



Advanced Electromagnetics:
21st Century Electromagnetics

Spatially-Variant Lattices

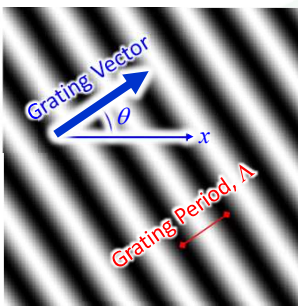
Lecture Outline

- Review of spatially-variant planar gratings
- Decomposition of lattices into planar gratings
- Synthesis algorithm for spatially-variant lattices
- Improving efficiency
- Extras
 - Deformation control
 - Arrays of discrete elements
 - Spatially-varying lattice symmetry
 - Spatial variance over curved surfaces

Review of Spatially-Variant Planar Gratings

Slide 3

The Grating Vector in Two Dimensions



- The grating vector \vec{K} is very similar to a wave vector \vec{k} .
- The direction of \vec{K} is perpendicular to the grating planes.
- The magnitude of \vec{K} is 2π divided by the period of the grating Λ .

$$|\vec{K}| = \frac{2\pi}{\Lambda}$$

- Given the slant angle θ , the grating vector is calculated as

$$\vec{K} = \frac{2\pi}{\Lambda} (\hat{a}_x \cos \theta + \hat{a}_y \sin \theta)$$

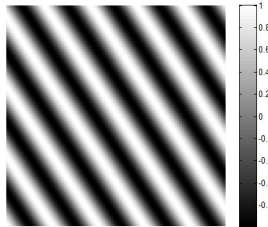
- This definition of \vec{K} allows a convenient calculation of the analog grating.

$$\cos(\vec{K} \cdot \vec{r})$$

Slide 4

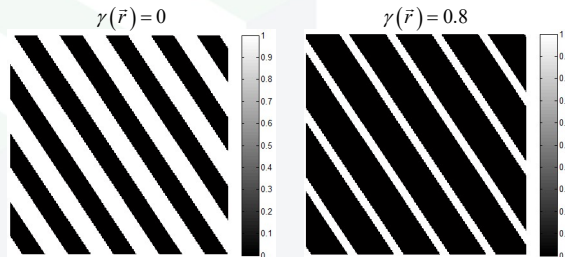
Analog Vs. Binary Gratings

Analog Gratings



$$\varepsilon_a(\vec{r}) = \cos(\vec{K} \cdot \vec{r})$$

Binary Gratings



$$\varepsilon_b(\vec{r}) = \begin{cases} \varepsilon_1 & \varepsilon_a(\vec{r}) \leq \gamma(\vec{r}) \\ \varepsilon_2 & \varepsilon_a(\vec{r}) > \gamma(\vec{r}) \end{cases}$$

$$\gamma(\vec{r}) \approx \cos[\pi f(\vec{r})]$$

Procedure for Generating Spatially-Variant Planar Gratings

1. Define the grating vector function $\vec{K}(\vec{r})$, or K -function.

$$\vec{K}(\vec{r}) = \frac{2\pi}{\Lambda(\vec{r})} \left\{ \hat{a}_x \cos[\theta(\vec{r})] + \hat{a}_y \sin[\theta(\vec{r})] \right\}$$

2. Calculate the grating phase $\Phi(\vec{r})$ from $\vec{K}(\vec{r})$.

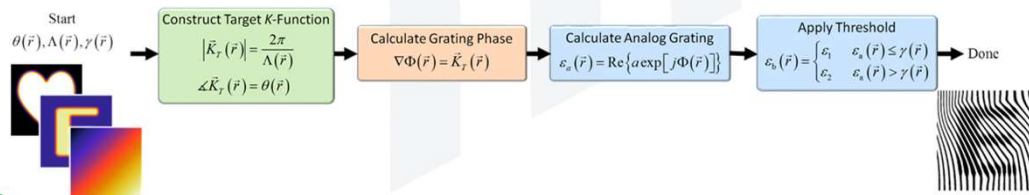
$$\nabla\Phi(\vec{r}) = \vec{K}(\vec{r})$$

