Transformation? (1 of 2)

Suppose the electromagnetic fields VERY far away from an object are to be calculated.

How can this be done? Should the simulation just use an extremely large grid?



NO!!

What is NF2FF Transformation? (2 of 2)

Instead, perform a small simulation and use the NF2FF technique to calculate the fields very far away. This is MUCH more efficient!

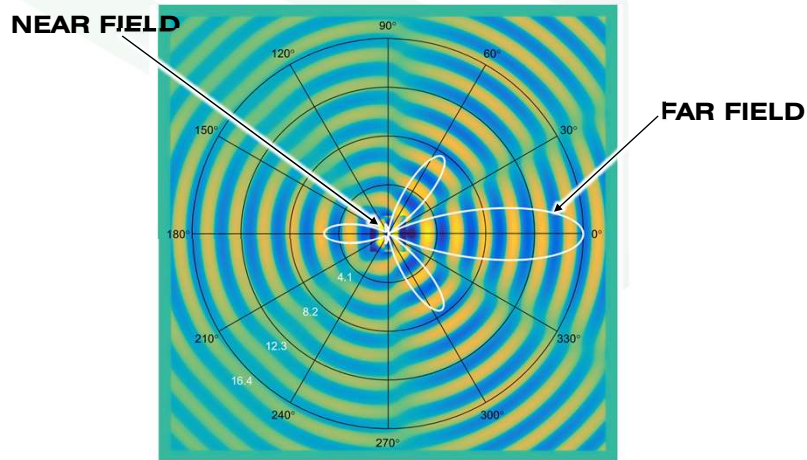


BIG SIMULATION!!

Little simulation + NF2FF

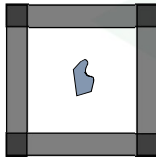
Representing the Far Field

Usually the far field is plotted in cylindrical (2D) or spherical (3D) coordinates.

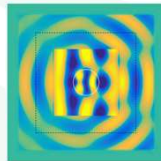


Basic Outline of Scattering Analysis

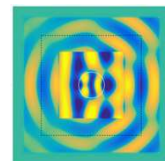
Step 1 –
Build object on grid



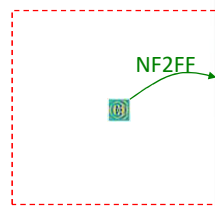
Step 2 –
Simulate object



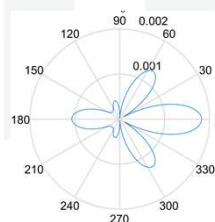
Step 3 –
Calculate steady-state field



Step 4 –
Perform NF2FF



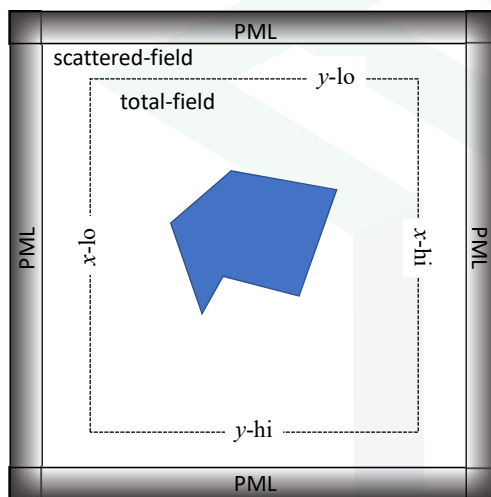
Step 5 –
Post process



FDTD for Scattering Simulations

Slide 9

Grid Scheme



The source is bounded on all sides so that the entire outer portion of the grid is scattered-field.

Recall the Curl Calculations

<u>E Mode</u>	<u>H Mode</u>
$C_x^E \Big _t^{i,j} = \frac{\tilde{E}_z \Big _t^{i,j+1} - \tilde{E}_z \Big _t^{i,j}}{\Delta y}$	$C_x^E \Big _t^{i,j} = \frac{\tilde{E}_z \Big _t^{i,j+1} - \tilde{E}_z \Big _t^{i,j}}{\Delta y}$
$C_y^E \Big _t^{i,j} = -\frac{\tilde{E}_z \Big _t^{i+1,j} - \tilde{E}_z \Big _t^{i,j}}{\Delta x}$	$C_y^E \Big _t^{i,j} = -\frac{\tilde{E}_z \Big _t^{i+1,j} - \tilde{E}_z \Big _t^{i,j}}{\Delta x}$
$C_z^H \Big _{t+\frac{\Delta x}{2}}^{i,j} = \frac{H_y \Big _{t+\frac{\Delta x}{2}}^{i,j} - H_y \Big _{t+\frac{\Delta x}{2}}^{i-1,j}}{\Delta x} - \frac{H_x \Big _{t+\frac{\Delta x}{2}}^{i,j} - H_x \Big _{t+\frac{\Delta x}{2}}^{i,j-1}}{\Delta y}$	$C_z^E \Big _t^{i,j} = \frac{\tilde{E}_y \Big _t^{i+1,j} - \tilde{E}_y \Big _t^{i,j}}{\Delta x} - \frac{\tilde{E}_x \Big _t^{i,j+1} - \tilde{E}_x \Big _t^{i,j}}{\Delta y}$

Note: the terms in red identify terms that require modification in the TF/SF framework.

x-lo Side

On the x-lo side, only C_{E_y} and C_{H_z} require modification.

$$C_y^E \Big|_t^{i-1,j,k} = -\frac{\tilde{E}_z \Big|_t^{i,j,k} - \tilde{E}_z \Big|_t^{i-1,j,k}}{\Delta x} = \left\{ -\frac{\tilde{E}_z \Big|_t^{i,j} - \tilde{E}_z \Big|_t^{i-1,j}}{\Delta x} \right\} + \frac{\tilde{E}_{z,\text{src}} \Big|_t^{i,j}}{\Delta x}$$

$$C_z^H \Big|_{t+\frac{\Delta x}{2}}^{i,j} = \frac{H_y \Big|_{t+\frac{\Delta x}{2}}^{i,j} - H_y \Big|_{t+\frac{\Delta x}{2}}^{i-1,j}}{\Delta x} - \frac{H_x \Big|_{t+\frac{\Delta x}{2}}^{i,j} - H_x \Big|_{t+\frac{\Delta x}{2}}^{i,j-1}}{\Delta y}$$

$$= \left\{ \frac{H_y \Big|_{t+\frac{\Delta x}{2}}^{i,j} - H_y \Big|_{t+\frac{\Delta x}{2}}^{i-1,j}}{\Delta x} - \frac{H_x \Big|_{t+\frac{\Delta x}{2}}^{i,j} - H_x \Big|_{t+\frac{\Delta x}{2}}^{i,j-1}}{\Delta y} \right\} - \frac{H_{y,\text{src}} \Big|_{t+\frac{\Delta x}{2}}^{i-1,j}}{\Delta x}$$

x-hi Side

On the x-hi side, only CE_y and CH_z require modification.

$$C_y^E \Big|_t^{i_2,j} = -\frac{\tilde{E}_z \Big|_t^{i_2+1,j} - \tilde{E}_z \Big|_t^{i_2,j}}{\Delta x} = \left\{ \frac{\tilde{E}_z \Big|_t^{i_2+1,j} - \tilde{E}_z \Big|_t^{i_2,j}}{\Delta x} \right\} - \frac{\tilde{E}_{z,\text{src}} \Big|_t^{i_2+1,j}}{\Delta x}$$

$$C_z^H \Big|_{t+\frac{\Delta t}{2}}^{i_2+1,j} = \frac{H_y \Big|_{t+\frac{\Delta t}{2}}^{i_2+1,j} - H_y \Big|_{t+\frac{\Delta t}{2}}^{i_2,j}}{\Delta x} - \frac{H_x \Big|_{t+\frac{\Delta t}{2}}^{i_2+1,j} - H_x \Big|_{t+\frac{\Delta t}{2}}^{i_2+1,j-1}}{\Delta y}$$

$$= \left\{ \frac{H_y \Big|_{t+\frac{\Delta t}{2}}^{i_2+1,j} - H_y \Big|_{t+\frac{\Delta t}{2}}^{i_2,j}}{\Delta x} - \frac{H_x \Big|_{t+\frac{\Delta t}{2}}^{i_2+1,j} - H_x \Big|_{t+\frac{\Delta t}{2}}^{i_2+1,j-1}}{\Delta y} \right\} + \frac{H_{y,\text{src}} \Big|_{t+\frac{\Delta t}{2}}^{i_2,j}}{\Delta x}$$

y-lo Side

On the y-lo side, only CE_x and CH_z require modification.

$$C_x^E \Big|_t^{i,j_1-1} = \frac{\tilde{E}_z \Big|_t^{i,j_1} - \tilde{E}_z \Big|_t^{i,j_1-1}}{\Delta y} = \left\{ \frac{\tilde{E}_z \Big|_t^{i,j_1} - \tilde{E}_z \Big|_t^{i,j_1-1}}{\Delta y} \right\} - \frac{\tilde{E}_{z,\text{src}} \Big|_t^{i,j_1}}{\Delta y}$$

$$C_z^H \Big|_{t+\frac{\Delta t}{2}}^{i,j_1} = \frac{H_y \Big|_{t+\frac{\Delta t}{2}}^{i,j_1} - H_y \Big|_{t+\frac{\Delta t}{2}}^{i-1,j_1}}{\Delta x} - \frac{H_x \Big|_{t+\frac{\Delta t}{2}}^{i,j_1} - H_x \Big|_{t+\frac{\Delta t}{2}}^{i,j_1-1}}{\Delta y}$$

$$= \left\{ \frac{H_y \Big|_{t+\frac{\Delta t}{2}}^{i,j_1} - H_y \Big|_{t+\frac{\Delta t}{2}}^{i-1,j_1}}{\Delta x} - \frac{H_x \Big|_{t+\frac{\Delta t}{2}}^{i,j_1} - H_x \Big|_{t+\frac{\Delta t}{2}}^{i,j_1-1}}{\Delta y} \right\} + \frac{H_{x,\text{src}} \Big|_{t+\frac{\Delta t}{2}}^{i,j_1-1}}{\Delta y}$$

y-hi Side

On the y-hi side, only CE_x and CH_z require modification.

$$C_x^E \Big|_t^{i,j_2} = \frac{\tilde{E}_z \Big|_t^{i,j_2+1} - \tilde{E}_z \Big|_t^{i,j_2}}{\Delta y} = \left\{ \frac{\tilde{E}_z \Big|_t^{i,j_2+1} - \tilde{E}_z \Big|_t^{i,j_2}}{\Delta y} \right\} + \frac{\tilde{E}_{z,\text{src}} \Big|_t^{i,j_2+1}}{\Delta y}$$

$$C_z^H \Big|_{t+\frac{\Delta t}{2}}^{i,j_2+1} = \frac{H_y \Big|_{t+\frac{\Delta t}{2}}^{i,j_2+1} - H_y \Big|_{t+\frac{\Delta t}{2}}^{i-1,j_2+1}}{\Delta x} - \frac{H_x \Big|_{t+\frac{\Delta t}{2}}^{i,j_2+1} - H_x \Big|_{t+\frac{\Delta t}{2}}^{i,j_2}}{\Delta y}$$

$$= \left\{ \frac{H_y \Big|_{t+\frac{\Delta t}{2}}^{i,j_2+1} - H_y \Big|_{t+\frac{\Delta t}{2}}^{i-1,j_2+1}}{\Delta x} - \frac{H_x \Big|_{t+\frac{\Delta t}{2}}^{i,j_2+1} - H_x \Big|_{t+\frac{\Delta t}{2}}^{i,j_2}}{\Delta y} \right\} - \frac{H_{x,\text{src}} \Big|_{t+\frac{\Delta t}{2}}^{i,j_2}}{\Delta y}$$

Deriving a Sinusoidal Source Function

The general expression for the normalized electric field component of a plane wave is

$$\vec{E}(x, y; t) = \vec{P} \cos(\omega t - \vec{k} \cdot \vec{r}) = \vec{P} \cos(\omega t - k_x x - k_y y)$$

The equivalent expression for the magnetic field can be derived by substituting the above expression into Faraday's law.

$$\vec{H} = \sqrt{\frac{\epsilon_r}{\mu_r}} (\hat{k} \times \vec{P}) \cos(\omega t - \vec{k} \cdot \vec{r}) = \sqrt{\frac{\epsilon_r}{\mu_r}} (\hat{k} \times \vec{P}) \cos(\omega t - k_x x - k_y y)$$

Sinusoidal Source Functions for the E Mode

The E mode has $P_x = P_y = 0$ and we set $P_z = 1$. Thus, the source functions reduce to

$$\tilde{E}_z(x, y; t) = \cos(\omega t - k_x x - k_y y)$$

$$H_x(x, y; t) = \hat{k}_y \sqrt{\frac{\epsilon_r}{\mu_r}} \cos(\omega t - k_x x - k_y y)$$

$$H_y(x, y; t) = -\hat{k}_x \sqrt{\frac{\epsilon_r}{\mu_r}} \cos(\omega t - k_x x - k_y y)$$