3. Calculate the analog grating $\varepsilon_a(\vec{r})$ from the grating phase $\Phi(\vec{r})$.

$$\varepsilon_a(\vec{r}) = \cos[\Phi(\vec{r})]$$

4. Calculate the binary grating $\varepsilon_b(\vec{r})$ from the analog grating $\varepsilon_a(\vec{r})$.

$$\varepsilon_b(\vec{r}) = \begin{cases} \varepsilon_1 & \varepsilon_a(\vec{r}) \leq \gamma(\vec{r}) \\ \varepsilon_2 & \varepsilon_a(\vec{r}) > \gamma(\vec{r}) \end{cases}$$



Decomposition of Lattices Into Planar Gratings

Slide 7

Lattices Can be Decomposed into a Set of Planar Gratings

Using the concept of the complex Fourier series, the unit cell is decomposed into a set of planar gratings. Each planar grating is described by a complex amplitude a_{pq} calculated from the Fourier series and grating vector \vec{K}_{pq} calculated analytically.

$$\sum_{p,q} \text{[Grating Pattern]} = \text{[Unit Cell]}$$

Recall Complex Fourier Series

Periodic functions can be expanded into a Fourier series.

For 1D periodic functions, this is

$$f(x) = \sum_{p=-\infty}^{\infty} a_p e^{j\frac{2\pi px}{\Lambda}} \quad a_p = \frac{1}{\Lambda} \int_{-\Lambda/2}^{\Lambda/2} f(x) e^{-j\frac{2\pi px}{\Lambda}} dx$$

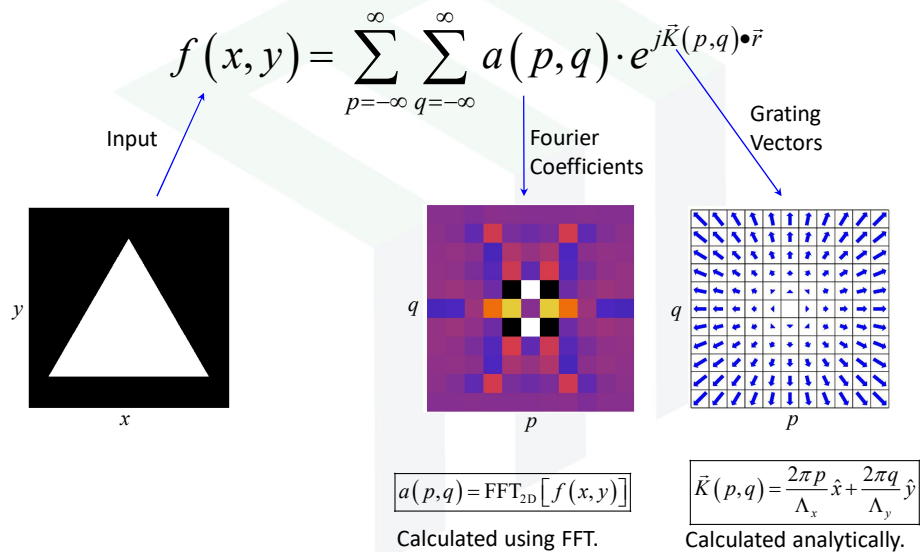
For 2D periodic functions, this is

$$f(x, y) = \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} a_{p,q} e^{j\left(\frac{2\pi px}{\Lambda_x} + \frac{2\pi qy}{\Lambda_y}\right)} \quad a_{p,q} = \frac{1}{A} \iint_A f(x, y) e^{-j\left(\frac{2\pi px}{\Lambda_x} + \frac{2\pi qy}{\Lambda_y}\right)} dA$$