Numerical Equations for Sinusoidal Source Functions for the E Mode

In terms of array indices, the source functions are

$$\tilde{E}_{z,\text{src}} \Big|_T^{i,j,k} = \cos(\omega T \Delta t - k_x i \Delta x - k_y j \Delta y)$$

$$H_{y,\text{src}} \Big|_{T+0.5}^{i-1,j} = -\hat{k}_x \sqrt{\frac{\epsilon_r}{\mu_r}} \cos[\omega(T+0.5)\Delta t - k_x(i-0.5)\Delta x - k_y j \Delta y]$$

$$\tilde{E}_{z,\text{src}} \Big|_T^{i_2+1,j} = \cos[\omega T \Delta t - k_x(i_2+1)\Delta x - k_y j \Delta y]$$

$$H_{y,\text{src}} \Big|_{T+0.5}^{i_2,j} = -\hat{k}_x \sqrt{\frac{\epsilon_r}{\mu_r}} \cos[\omega(T+0.5)\Delta t - k_x(i_2+0.5)\Delta x - k_y j \Delta y]$$

$$\tilde{E}_{z,\text{src}} \Big|_T^{i,j_1} = \cos(\omega T \Delta t - k_x i \Delta x - k_y j_1 \Delta y)$$

$$H_{x,\text{src}} \Big|_{T+0.5}^{i,j_1-1} = \hat{k}_y \sqrt{\frac{\epsilon_r}{\mu_r}} \cos[\omega(T+0.5)\Delta t - k_x i \Delta x - k_y(j_1-0.5)\Delta y]$$

$$\tilde{E}_{z,\text{src}} \Big|_T^{i,j_2+1} = \cos[\omega T \Delta t - k_x i \Delta x - k_y(j_2+1)\Delta y]$$

$$H_{x,\text{src}} \Big|_{T+0.5}^{i,j_2} = \hat{k}_y \sqrt{\frac{\epsilon_r}{\mu_r}} \cos[\omega(T+0.5)\Delta t - k_x i \Delta x - k_y(j_2+0.5)\Delta y]$$

Code Snippets for Sinusoidal Source

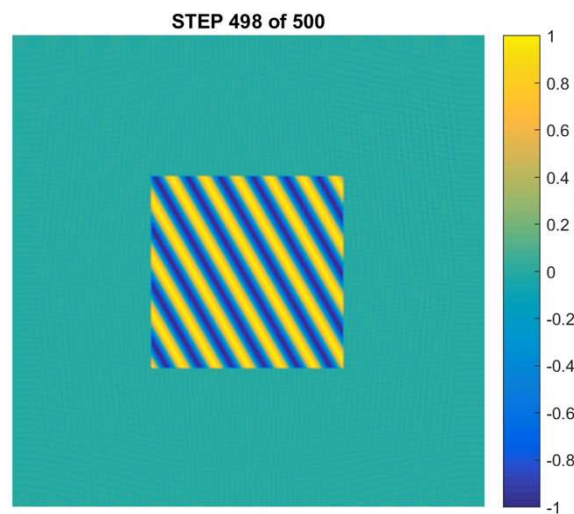
```

% TF/SF FOR CURL OF E
Ez0 = RAMP(T);
%xlo
Ezsrc = Ez0*cos(2*pi*f0*T*dt - kx*nxlo*dx - ky*[nylo:nyhi]*dy);
CEy(nxlo-1,nylo:nyhi) = CEy(nxlo-1,nylo:nyhi) + Ezsrc/dx;
%xhi
Ezsrc = Ez0*cos(2*pi*f0*T*dt - kx*(nxhi + 1)*dx - ky*[nylo:nyhi]*dy);
CEy(nxhi,nylo:nyhi) = CEy(nxhi,nylo:nyhi) - Ezsrc/dx;
%ylo
Ezsrc = Ez0*cos(2*pi*f0*T*dt - kx*[nxlo:nxhi]*dx - ky*nylo*dy);
CEx(nxlo:nxhi,nylo-1) = CEx(nxlo:nxhi,nylo-1) - Ezsrc/dy;
%yhi
Ezsrc = Ez0*cos(2*pi*f0*T*dt - kx*[nxlo:nxhi]*dx - ky*(nyhi + 1)*dy);
CEx(nxlo:nxhi,nyhi) = CEx(nxlo:nxhi,nyhi) + Ezsrc/dy;

% TF/SF FOR CURL OF H
Hx0 = + (ky/k0)*RAMP(T);
Hy0 = - (kx/k0)*RAMP(T);
%xlo
Hysrc = Hy0*cos(2*pi*f0*(T + 0.5)*dt - kx*(nxlo - 0.5)*dx - ky*[nylo:nyhi]*dy);
CHz(nxlo,nylo:nyhi) = CHz(nxlo,nylo:nyhi) - Hysrc/dx;
%xhi
Hysrc = Hy0*cos(2*pi*f0*(T + 0.5)*dt - kx*(nxhi + 0.5)*dx - ky*[nylo:nyhi]*dy);
CHz(nxhi+1,nylo:nyhi) = CHz(nxhi+1,nylo:nyhi) + Hysrc/dx;
%ylo
Hxsrc = Hx0*cos(2*pi*f0*(T + 0.5)*dt - kx*[nxlo:nxhi]*dx - ky*(nylo - 0.5)*dy);
CHz(nxlo:nxhi,nylo) = CHz(nxlo:nxhi,nylo) + Hxsrc/dy;
%yhi
Hxsrc = Hx0*cos(2*pi*f0*(T + 0.5)*dt - kx*[nxlo:nxhi]*dx - ky*(nyhi + 0.5)*dy);
CHz(nxlo:nxhi,nyhi+1) = CHz(nxlo:nxhi,nyhi+1) - Hxsrc/dy;

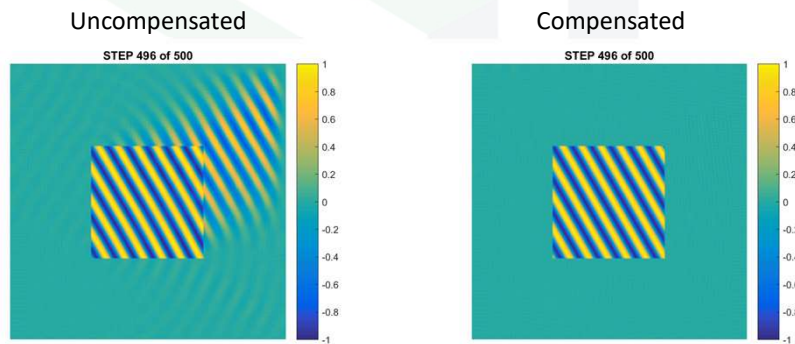
```

Animation of CW Source



Calculating k_x and k_y (1 of 2)

The TF/SF spans a large enough amount of space that the numerical dispersion due to the Yee grid causes problems. You must compensate for this.



Calculating k_x and k_y (2 of 2)

Step 1 – Define θ in dashboard.

Step 2 – Calculate the refractive index where source is injected.

$$n_{\text{src}} = \sqrt{\mu_{r,\text{src}} \epsilon_{r,\text{src}}}$$

Step 3 – Calculate k_x and k_y .

$$k_x = k_0 n_{\text{src}} \cos \theta \quad k_y = k_0 n_{\text{src}} \sin \theta$$

Step 3 – Calculate “fudge factor” f .

$$f = \frac{c_0 \Delta t}{n_{\text{src}} \sin(\pi f_0 \Delta t)} \sqrt{\left[\left(\frac{1}{\Delta x} \right) \sin(k_x \Delta x / 2) \right]^2 + \left[\left(\frac{1}{\Delta y} \right) \sin(k_y \Delta y / 2) \right]^2}$$

Step 4 – Adjust k_x and k_y .

$$\tilde{k}_x = f k_x \quad \tilde{k}_y = f k_y$$

Note: An alternate method is described for the Gaussian source that usually works better.

Analytical Equations for a Gaussian Pulse

For a Gaussian pulse, the field components are

$$\tilde{E}_z(x, y; t) = \exp\left[-\left(\frac{t - \delta t}{\tau}\right)^2\right]$$

$$H_x(x, y; t) = \hat{k}_y \exp\left[-\left(\frac{t - \delta t}{\tau}\right)^2\right]$$

$$H_y(x, y; t) = -\hat{k}_x \exp\left[-\left(\frac{t - \delta t}{\tau}\right)^2\right]$$

$$\delta t = (\hat{k}_x x + \hat{k}_y y) / c_0 \quad \text{This time delay ensures the pulse begins just outside of the TF/SF region regardless of the angle of incidence.}$$

Numerical Equations for a Gaussian Pulse

For a Gaussian pulse, the field components are

$$\tilde{E}_z(i, j; T) = \exp\left[-\left(\frac{T\Delta t - (\hat{k}_x i\Delta x + \hat{k}_y j\Delta y) / c_0}{\tau}\right)^2\right]$$

$$H_x(i, j; T) = \hat{k}_y \exp\left[-\left(\frac{(T + 0.5)\Delta t - [\hat{k}_x i\Delta x + \hat{k}_y (j + 0.5)\Delta y] / c_0}{\tau}\right)^2\right]$$

$$H_y(i, j; T) = -\hat{k}_x \exp\left[-\left(\frac{(T + 0.5)\Delta t - [\hat{k}_x (i + 0.5)\Delta x + \hat{k}_y j\Delta y] / c_0}{\tau}\right)^2\right]$$

Code Snippets for Gaussian Source

```

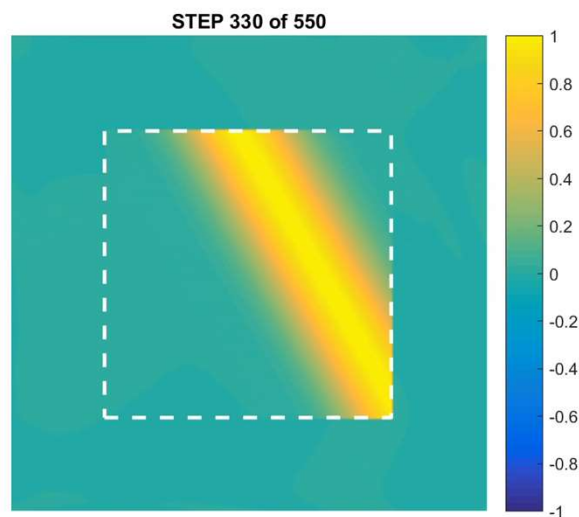
% TF/SF FOR CURL OF E
Ez0 = exp(-((T*dt - T0)/tau).^2);
Ez0 = Ez0(1:2:Nx2,1:2:Ny2);
%xlo
Ezsrc = Ez0(nxlo,nylo:nyhi);
CEy(nxlo-1,nylo:nyhi) = CEy(nxlo-1,nylo:nyhi) + Ezsrc/dx;
%xhi
Ezsrc = Ez0(nxhi+1,nylo:nyhi);
CEy(nxhi,nylo:nyhi) = CEy(nxhi,nylo:nyhi) - Ezsrc/dx;
%ylo
Ezsrc = Ez0(nxlo:nxhi,nylo);
CEx(nxlo:nxhi,nylo-1) = CEx(nxlo:nxhi,nylo-1) - Ezsrc/dy;
%yhi
Ezsrc = Ez0(nxlo:nxhi,nyhi+1);
CEx(nxlo:nxhi,nyhi) = CEx(nxlo:nxhi,nyhi) + Ezsrc/dy;

% TF/SF
H0 = exp(-((T + 0.5)*dt - T0)/tau).^2);
Hx0 = + (ky/k0)*H0(2:2:Nx2,1:2:Ny2);
Hy0 = - (kx/k0)*H0(1:2:Nx2,2:2:Ny2);
%xlo
Hysrc = Hy0(nxlo-1,nylo:nyhi);
CHz(nxlo,nylo:nyhi) = CHz(nxlo,nylo:nyhi) - Hysrc/dx;
%xhi
Hysrc = Hy0(nxhi,nylo:nyhi);
CHz(nxhi+1,nylo:nyhi) = CHz(nxhi+1,nylo:nyhi) + Hysrc/dx;
%ylo
Hxsrc = Hx0(nxlo:nxhi,nylo - 1);
CHz(nxlo:nxhi,nylo) = CHz(nxlo:nxhi,nylo) + Hxsrc/dy;
%yhi
Hxsrc = Hx0(nxlo:nxhi,nyhi);
CHz(nxlo:nxhi,nyhi+1) = CHz(nxlo:nxhi,nyhi+1) - Hxsrc/dy;

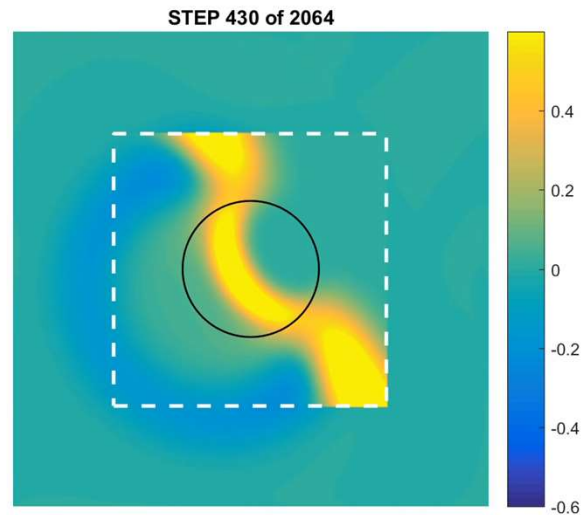
% CALCULATE TIME GRADIENT ACROSS ZX GRID
delx = Sx*cos(theta);
dely = Sy*sin(theta);
T0 = (cos(theta)*(X2 + delx) + sin(theta)*(Y2 + dely))/c0;