For 3D periodic functions, this is

$$f(x, y, z) = \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \sum_{r=-\infty}^{\infty} a_{p,q,r} e^{j\left(\frac{2\pi px}{\Lambda_x} + \frac{2\pi qy}{\Lambda_y} + \frac{2\pi rz}{\Lambda_z}\right)} \quad a_{p,q,r} = \frac{1}{V} \iiint_V f(x, y, z) e^{-j\left(\frac{2\pi px}{\Lambda_x} + \frac{2\pi qy}{\Lambda_y} + \frac{2\pi rz}{\Lambda_z}\right)} dV$$

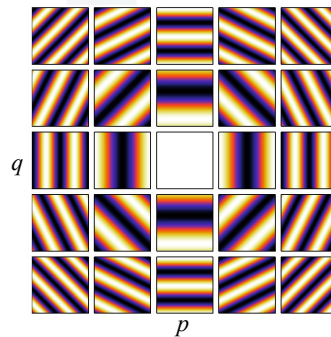
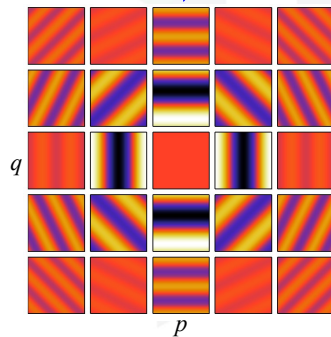
Two Parts to the Decomposition



Visualize the Terms



$$f(x, y) = \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} a(p, q) \cdot e^{j\vec{K}(p, q) \cdot \vec{r}}$$

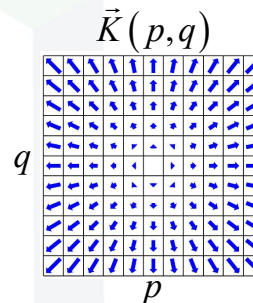


Grating Vector Part of the Fourier Expansion

$$f(x, y) = \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} a_{p, q} e^{j \left[\frac{2\pi p x}{\Lambda_x} + \frac{2\pi q y}{\Lambda_y} \right]}$$

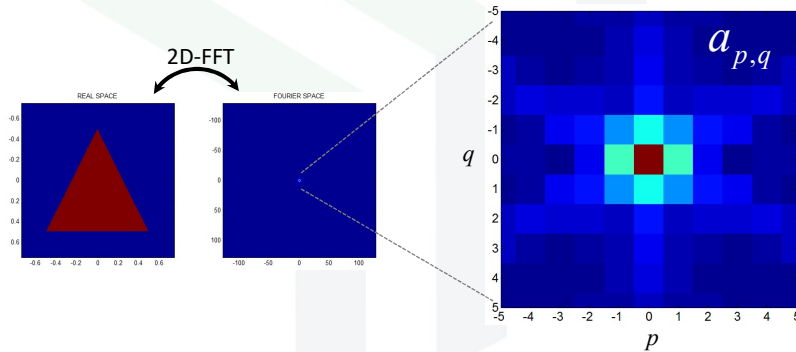
$$K_x(p, q) = \frac{2\pi p}{\Lambda_x}$$