```

Animation of Gaussian Pulse Source

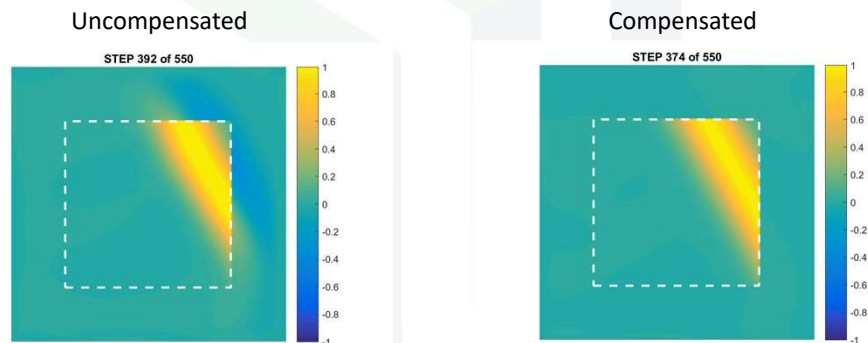


Animation of Gaussian Pulse Source With a Scattering Object



Compensating for Dispersion for a Gaussian Source (1 of 2)

The TF/SF spans a large enough amount of space that the numerical dispersion due to the Yee grid causes problems. You must compensate for this.



Compensating for Dispersion for a Gaussian Source (2 of 2)

Step 1 – Define θ in dashboard.

Step 2 – Calculate the refractive index where source is injected.

$$n_{\text{src}} = \sqrt{\mu_{r,\text{src}} \epsilon_{r,\text{src}}}$$

Step 3 – Calculate k_x and k_y .

$$k_x = k_0 n_{\text{src}} \cos \theta \quad k_y = k_0 n_{\text{src}} \sin \theta$$

Step 4 – Calculate “fudge factor” f .

$$f = \frac{c_0 \Delta t}{n_{\text{src}} \sin(\pi f_0 \Delta t)} \sqrt{\left[\left(\frac{1}{\Delta x} \right) \sin(k_x \Delta x / 2) \right]^2 + \left[\left(\frac{1}{\Delta y} \right) \sin(k_y \Delta y / 2) \right]^2}$$

Step 5 – Adjust material arrays ER2 and UR2 according to f .

$$\text{UR2} = f * \text{UR2};$$

$$\text{ER2} = f * \text{ER2};$$

Note: This method of compensate works for the sinusoidal source as well.

Formulation of NF2FF Transformation

Electromagnetic Boundary Conditions

Magnetic Field at an Interface

$$\vec{H}_{2,T} = \vec{H}_{1,T} + \vec{J}_s$$

$$\vec{H}_{2,T} - \vec{H}_{1,T} = \vec{J}_s$$

$$\hat{n} \times \vec{H}_2 - \hat{n} \times \vec{H}_1 = \vec{J}_s$$

$$\hat{n} \times (\vec{H}_2 - \vec{H}_1) = \vec{J}_s$$

$$\vec{J}_s = \hat{n} \times (\vec{H}_2 - \vec{H}_1)$$

$\hat{n} \equiv$ surface normal pointing from 1 to 2

Electric Field at an Interface

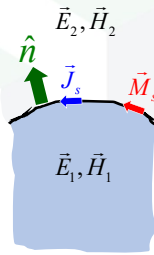
$$\vec{E}_{2,T} = \vec{E}_{1,T} - \vec{M}_s$$

$$\vec{E}_{2,T} - \vec{E}_{1,T} = -\vec{M}_s$$

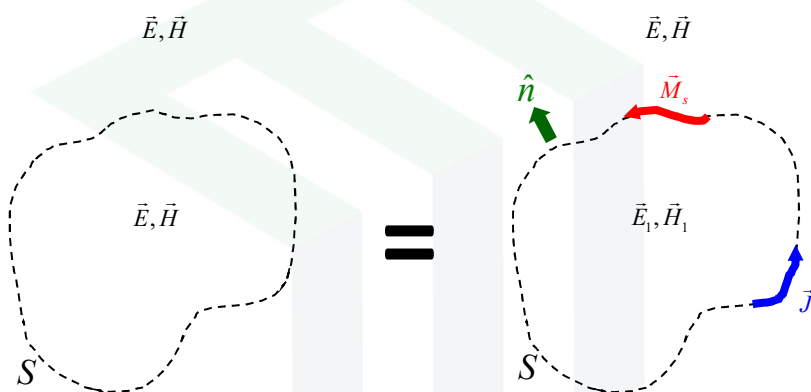
$$\hat{n} \times \vec{E}_2 - \hat{n} \times \vec{E}_1 = -\vec{M}_s$$

$$\hat{n} \times (\vec{E}_2 - \vec{E}_1) = -\vec{M}_s$$

$$\vec{M}_s = -\hat{n} \times (\vec{E}_2 - \vec{E}_1)$$



Surface Equivalence Theorem



The far field \vec{E} and \vec{H} is completely described by the surface currents \vec{J}_s and \vec{M}_s flowing around a closed surface S .

Calculating the Surface Currents

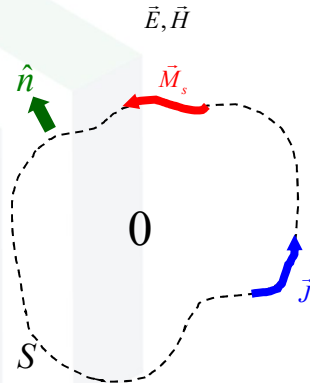
Since only the far fields \vec{E} and \vec{H} are of interest, \vec{E}_1 and \vec{H}_1 can be set to whatever value is convenient.

Let $\vec{E}_1 = \vec{H}_1 = 0$.

The surface currents are

$$\vec{J}_s = \hat{n} \times \vec{H}$$

$$\vec{M}_s = -\hat{n} \times \vec{E}$$



Note: Calculating \vec{J}_s and \vec{M}_s in this way makes it only possible to calculate fields outside of the surface.

Vector Potentials

It is desired to calculate the fields very far from the surface S . It is useful to do this using the vector potentials \vec{A} and \vec{F} .

$$\vec{A} = \frac{\mu}{4\pi} \iint_S \vec{J}_s \frac{e^{jkR}}{R} ds'$$

$$\vec{F} = \frac{\varepsilon}{4\pi} \iint_S \vec{M}_s \frac{e^{jkR}}{R} ds'$$

$R \equiv$ distance from point on surface s' to the observation point.

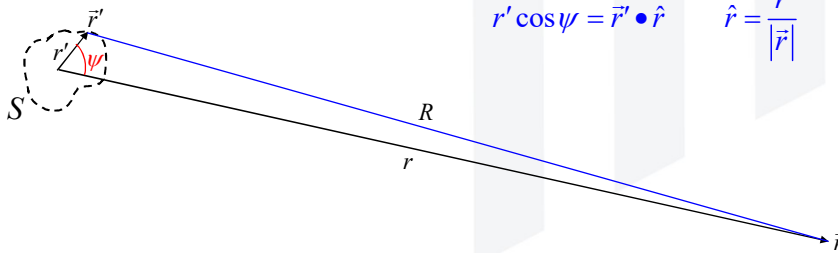
Far Field Approximation

In the far field, R is extremely large.

$$R = |\vec{r} - \vec{r}'| \approx \begin{cases} r - r' \cos \psi & \text{for phase calculations} \\ r & \text{for amplitude calculations} \end{cases}$$

Note:

$$r' \cos \psi = \vec{r}' \cdot \hat{r} \quad \hat{r} = \frac{\vec{r}}{|\vec{r}|}$$



Vector Potentials in the Far Field

Using the far-field approximation for R , the equations for calculating the vector potentials become

$$\begin{aligned} \vec{A} &\cong \frac{\mu}{4\pi} \iint_S \vec{J}_S \frac{e^{jk(r-r'\cos\psi)}}{r - r'\cos\psi} ds' & \vec{F} &\cong \frac{\varepsilon}{4\pi} \iint_S \vec{M}_S \frac{e^{jk(r-r'\cos\psi)}}{r - r'\cos\psi} ds' \\ &\cong \frac{\mu e^{jkr}}{4\pi r} \iint_S \vec{J}_S e^{-jkr'\cos\psi} ds' & &\cong \frac{\varepsilon e^{jkr}}{4\pi r} \iint_S \vec{M}_S e^{-jkr'\cos\psi} ds' \end{aligned}$$

Last, these equations in spherical coordinates are

$$\begin{aligned} \vec{A}(r, \theta, \phi) &\cong \frac{\mu e^{jkr}}{4\pi r} \vec{N}(\theta, \phi) & \vec{F}(r, \theta, \phi) &\cong \frac{\varepsilon e^{jkr}}{4\pi r} \vec{L}(\theta, \phi) \\ \vec{N}(\theta, \phi) &\cong \iint_S \vec{J}_S e^{-jkr'\cos\psi} ds' & \vec{L}(\theta, \phi) &\cong \iint_S \vec{M}_S e^{-jkr'\cos\psi} ds' \end{aligned}$$

It is important to note the r , ϕ , and θ dependencies in these equations.

Cartesian to Spherical Coordinates (1 of 2)

Standard FDTD produces field quantities in Cartesian coordinates. This means the current terms \vec{J}_S and \vec{M}_S will be in Cartesian coordinates. Usually far-field analysis is calculated in spherical coordinates so these functions must be converted to spherical coordinates.

$$\begin{bmatrix} J_r \\ J_\theta \\ J_\phi \end{bmatrix} = \begin{bmatrix} \sin \theta \cos \phi & \sin \theta \sin \phi & \cos \theta \\ \cos \theta \cos \phi & \cos \theta \sin \phi & -\sin \theta \\ -\sin \phi & \cos \phi & 0 \end{bmatrix} \begin{bmatrix} J_x \\ J_y \\ J_z \end{bmatrix} \quad \vec{N} = N_r \hat{a}_r + N_\theta \hat{a}_\theta + N_\phi \hat{a}_\phi$$

$$N_r = \iint_S \underbrace{(J_x \sin \theta \cos \phi + J_y \sin \theta \sin \phi + J_z \cos \theta)}_{J_r} e^{-jkr' \cos \psi} ds'$$

$$N_\theta = \iint_S \underbrace{(J_x \cos \theta \cos \phi + J_y \cos \theta \sin \phi - J_z \sin \theta)}_{J_\theta} e^{-jkr' \cos \psi} ds'$$

$$N_\phi = \iint_S \underbrace{(-J_x \sin \phi + J_y \cos \phi)}_{J_\phi} e^{-jkr' \cos \psi} ds'$$

Cartesian to Spherical Coordinates (2 of 2)

The calculation for $\vec{L}(\theta, \phi)$ is

$$\begin{bmatrix} M_r \\ M_\theta \\ M_\phi \end{bmatrix} = \begin{bmatrix} \sin \theta \cos \phi & \sin \theta \sin \phi & \cos \theta \\ \cos \theta \cos \phi & \cos \theta \sin \phi & -\sin \theta \\ -\sin \phi & \cos \phi & 0 \end{bmatrix} \begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix} \quad \vec{L} = L_r \hat{a}_r + L_\theta \hat{a}_\theta + L_\phi \hat{a}_\phi$$

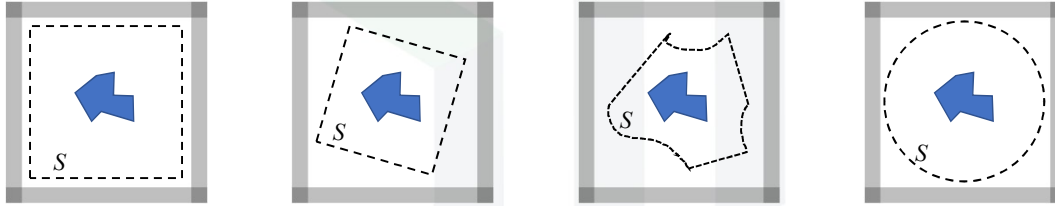
$$L_r = \iint_S \underbrace{(M_x \sin \theta \cos \phi + M_y \sin \theta \sin \phi + M_z \cos \theta)}_{M_r} e^{-jkr' \cos \psi} ds'$$

$$L_\theta = \iint_S \underbrace{(M_x \cos \theta \cos \phi + M_y \cos \theta \sin \phi - M_z \sin \theta)}_{M_\theta} e^{-jkr' \cos \psi} ds'$$

$$L_\phi = \iint_S \underbrace{(-M_x \sin \phi + M_y \cos \phi)}_{M_\phi} e^{-jkr' \cos \psi} ds'$$

N and L Only Depend on θ and ϕ (1 of 3)

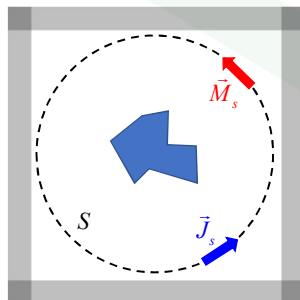
$\vec{N}(\theta, \phi)$ and $\vec{L}(\theta, \phi)$ are far field quantities so they are independent of the choice of the surface S .