$$K_y(p, q) = \frac{2\pi q}{\Lambda_y}$$

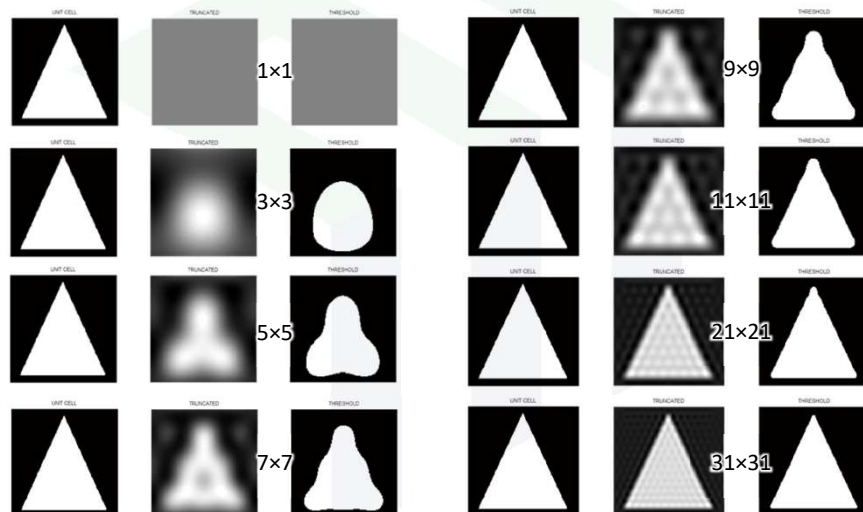


$$f(x, y) = \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} a_{p, q} e^{j[K_x(p, q)x + K_y(p, q)y]} = \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} a(p, q) \cdot e^{j\vec{K}(p, q) \cdot \vec{r}}$$

Complex Amplitude Part of the Fourier Expansion



Impact of Number of Planar Gratings



3D Decomposition

3D Unit Cell

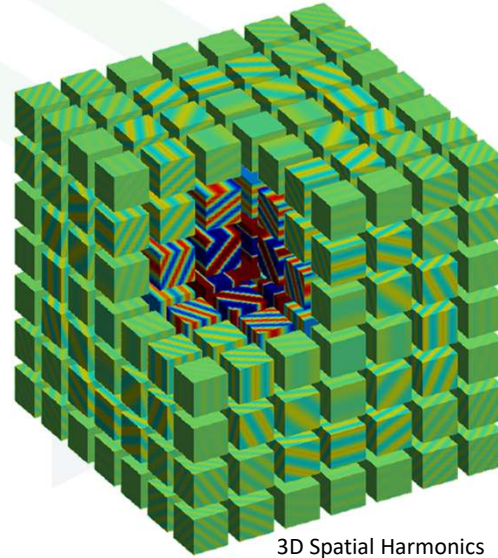


$$\vec{K} = K_x \hat{x} + K_y \hat{y} + K_z \hat{z}$$

$$K_x = \frac{2\pi p}{\Lambda_x} \quad p \equiv \text{integer}$$

$$K_y = \frac{2\pi q}{\Lambda_y} \quad q \equiv \text{integer}$$

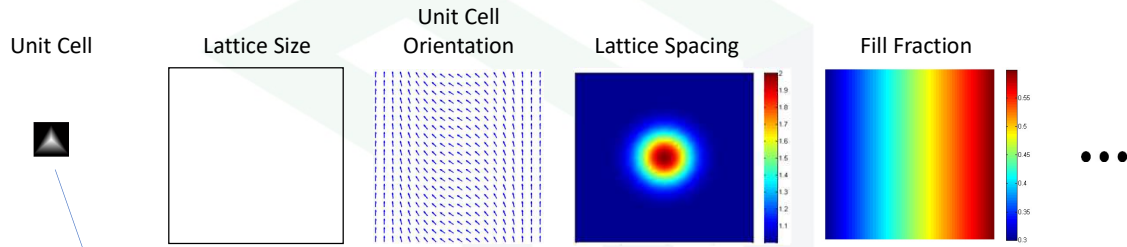
$$K_z = \frac{2\pi r}{\Lambda_z} \quad r \equiv \text{integer}$$