Pick this one for now.

N and L Only Depend on θ and ϕ (2 of 3)

The current terms \vec{J}_S and \vec{M}_S are surface currents and always tangential to the surface.



So when we choose S to be a sphere, the surface currents have no radial components.

$$J_r = M_r = 0 \quad \text{when } S \text{ is a sphere}$$

\vec{N} and \vec{L} Only Depend on θ and ϕ (3 of 3)

From the following equations, it is observed that \vec{N} and \vec{L} take on the same vector components as \vec{J}_S and \vec{M}_S .

$$\vec{N}(\theta, \phi) \cong \iint_S \vec{J}_S e^{-jkr' \cos \psi} ds' \quad \vec{L}(\theta, \phi) \cong \iint_S \vec{M}_S e^{-jkr' \cos \psi} ds'$$

Therefore, \vec{N} and \vec{L} have no radial components.

Also recognize that this will always be the case since the far-field is fixed and independent of the choice of S .

$$N_r = L_r = 0 \quad \text{always}$$

Final Expressions for \vec{N} and \vec{L}

$$\begin{aligned} N_r(\theta, \phi) &= 0 \\ N_\theta(\theta, \phi) &= \iint_S (J_x \cos \theta \cos \phi + J_y \cos \theta \sin \phi - J_z \sin \theta) e^{-jkr' \cos \psi} ds' \\ N_\phi(\theta, \phi) &= \iint_S (-J_x \sin \phi + J_y \cos \phi) e^{-jkr' \cos \psi} ds' \end{aligned}$$

$$\begin{aligned} L_r(\theta, \phi) &= 0 \\ L_\theta(\theta, \phi) &= \iint_S (M_x \cos \theta \cos \phi + M_y \cos \theta \sin \phi - M_z \sin \theta) e^{-jkr' \cos \psi} ds' \\ L_\phi(\theta, \phi) &= \iint_S (-M_x \sin \phi + M_y \cos \phi) e^{-jkr' \cos \psi} ds' \end{aligned}$$

Calculating the Far Fields (1 of 3)

The electric and magnetic fields, \vec{E} and \vec{H} , are calculated from the vector potentials according to

$$\vec{H} = \frac{1}{\mu} \nabla \times \vec{A} + j\omega \vec{F} + \frac{j}{\omega\mu\epsilon} \nabla(\nabla \cdot \vec{F}) \quad \vec{E} = -\frac{1}{\epsilon} \nabla \times \vec{F} + j\omega \vec{A} + \frac{j}{\omega\mu\epsilon} \nabla(\nabla \cdot \vec{A})$$

The $\nabla(\nabla \cdot ?)$ terms produce equations with $1/r^2$ dependence. In the far field, r is very large so these terms vanish.

$$\vec{H} \cong \frac{1}{\mu} \nabla \times \vec{A} + j\omega \vec{F} \quad \vec{E} \cong -\frac{1}{\epsilon} \nabla \times \vec{F} + j\omega \vec{A}$$

Calculating the Far Fields (2 of 3)

$$\vec{H} = \frac{1}{\mu} \nabla \times \vec{A} + j\omega \vec{F} \quad \vec{E} = -\frac{1}{\epsilon} \nabla \times \vec{F} + j\omega \vec{A}$$

The curl terms become

$$\begin{aligned} \nabla \times \vec{A} &= \nabla \times \left(\frac{\mu e^{jkr}}{4\pi r} \vec{N} \right) \\ &= \frac{\mu}{4\pi} \nabla \times \left(\frac{e^{jkr}}{r} \vec{N} \right) \\ &= \frac{\mu}{4\pi} \frac{jke^{jkr}}{r} (N_\theta \hat{a}_\phi - N_\phi \hat{a}_\theta) \\ &= jk (A_\theta \hat{a}_\phi - A_\phi \hat{a}_\theta) \end{aligned} \quad \begin{aligned} \nabla \times \vec{F} &= \nabla \times \left(\frac{\epsilon e^{jkr}}{4\pi r} \vec{N} \right) \\ &= \frac{\epsilon}{4\pi} \nabla \times \left(\frac{e^{jkr}}{r} \vec{N} \right) \\ &= \frac{\epsilon}{4\pi} \frac{jke^{jkr}}{r} (L_\theta \hat{a}_\phi - L_\phi \hat{a}_\theta) \\ &= jk (F_\theta \hat{a}_\phi - F_\phi \hat{a}_\theta) \end{aligned}$$

The electric and magnetic fields are now

$$\vec{H} = \frac{jk}{\mu} (A_\theta \hat{a}_\phi - A_\phi \hat{a}_\theta) + j\omega \vec{F} \quad \vec{E} = -\frac{jk}{\epsilon} (F_\theta \hat{a}_\phi - F_\phi \hat{a}_\theta) + j\omega \vec{A}$$

Calculating the Far Fields (3 of 3)

$$\vec{H} = \frac{jk}{\mu} (A_\theta \hat{a}_\phi - A_\phi \hat{a}_\theta) + j\omega \vec{F} \qquad \vec{E} = -\frac{jk}{\varepsilon} (F_\theta \hat{a}_\phi - F_\phi \hat{a}_\theta) + j\omega \vec{A}$$

Combine vector components.

$$\vec{H} = -j\omega \left(\frac{1}{\eta} A_\phi - F_\theta \right) \hat{a}_\theta + j\omega \left(\frac{1}{\eta} A_\theta + F_\phi \right) \hat{a}_\phi$$

$$\vec{E} = j\omega (A_\phi + \eta F_\theta) \hat{a}_\theta + j\omega (A_\theta - \eta F_\phi) \hat{a}_\phi$$

Last, put these equations in terms of \vec{N} and \vec{L} .

$$H_r = 0$$

$$H_\theta = -\frac{jke^{jkr}}{4\pi r} \left(N_\phi - \frac{1}{\eta} L_\theta \right)$$

$$H_\phi = \frac{jke^{jkr}}{4\pi r} \left(N_\theta + \frac{1}{\eta} L_\phi \right)$$

$$E_r = 0$$

$$E_\theta = \frac{jke^{jkr}}{4\pi r} (\eta N_\theta + L_\phi)$$

$$E_\phi = \frac{jke^{jkr}}{4\pi r} (\eta N_\phi - L_\theta)$$

Impedance of the
far-field medium:

$$\eta = \sqrt{\frac{\mu}{\varepsilon}}$$

Radar Cross Section (RCS)

The power in the radiated field can be calculated from the far-field quantities as

$$P_{\text{rad}} = \frac{1}{2} \text{Re} [E_\theta H_\phi^*] + \frac{1}{2} \text{Re} [-E_\phi H_\theta^*]$$

$$= \frac{1}{2\eta} \left(\frac{k}{4\pi r} \right)^2 \left(|\eta N_\theta + L_\phi|^2 + |\eta N_\phi - L_\theta|^2 \right)$$

The radar cross section (RCS) in three dimensions is

$$\text{RCS}(\theta, \phi) = \lim_{r \rightarrow \infty} \left(4\pi r^2 \frac{P_{\text{rad}}}{P_{\text{inc}}} \right)$$

$$= \frac{k^2}{8\pi\eta P_{\text{inc}}} \left(|\eta N_\theta + L_\phi|^2 + |\eta N_\phi - L_\theta|^2 \right)$$

Flow of Equations for 3D NF2FF

Step 1 – Calculate current terms around a closed surface.

$$\vec{J}_s = \hat{n} \times \vec{H} \quad \vec{M}_s = -\hat{n} \times \vec{E}$$

Step 2 – Loop over all values of θ and ϕ

a) Calculate \vec{N} and \vec{L} in the far field.

$$N_\theta = \iint_S (J_x \cos \theta \cos \phi + J_y \cos \theta \sin \phi - J_z \sin \theta) e^{-jk' r \cos \theta} ds'$$

$$N_\phi = \iint_S (-J_x \sin \phi + J_y \cos \phi) e^{-jk' r \cos \theta} ds'$$

$$L_\theta = \iint_S (M_x \cos \theta \cos \phi + M_y \cos \theta \sin \phi - M_z \sin \theta) e^{-jk' r \cos \theta} ds'$$

$$L_\phi = \iint_S (-M_x \sin \phi + M_y \cos \phi) e^{-jk' r \cos \theta} ds'$$

b) Calculate \vec{E} and \vec{H} in the far field.

$$H_r = 0 \quad E_r = 0$$

$$H_\theta = -\frac{jke^{jk'r}}{4\pi r} \left(N_\phi - \frac{1}{\eta} L_\theta \right) \quad E_\theta = \frac{jke^{jk'r}}{4\pi r} (\eta N_\theta + L_\phi)$$

$$H_\phi = \frac{jke^{jk'r}}{4\pi r} \left(N_\theta + \frac{1}{\eta} L_\phi \right) \quad E_\phi = \frac{jke^{jk'r}}{4\pi r} (\eta N_\phi - L_\theta)$$

c) Calculate RCS or whatever else is needed.

$$\text{RCS}(\theta, \phi) = \frac{k^2}{8\pi\eta P_{\text{inc}}} \left(|\eta N_\theta + L_\phi|^2 + |\eta N_\phi - L_\theta|^2 \right)$$

NF2FF in Two Dimensions

Surface Currents (1 of 2)

Restrict the analysis to just the x - y plane.

This means the normal vector \hat{n} is restricted to just the x - y plane.

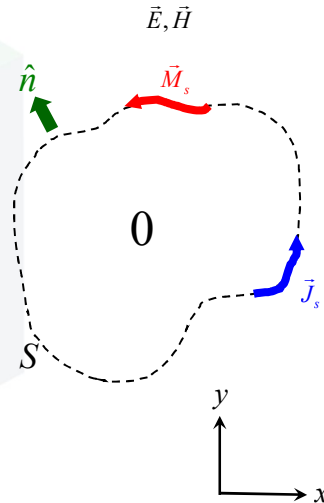
$$\hat{n} = n_x \hat{a}_x + n_y \hat{a}_y$$

This also means that the spherical coordinate parameter θ is zero.

$$\theta = 0$$

Therefore, \vec{N} and \vec{L} are only written as functions of ϕ .

$$N_\theta(\phi), N_\phi(\phi), L_\theta(\phi), L_\phi(\phi)$$



Surface Integrals to Line Integrals (1 of 2)

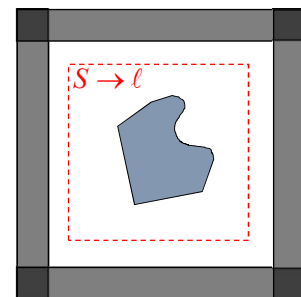
For the general case, the current terms \vec{J}_S and \vec{M}_S are integrated over a closed surface to obtain $\vec{N}(\phi)$ and $\vec{L}(\phi)$ in the far field.

$$N_\theta(\phi) = \oiint_S (\dots) ds' \quad L_\theta(\phi) = \oiint_S (\dots) ds'$$

$$N_\phi(\phi) = \oiint_S (\dots) ds' \quad L_\phi(\phi) = \oiint_S (\dots) ds'$$

In FDTD with a Cartesian grid, it is most convenient to setup the integrating surface S to be a rectangle.

In 2D, the closed surface becomes a closed line.



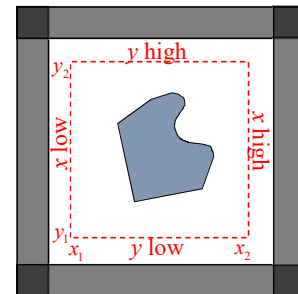
Surface Integrals to Line Integrals (2 of 2)

The integrations must be written as closed line integrals.

$$\oiint_S (\dots) ds' \rightarrow \oint_{\ell} (\dots) d\ell'$$

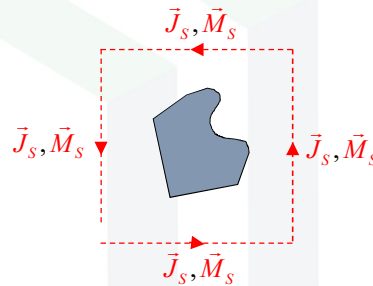
Given that the integrating line is a rectangle, it is divided into four separate ordinary integrals.

$$\begin{aligned} \oint_{\ell} (\dots) d\ell' &= \int_{x_1}^{x_2} (\dots) dx + \int_{y_1}^{y_2} (\dots) dy \\ &\quad \underbrace{\hspace{1.5cm}}_{y \text{ low side}} \quad \underbrace{\hspace{1.5cm}}_{x \text{ high side}} \\ &+ \int_{x_2}^{x_1} (\dots) dx + \int_{y_2}^{y_1} (\dots) dy \\ &\quad \underbrace{\hspace{1.5cm}}_{y \text{ high side}} \quad \underbrace{\hspace{1.5cm}}_{x \text{ low side}} \end{aligned}$$



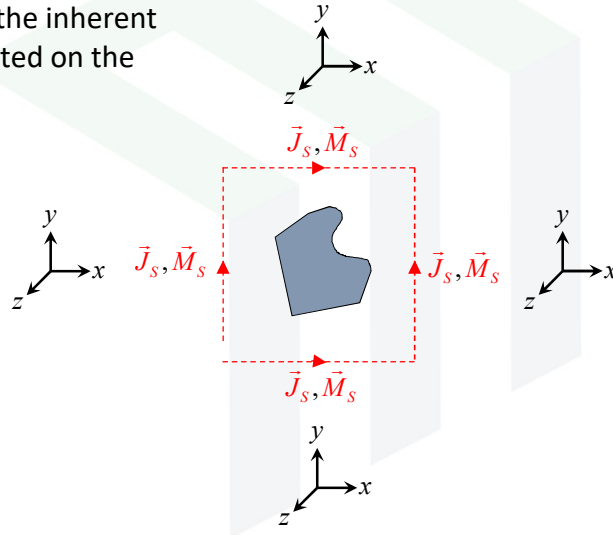
Orientation of the Surface Currents (1 of 3)

The surface currents have been defined to be flowing in a common direction.



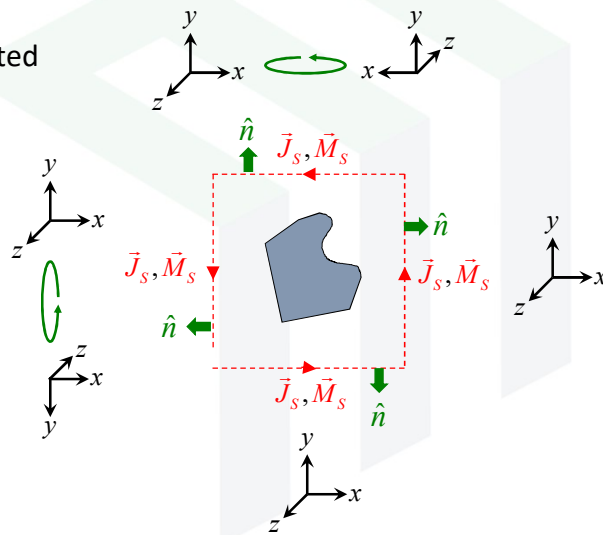
Orientation of the Surface Currents (2 of 3)

However, this is not the inherent directionality calculated on the Yee grid.



Orientation of the Surface Currents (3 of 3)

The surface current vectors must be rotated about the surface normal.



Rotation of Surface Current Terms

Based on this line of reasoning, the surface current terms in two of the line segment integrals must be rotated.

$$\oint_{\ell} (\dots) d\ell' = \int_{x_1}^{x_2} (\dots) dx + \int_{y_1}^{y_2} (\dots) dy + \int_{x_2}^{x_1} (\dots) dx + \int_{y_2}^{y_1} (\dots) dy$$

$\underbrace{\hspace{2cm}}_{y \text{ low side}} \quad \underbrace{\hspace{2cm}}_{x \text{ high side}} \quad \underbrace{\hspace{2cm}}_{y \text{ high side}} \quad \underbrace{\hspace{2cm}}_{x \text{ low side}}$
 $\hspace{10cm} R_y(180^\circ) \hspace{10cm} R_x(180^\circ)$

Flip sign of J_x, J_z, M_x and M_z
 Flip sign of J_y, J_z, M_y and M_z

Write Integrals as Standard Integrals

The last two integrals are calculated from b to a . To arrive at a more straightforward numerical implementation, the order of integration is reversed. When this is done, the sign of the integral changes.

$$\oint_{\ell} (\dots) d\ell' = \int_{x_1}^{x_2} (\dots) dx + \int_{y_1}^{y_2} (\dots) dy + \int_{x_2}^{x_1} (\dots) dx + \int_{y_2}^{y_1} (\dots) dy$$

$\underbrace{\hspace{2cm}}_{y \text{ low side}} \quad \underbrace{\hspace{2cm}}_{x \text{ high side}} \quad \underbrace{\hspace{2cm}}_{y \text{ high side}} \quad \underbrace{\hspace{2cm}}_{x \text{ low side}}$
 $\hspace{10cm} R_y(180^\circ) \hspace{10cm} R_x(180^\circ)$

↓

$$\oint_{\ell} (\dots) d\ell' = \int_{x_1}^{x_2} (\dots) dx + \int_{y_1}^{y_2} (\dots) dy - \int_{x_1}^{x_2} (\dots) dx - \int_{y_1}^{y_2} (\dots) dy$$

$\underbrace{\hspace{2cm}}_{y \text{ low side}} \quad \underbrace{\hspace{2cm}}_{x \text{ high side}} \quad \underbrace{\hspace{2cm}}_{y \text{ high side}} \quad \underbrace{\hspace{2cm}}_{x \text{ low side}}$
 $\hspace{10cm} R_y(180^\circ) \hspace{10cm} R_x(180^\circ)$

Final Equations for 2D Analysis

Applying all of this to the original surface integral equations for $\vec{N}(\phi)$ and $\vec{L}(\phi)$ gives

$$\begin{aligned}
 N_{\theta}(\phi) &= -\int_{x_1}^{x_2} \underbrace{J_z e^{-jkr' \cos \psi}}_{y \text{ low side}} dx - \int_{y_1}^{y_2} \underbrace{J_z e^{-jkr' \cos \psi}}_{x \text{ high side}} dy - \int_{x_1}^{x_2} \underbrace{J_z e^{-jkr' \cos \psi}}_{y \text{ high side}} dx - \int_{y_1}^{y_2} \underbrace{J_z e^{-jkr' \cos \psi}}_{x \text{ low side}} dy \\
 N_{\phi}(\phi) &= \int_{x_1}^{x_2} \underbrace{(J_y \cos \phi - J_x \sin \phi) e^{-jkr' \cos \psi}}_{y \text{ low side}} dx + \int_{y_1}^{y_2} \underbrace{(J_y \cos \phi - J_x \sin \phi) e^{-jkr' \cos \psi}}_{x \text{ high side}} dy \\
 &\quad - \int_{x_1}^{x_2} \underbrace{(J_y \cos \phi + J_x \sin \phi) e^{-jkr' \cos \psi}}_{y \text{ high side}} dx + \int_{y_1}^{y_2} \underbrace{(J_y \cos \phi + J_x \sin \phi) e^{-jkr' \cos \psi}}_{x \text{ low side}} dy \\
 L_{\theta}(\phi) &= -\int_{x_1}^{x_2} \underbrace{M_z e^{-jkr' \cos \psi}}_{y \text{ low side}} dx - \int_{y_1}^{y_2} \underbrace{M_z e^{-jkr' \cos \psi}}_{x \text{ high side}} dy - \int_{x_1}^{x_2} \underbrace{M_z e^{-jkr' \cos \psi}}_{y \text{ high side}} dx - \int_{y_1}^{y_2} \underbrace{M_z e^{-jkr' \cos \psi}}_{x \text{ low side}} dy \\
 L_{\phi}(\phi) &= \int_{x_1}^{x_2} \underbrace{(M_y \cos \phi - M_x \sin \phi) e^{-jkr' \cos \psi}}_{y \text{ low side}} dx + \int_{y_1}^{y_2} \underbrace{(M_y \cos \phi - M_x \sin \phi) e^{-jkr' \cos \psi}}_{x \text{ high side}} dy \\
 &\quad - \int_{x_1}^{x_2} \underbrace{(M_y \cos \phi + M_x \sin \phi) e^{-jkr' \cos \psi}}_{y \text{ high side}} dx + \int_{y_1}^{y_2} \underbrace{(M_y \cos \phi + M_x \sin \phi) e^{-jkr' \cos \psi}}_{x \text{ low side}} dy
 \end{aligned}$$

Simplifications for E and H Modes

E-Mode

$$\vec{E} = E_z \hat{a}_z$$

$$\vec{H} = H_x \hat{a}_x + H_y \hat{a}_y$$

$$E_x = E_y = H_z = 0$$

$$\vec{J}_s = (n_x \hat{a}_x + n_y \hat{a}_y) \times (H_x \hat{a}_x + H_y \hat{a}_y)$$

$$\vec{M}_s = -(n_x \hat{a}_x + n_y \hat{a}_y) \times (E_z \hat{a}_z)$$

$$\vec{J}_s = (n_x H_y - n_y H_x) \hat{a}_z$$

$$\vec{M}_s = -n_y E_z \hat{a}_x + n_x E_z \hat{a}_y$$

$$J_x = J_y = M_z = 0$$

H-Mode

$$\vec{E} = E_x \hat{a}_x + E_y \hat{a}_y$$

$$\vec{H} = H_z \hat{a}_z$$

$$H_x = H_y = E_z = 0$$

$$\vec{J}_s = (n_x \hat{a}_x + n_y \hat{a}_y) \times (H_z \hat{a}_z)$$

$$\vec{M}_s = -(n_x \hat{a}_x + n_y \hat{a}_y) \times (E_x \hat{a}_x + E_y \hat{a}_y)$$

$$\vec{J}_s = n_y H_z \hat{a}_x - n_x H_z \hat{a}_y$$

$$\vec{M}_s = (n_y E_x - n_x E_y) \hat{a}_z$$

$$M_x = M_y = J_z = 0$$

$\vec{N}(\theta, \phi)$ and $\vec{L}(\theta, \phi)$ for the E Mode

Given all of the simplifications on the previous slide, the expressions for $\vec{N}(\theta, \phi)$ and $\vec{L}(\theta, \phi)$ reduce to

$$N_{\theta}(\phi) = - \underbrace{\int_{x_1}^{x_2} H_x e^{-jkr' \cos \psi} dx}_{y \text{ low side}} - \underbrace{\int_{y_1}^{y_2} H_y e^{-jkr' \cos \psi} dy}_{x \text{ high side}} + \underbrace{\int_{x_1}^{x_2} H_x e^{-jkr' \cos \psi} dx}_{y \text{ high side}} + \underbrace{\int_{y_1}^{y_2} H_y e^{-jkr' \cos \psi} dy}_{x \text{ low side}}$$

$$N_{\phi}(\phi) = 0$$

$$L_{\theta}(\phi) = 0$$

$$L_{\phi}(\phi) = - \underbrace{\sin \phi \int_{x_1}^{x_2} E_z e^{-jkr' \cos \psi} dx}_{y \text{ low side}} + \underbrace{\cos \phi \int_{y_1}^{y_2} E_z e^{-jkr' \cos \psi} dy}_{x \text{ high side}}$$

$$+ \underbrace{\sin \phi \int_{x_1}^{x_2} E_z e^{-jkr' \cos \psi} dx}_{y \text{ high side}} - \underbrace{\cos \phi \int_{y_1}^{y_2} E_z e^{-jkr' \cos \psi} dy}_{x \text{ low side}}$$

$\vec{N}(\theta, \phi)$ and $\vec{L}(\theta, \phi)$ for the H Mode

Given all of the simplifications on the previous slide, the expressions for $\vec{N}(\theta, \phi)$ and $\vec{L}(\theta, \phi)$ reduce to

$$N_{\theta}(\phi) = 0$$

$$N_{\phi}(\phi) = \underbrace{\sin \phi \int_{x_1}^{x_2} H_z e^{-jkr' \cos \psi} dx}_{y \text{ low side}} - \underbrace{\cos \phi \int_{y_1}^{y_2} H_z e^{-jkr' \cos \psi} dy}_{x \text{ high side}}$$

$$- \underbrace{\sin \phi \int_{x_1}^{x_2} H_z e^{-jkr' \cos \psi} dx}_{y \text{ high side}} + \underbrace{\cos \phi \int_{y_1}^{y_2} H_z e^{-jkr' \cos \psi} dy}_{x \text{ low side}}$$

$$L_{\theta}(\phi) = + \underbrace{\int_{x_1}^{x_2} E_x e^{-jkr' \cos \psi} dx}_{y \text{ low side}} + \underbrace{\int_{y_1}^{y_2} E_y e^{-jkr' \cos \psi} dy}_{x \text{ high side}} - \underbrace{\int_{x_1}^{x_2} E_x e^{-jkr' \cos \psi} dx}_{y \text{ high side}} - \underbrace{\int_{y_1}^{y_2} E_y e^{-jkr' \cos \psi} dy}_{x \text{ low side}}$$

$$L_{\phi}(\phi) = 0$$

$\vec{E}(\theta, \phi)$ and $\vec{H}(\theta, \phi)$ for 2D Analysis

Given that $N_\phi = L_\theta = 0$ for the E mode and $N_\theta = L_\phi = 0$ for the H mode, the electromagnetic far-fields for each mode are calculated as

E Mode

$$E_\theta = \frac{jke^{jkr}}{4\pi r} (\eta N_\theta + L_\phi)$$

$$E_\phi = 0$$

$$H_\theta = 0$$

$$H_\phi = \frac{jke^{jkr}}{4\pi r} \left(\frac{L_\phi}{\eta} + N_\theta \right)$$

H Mode

$$E_\theta = 0$$

$$E_\phi = \frac{jke^{jkr}}{4\pi r} (\eta N_\phi - L_\theta)$$

$$H_\theta = \frac{jke^{jkr}}{4\pi r} \left(\frac{L_\theta}{\eta} - N_\phi \right)$$

$$H_\phi = 0$$

$P_{\text{rad}}(\theta, \phi)$ and $\text{RCS}(\theta, \phi)$ for 2D Analysis

E Mode

$$P_{\text{rad}}(\phi) = \frac{1}{2\eta} \left(\frac{k}{4\pi r} \right)^2 |\eta N_\phi - L_\theta|^2$$

$$\text{RCS}(\phi) = \frac{k^2}{8\pi\eta P_{\text{inc}}} |\eta N_\phi - L_\theta|^2$$

H Mode

$$P_{\text{rad}}(\phi) = \frac{1}{2\eta} \left(\frac{k}{4\pi r} \right)^2 |\eta N_\theta + L_\phi|^2$$

$$\text{RCS}(\phi) = \frac{k^2}{8\pi\eta P_{\text{inc}}} |\eta N_\theta + L_\phi|^2$$

Incident Power P_{inc}

The power in a uniform plane wave is calculated as

$$P_{\text{inc}} = \frac{1}{2} \frac{|\vec{E}|^2}{\eta} = \frac{1}{2} |\vec{H}|^2 \eta$$

Assuming a unit amplitude plane wave source, P_{inc} reduces to

$$P_{\text{inc}} = \begin{cases} 0.5/\eta & \text{for } |\vec{E}|=1 \\ 0.5\eta & \text{for } |\vec{H}|=1 \end{cases}$$

$$P_{\text{inc}} = \frac{1}{2\eta_0} \sqrt{\frac{\epsilon_r}{\mu_r}}$$

E Mode with $|\vec{E}|=1$

$$P_{\text{inc}} = \frac{\eta_0}{2} \sqrt{\frac{\mu_r}{\epsilon_r}}$$

H Mode with $|\vec{H}|=1$

Flow of Equations for E-Mode NF2FF

Loop over all values of ϕ

a) Integrate to calculate \vec{N} and \vec{L} in the far field

$$N_\theta(\phi) = - \underbrace{\int_{x_1}^{x_2} H_x e^{-jkr' \cos \psi} dx}_{y \text{ low side}} - \underbrace{\int_{y_1}^{y_2} H_y e^{-jkr' \cos \psi} dy}_{x \text{ high side}} + \underbrace{\int_{x_1}^{x_2} H_x e^{-jkr' \cos \psi} dx}_{y \text{ high side}} + \underbrace{\int_{y_1}^{y_2} H_y e^{-jkr' \cos \psi} dy}_{x \text{ low side}}$$

$$L_\phi(\phi) = - \sin \phi \underbrace{\int_{x_1}^{x_2} E_z e^{-jkr' \cos \psi} dx}_{y \text{ low side}} + \cos \phi \underbrace{\int_{y_1}^{y_2} E_z e^{-jkr' \cos \psi} dy}_{x \text{ high side}} + \sin \phi \underbrace{\int_{x_1}^{x_2} E_z e^{-jkr' \cos \psi} dx}_{y \text{ high side}} - \cos \phi \underbrace{\int_{y_1}^{y_2} E_z e^{-jkr' \cos \psi} dy}_{x \text{ low side}}$$

b) Calculate \vec{E} and \vec{H} in the far field from \vec{N} and \vec{L} (if desired)

$$E_\theta = \frac{jke^{jkr}}{4\pi r} (\eta N_\theta + L_\phi) \quad H_\phi = \frac{jke^{jkr}}{4\pi r} \left(\frac{L_\theta}{\eta} + N_\theta \right)$$

c) Calculate P_{rad} and RCS from \vec{N} and \vec{L} (if desired)

$$P_{\text{rad}}(\phi) = \frac{1}{2\eta} \left(\frac{k}{4\pi r} \right)^2 |\eta N_\theta - L_\phi|^2 \quad \text{RCS}(\phi) = \frac{k^2}{8\pi\eta P_{\text{inc}}} |\eta N_\theta - L_\phi|^2$$

Flow of Equations for H-Mode NF2FF

Loop over all values of ϕ

a) Integrate to calculate \vec{N} and \vec{L} in the far field

$$N_\phi(\phi) = \underbrace{\sin\phi \int_{x_1}^{x_2} H_z e^{-jkr' \cos\psi} dx}_{y \text{ low side}} - \underbrace{\cos\phi \int_{y_1}^{y_2} H_z e^{-jkr' \cos\psi} dy}_{x \text{ high side}} - \underbrace{\sin\phi \int_{x_1}^{x_2} H_x e^{-jkr' \cos\psi} dx}_{y \text{ high side}} + \underbrace{\cos\phi \int_{y_1}^{y_2} H_x e^{-jkr' \cos\psi} dy}_{x \text{ low side}}$$

$$L_\theta(\phi) = + \underbrace{\int_{x_1}^{x_2} E_x e^{-jkr' \cos\psi} dx}_{y \text{ low side}} + \underbrace{\int_{y_1}^{y_2} E_y e^{-jkr' \cos\psi} dy}_{x \text{ high side}} - \underbrace{\int_{x_1}^{x_2} E_x e^{-jkr' \cos\psi} dx}_{y \text{ high side}} - \underbrace{\int_{y_1}^{y_2} E_y e^{-jkr' \cos\psi} dy}_{x \text{ low side}}$$

b) Calculate \vec{E} and \vec{H} in the far field from \vec{N} and \vec{L} (if desired)

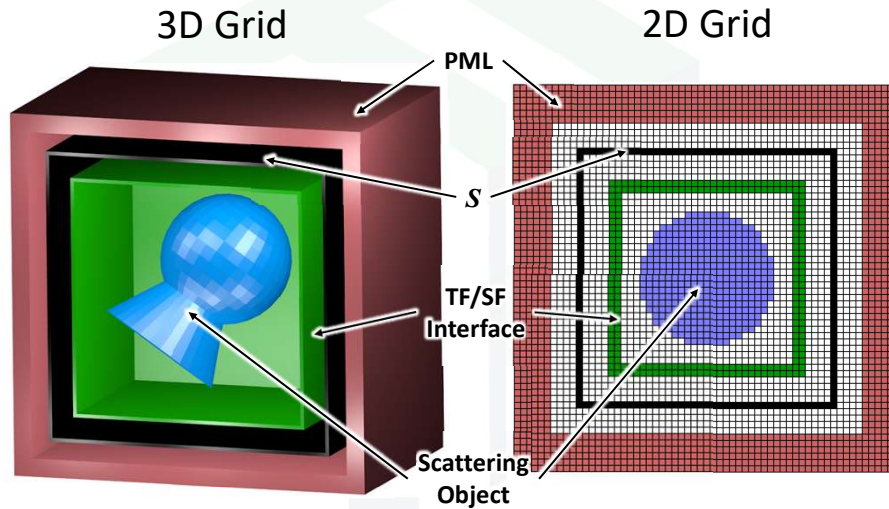
$$E_\phi = \frac{jke^{jkr}}{4\pi r} (\eta N_\phi - L_\theta) \quad H_\theta = \frac{jke^{jkr}}{4\pi r} \left(\frac{L_\theta}{\eta} - N_\phi \right)$$

c) Calculate P_{rad} and RCS from \vec{N} and \vec{L} (if desired)

$$P_{\text{rad}}(\phi) = \frac{1}{2\eta} \left(\frac{k}{4\pi r} \right)^2 |\eta N_\phi + L_\theta|^2 \quad \text{RCS}(\phi) = \frac{k^2}{8\pi\eta P_{\text{inc}}} |\eta N_\phi + L_\theta|^2$$

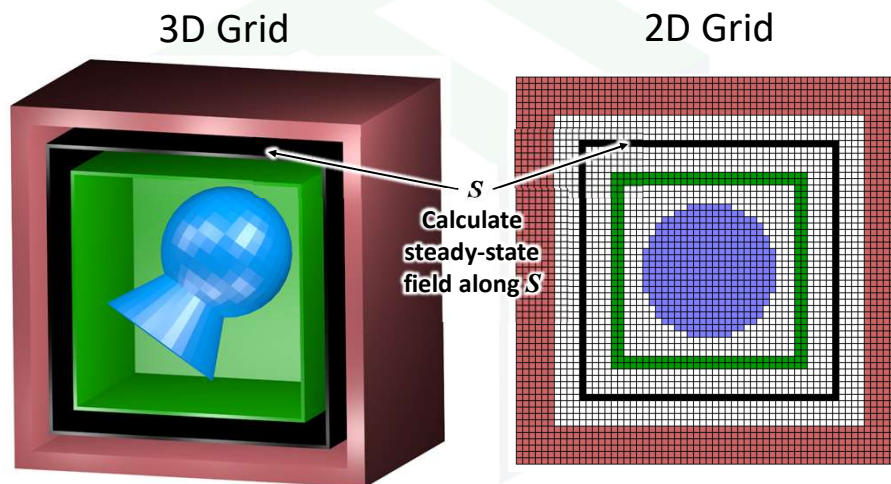
Implementation

Step 1 – Setup the Grid



Note that the near-field surface S is located within the scattered-field region.

Step 2 – Simulate to Get Steady-State Field



The steady-state field from FDTD is the same data that would be generated if simulated by finite-difference frequency-domain (FDFD).

Step 2 – Simulate to Get Steady-State Field

Below is MATLAB code from inside the main FDTD loop that updates the Fourier transform across the entire grid at a single frequency defined by the kernel K . The Fourier transform of the source at this same frequency is also calculated.

```
% Update Fourier Transforms
FT_Ez = FT_Ez + (K^T*dt)*Ez;
FT_Hx = FT_Hx + (K^T*dt)*Hx;
FT_Hy = FT_Hy + (K^T*dt)*Hy;
FT_S = FT_S + (K^T*dt)*H0 (nx2c,ny2c);
```

After the main FDTD loop has finished, the Fourier transforms are normalized to source using the follow code:

```
% NORMALIZE FOURIER TRANSFORMS
FT_Ez = FT_Ez/FT_S;
FT_Hx = FT_Hx/FT_S;
FT_Hy = FT_Hy/FT_S;
```

Step 3 – Interpolate Field Quantities Along S

The field quantities in each Yee cell must be interpolated at a common point. It is convenient to choose the origin of each cell as the common point.

$$\hat{E}_x(i, j, k) = \frac{E_x(i-1, j, k) + E_x(i, j, k)}{2}$$

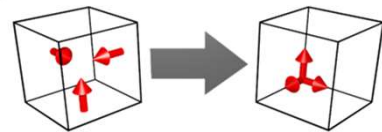
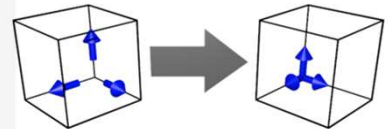
$$\hat{E}_y(i, j, k) = \frac{E_y(i, j-1, k) + E_y(i, j, k)}{2}$$

$$\hat{E}_z(i, j, k) = \frac{E_z(i, j, k-1) + E_z(i, j, k)}{2}$$

$$\hat{H}_x(i, j, k) = \frac{H_x(i, j-1, k-1) + H_x(i, j-1, k) + H_x(i, j, k-1) + H_x(i, j, k)}{4}$$

$$\hat{H}_y(i, j, k) = \frac{H_y(i-1, j, k-1) + H_y(i-1, j, k) + H_y(i, j, k-1) + H_y(i, j, k)}{4}$$

$$\hat{H}_z(i, j, k) = \frac{H_z(i-1, j-1, k) + H_z(i-1, j, k) + H_z(i, j-1, k) + H_z(i, j, k)}{4}$$



Step 3 – Interpolate Field Quantities Along S

For 2D simulations, the interpolation equations reduce to

E Mode

$$\hat{E}_z(i, j) = E_z(i, j)$$

$$\hat{H}_x(i, j) = \frac{H_x(i, j-1) + H_x(i, j)}{2}$$

$$\hat{H}_y(i, j) = \frac{H_y(i-1, j) + H_y(i, j)}{2}$$

```
% INTERPOLATE FIELDS TO ORIGIN OF YEE CELL
iEz      = FT_Ez;
iHx      = 0.5*FT_Hx;
iHx(:, 2:Ny) = 0.5*(FT_Hx(:, 1:Ny-1) ...
+ FT_Hx(:, 2:Ny));
iHy      = 0.5*FT_Hy;
iHy(2:Nx, :) = 0.5*(FT_Hy(1:Nx-1, :) ...
+ FT_Hy(2:Nx, :));
```

H Mode

$$\hat{H}_z(i, j) = \frac{H_z(i-1, j-1) + H_z(i-1, j) + H_z(i, j-1) + H_z(i, j)}{4}$$

$$\hat{E}_x(i, j) = \frac{E_x(i-1, j) + E_x(i, j)}{2}$$

$$\hat{E}_y(i, j) = \frac{E_y(i, j-1) + E_y(i, j)}{2}$$

Step 4 – Calculate \vec{N} and \vec{L} by Integrating Fields Along S

Pick a range of values for ϕ , usually $-\pi < \theta < \pi$. For each value of ϕ we integrate \vec{E} and \vec{H} around the contour S .

E Mode

$$N_\theta(\phi) = - \underbrace{\int_{x_1}^{x_2} H_x e^{-jkr' \cos \psi} dx}_{y \text{ low side}} - \underbrace{\int_{y_1}^{y_2} H_y e^{-jkr' \cos \psi} dy}_{x \text{ high side}} + \underbrace{\int_{x_1}^{x_2} H_x e^{-jkr' \cos \psi} dx}_{y \text{ high side}} + \underbrace{\int_{y_1}^{y_2} H_y e^{-jkr' \cos \psi} dy}_{x \text{ low side}}$$

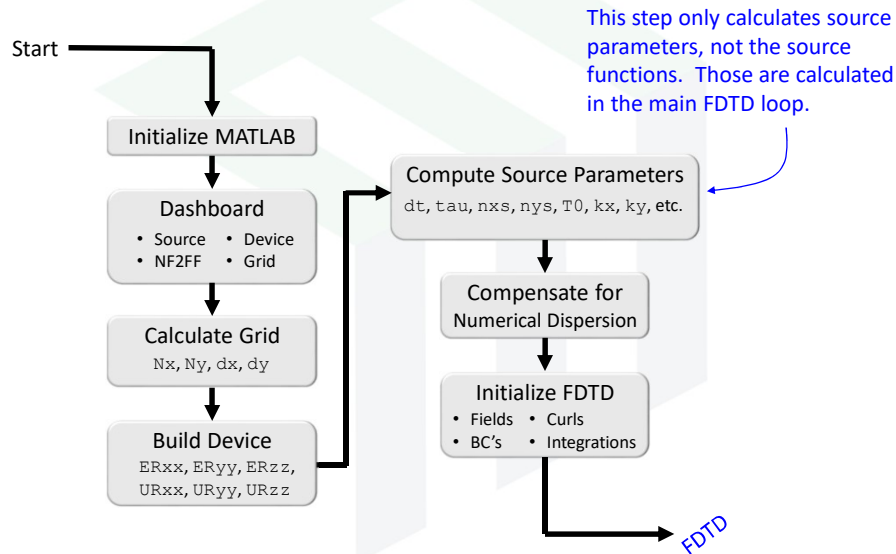
$$L_\phi(\phi) = - \sin \phi \underbrace{\int_{x_1}^{x_2} E_z e^{-jkr' \cos \psi} dx}_{y \text{ low side}} + \cos \phi \underbrace{\int_{y_1}^{y_2} E_z e^{-jkr' \cos \psi} dy}_{x \text{ high side}} + \sin \phi \underbrace{\int_{x_1}^{x_2} E_z e^{-jkr' \cos \psi} dx}_{y \text{ high side}} - \cos \phi \underbrace{\int_{y_1}^{y_2} E_z e^{-jkr' \cos \psi} dy}_{x \text{ low side}}$$

H Mode

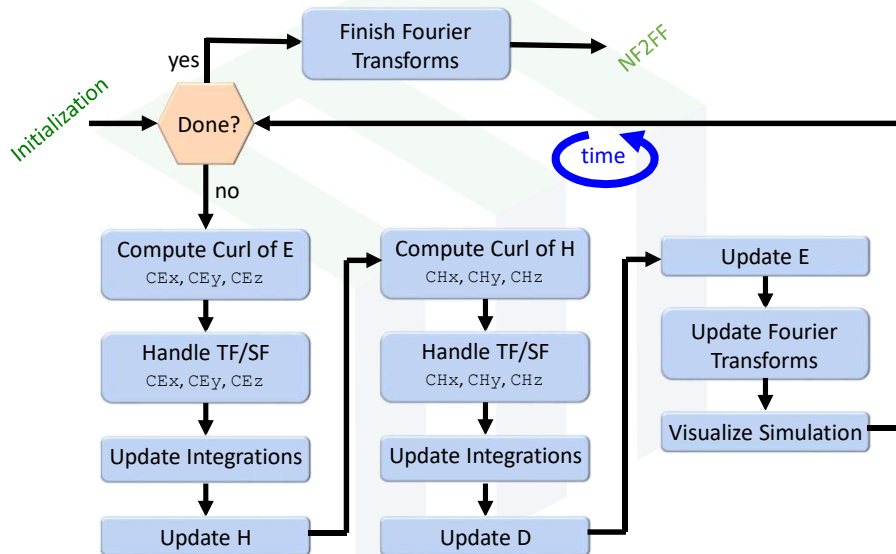
$$N_\theta(\phi) = \sin \phi \underbrace{\int_{x_1}^{x_2} H_z e^{-jkr' \cos \psi} dx}_{y \text{ low side}} - \cos \phi \underbrace{\int_{y_1}^{y_2} H_z e^{-jkr' \cos \psi} dy}_{x \text{ high side}} - \sin \phi \underbrace{\int_{x_1}^{x_2} H_z e^{-jkr' \cos \psi} dx}_{y \text{ high side}} + \cos \phi \underbrace{\int_{y_1}^{y_2} H_z e^{-jkr' \cos \psi} dy}_{x \text{ low side}}$$

$$L_\theta(\phi) = + \underbrace{\int_{x_1}^{x_2} E_x e^{-jkr' \cos \psi} dx}_{y \text{ low side}} + \underbrace{\int_{y_1}^{y_2} E_y e^{-jkr' \cos \psi} dy}_{x \text{ high side}} - \underbrace{\int_{x_1}^{x_2} E_x e^{-jkr' \cos \psi} dx}_{y \text{ high side}} - \underbrace{\int_{y_1}^{y_2} E_y e^{-jkr' \cos \psi} dy}_{x \text{ low side}}$$

Block Diagram for Scattering Analysis (1 of 3)

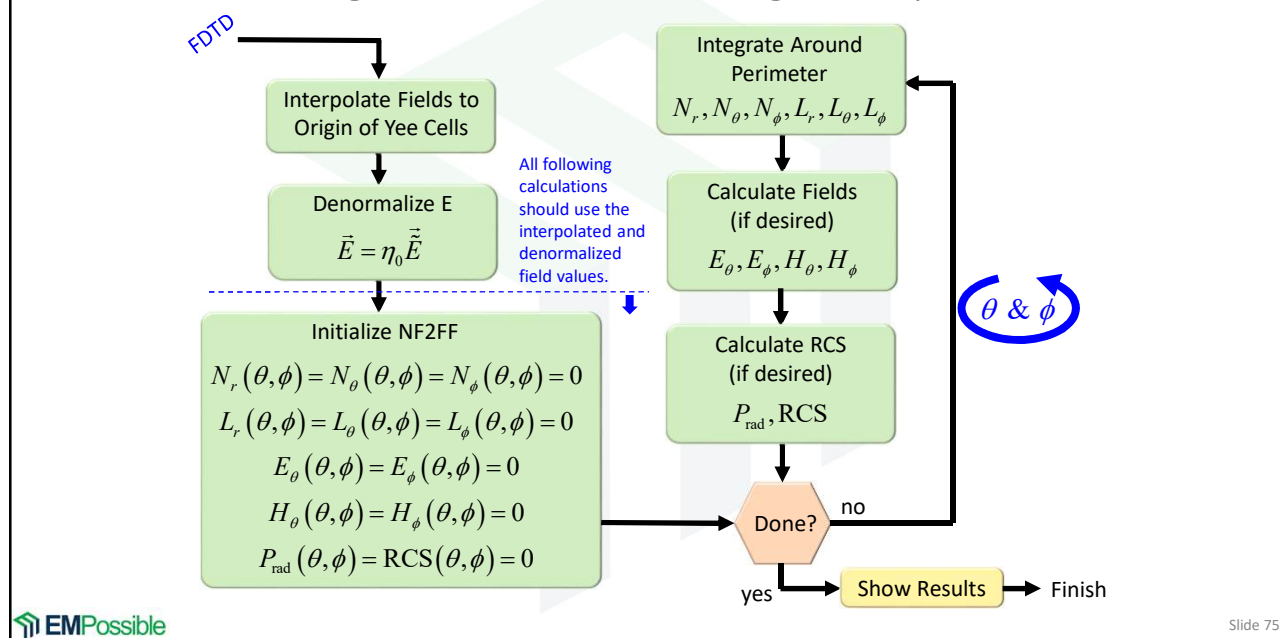


Block Diagram for Scattering Analysis (2 of 3)



This is the ordinary main loop for FDTD except the TF/SF and calculating Fourier transforms where the NF2FF is to be calculated.

Block Diagram for Scattering Analysis (3 of 3)



High-Level Code for ϕ Loop

```

% CALCULATE N AND L
r = 10*max(Sx,Sy);
PHI = linspace(-pi,pi,NPHI);
for nphi = 1 : NPHI

    % Next Phi Angle
    phi = PHI(nphi);
    rn = [ cos(phi) ; sin(phi) ];

    % ylo trapezoidal integration
    ...

    % xhi trapezoidal integration
    ...

    % yhi trapezoidal integration
    ...

    % xlo trapezoidal integration
    ...

end
  
```

Example Code for Integration

$$N_{\theta}(\phi) = \underbrace{-\int_{x_1}^{x_2} H_x e^{-jkr' \cos \psi} dx}_{y \text{ low side}} - \underbrace{\int_{y_1}^{y_2} H_y e^{-jkr' \cos \psi} dy}_{x \text{ high side}} + \underbrace{\int_{x_1}^{x_2} H_x e^{-jkr' \cos \psi} dx}_{y \text{ high side}} + \underbrace{\int_{y_1}^{y_2} H_y e^{-jkr' \cos \psi} dy}_{x \text{ low side}}$$

$$L_{\phi}(\phi) = \underbrace{-\sin \phi \int_{x_1}^{x_2} E_z e^{-jkr' \cos \psi} dx}_{y \text{ low side}} + \underbrace{\cos \phi \int_{y_1}^{y_2} E_z e^{-jkr' \cos \psi} dy}_{x \text{ high side}} + \underbrace{\sin \phi \int_{x_1}^{x_2} E_z e^{-jkr' \cos \psi} dx}_{y \text{ high side}} - \underbrace{\cos \phi \int_{y_1}^{y_2} E_z e^{-jkr' \cos \psi} dy}_{x \text{ low side}}$$

Note: nxa and nxb are the left and right array indices of S. nya and nyb are the top and bottom array indices of S.

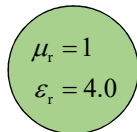
```
% ylo trapezoidal integration
N = 0;
L = 0;
ny = nya;
for nx = nxa : nxb
    rp = [ xa(nx) ; ya(ny) ];
    et = exp(-1i*k0*dot(rn,rp));
    iw = 1 - ((nx==nxa) + (nx==nxb))/2;
    N = N + iw*Hx(nx,ny)*et;
    L = L + iw*Ez(nx,ny)*et;
end
Nth(nphi) = Nth(nphi) - N*dx;
Lphi(nphi) = Lphi(nphi) - sin(phi)*L*dx;
```

Simulation Examples

Example #1: Scattering from a Dielectric Cylinder

The Problem

What is the RCS of a dielectric cylinder in air with radius $0.5\lambda_0$ and dielectric constant of $\epsilon_r = 4.0$?

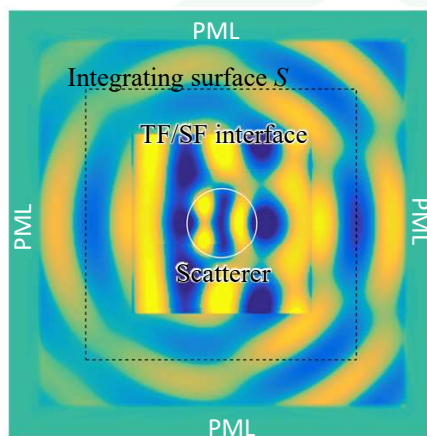


$$\mu_r = 1$$

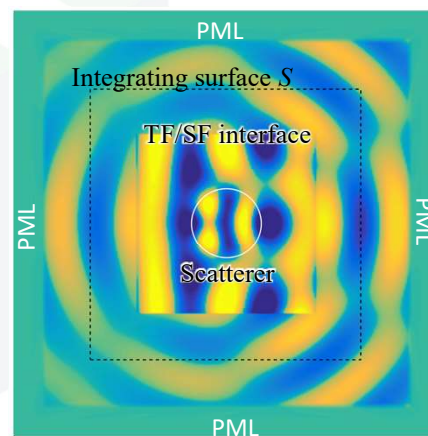
$$\epsilon_r = 1$$

Example #1: Scattering from a Dielectric Cylinder

The Simulation (E-Mode)

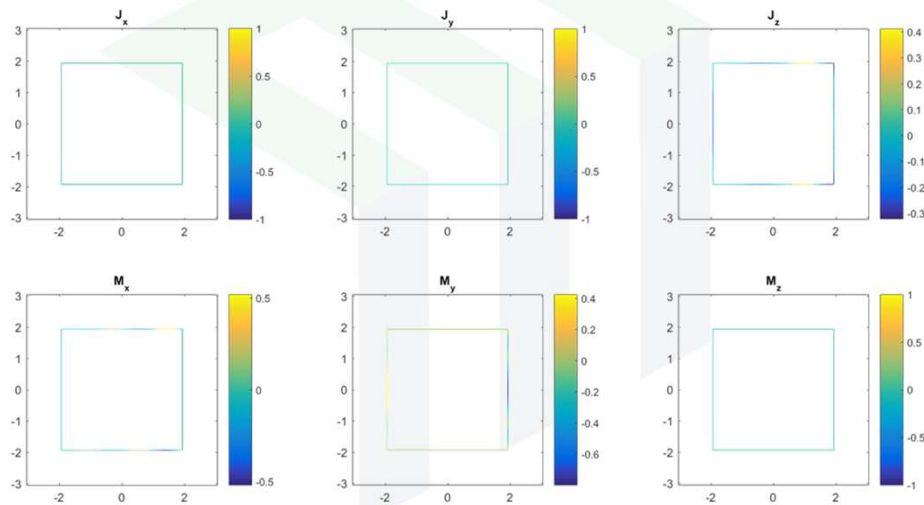


The Simulation (H-Mode)



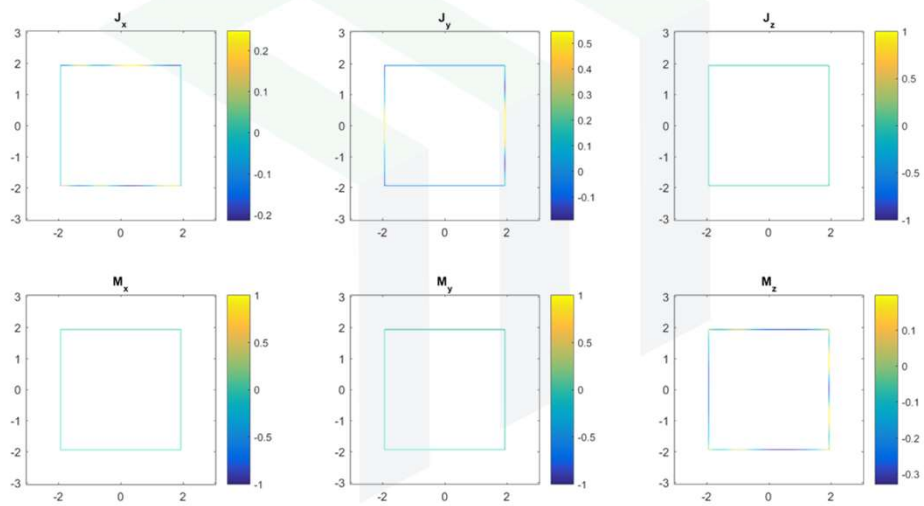
Example #1: Scattering from a Dielectric Cylinder

The Surface Currents (E-Mode)



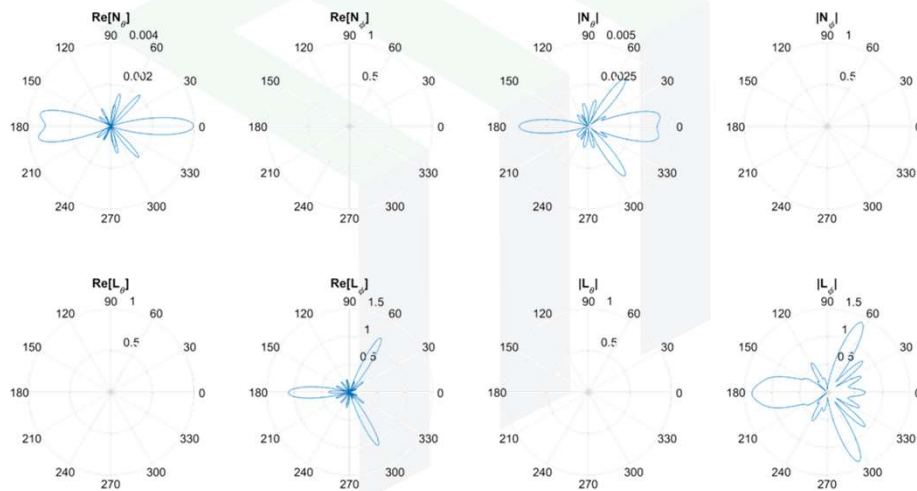
Example #1: Scattering from a Dielectric Cylinder

The Surface Currents (H-Mode)



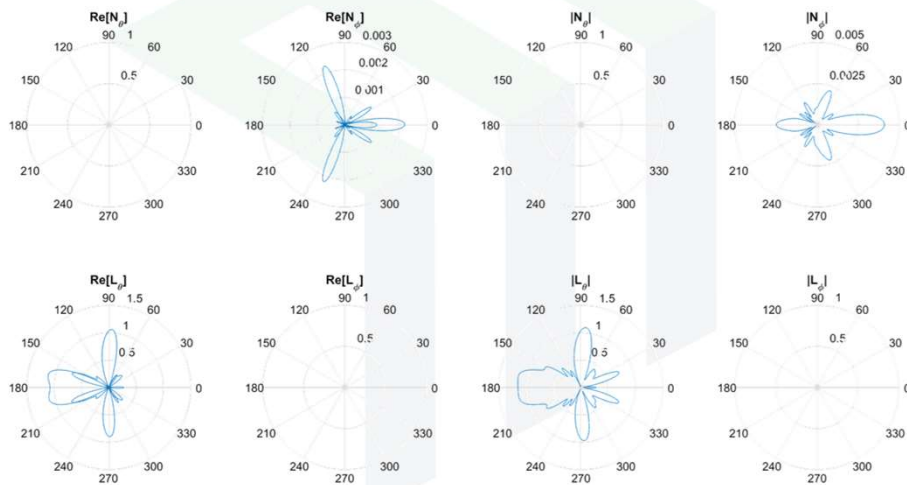
Example #1: Scattering from a Dielectric Cylinder

\vec{N} and \vec{L} Functions (E-Mode)



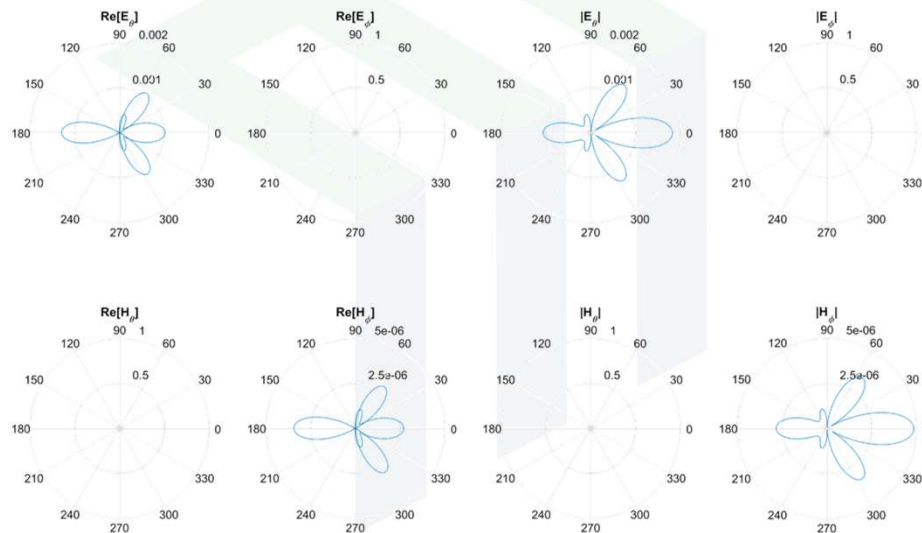
Example #1: Scattering from a Dielectric Cylinder

\vec{N} and \vec{L} Functions (H-Mode)



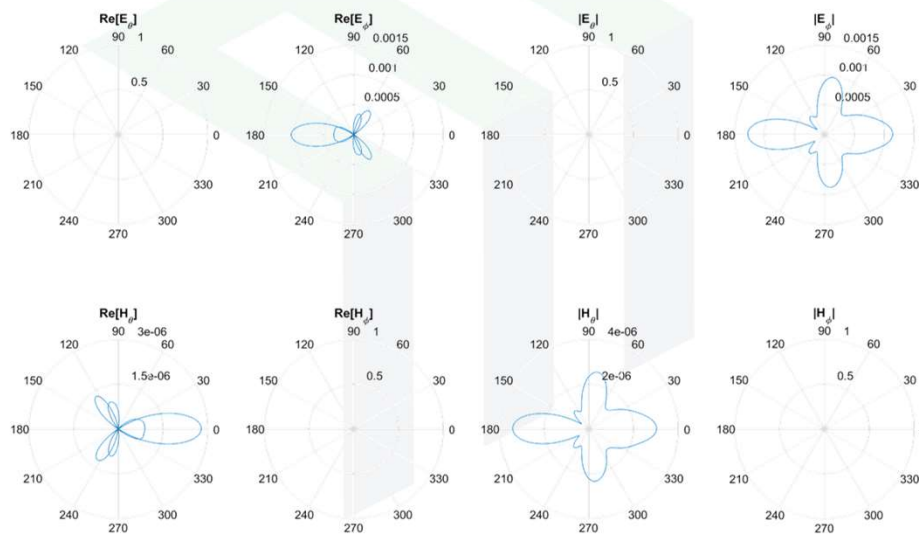
Example #1: Scattering from a Dielectric Cylinder

\vec{E} and \vec{H} Fields (E-Mode)



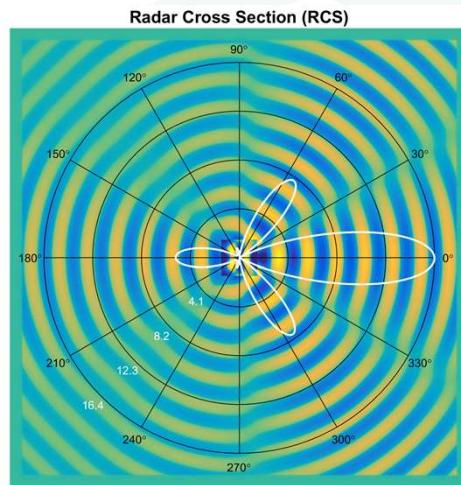
Example #1: Scattering from a Dielectric Cylinder

\vec{E} and \vec{H} Fields (H-Mode)



Example #1: Scattering from a Dielectric Cylinder

RCS (E-Mode)



RCS (H-Mode)