3D Spatial Harmonics

Algorithm for Synthesis of Spatially-Variant Lattices

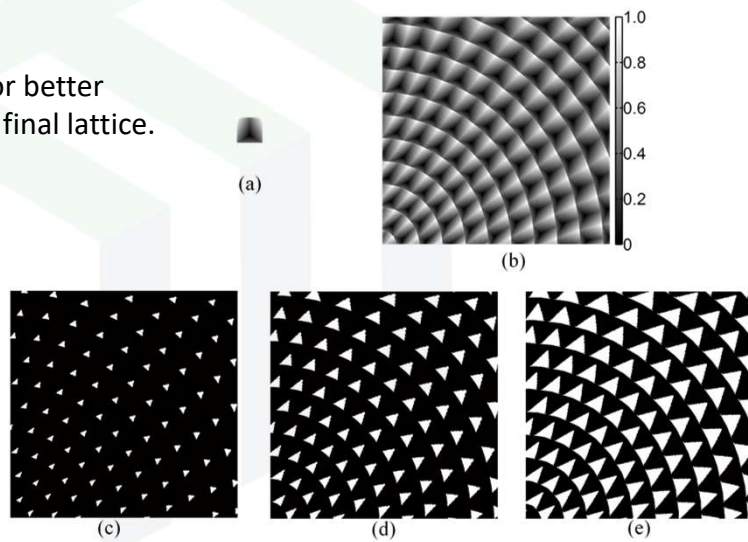
Step 1 – Generate Input to Algorithm



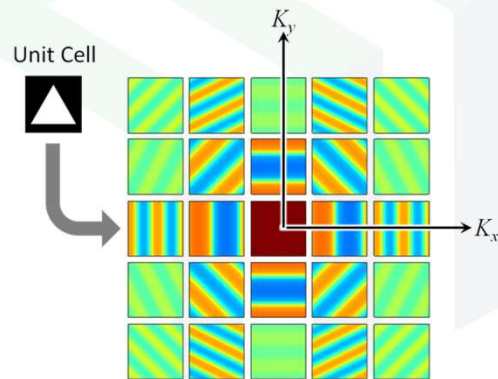
A grayscale unit cell allows easier adjustment of fill fraction.

Step 2 – Build a Grayscale Unit Cell

A grayscale unit cell allows for better control of fill fraction for the final lattice.

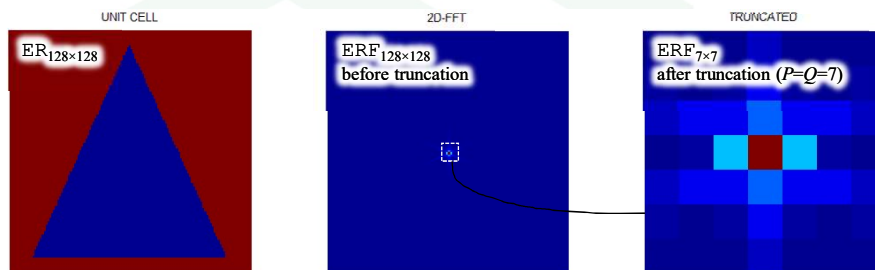


Step 3 – Decompose Unit Cell Into a Set of Planar Gratings



Step 4 – Truncate the Set of Planar Gratings

Retain only a minimum number of spatial harmonics.



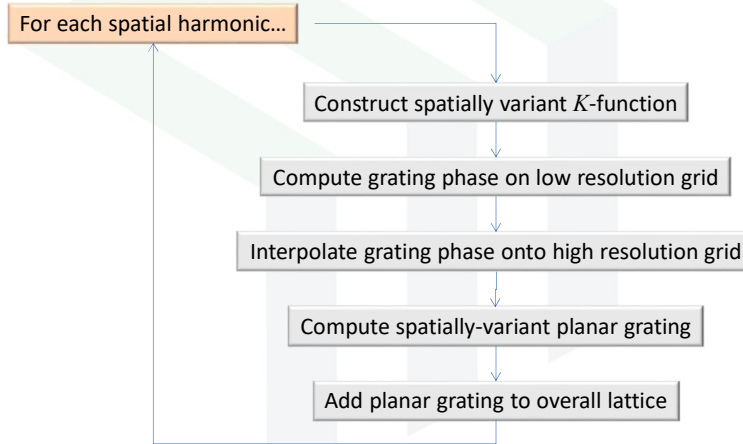
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% COMPUTE 2D-FFT
ERF = fftshift(fft2(ER))/(Nx*Ny);

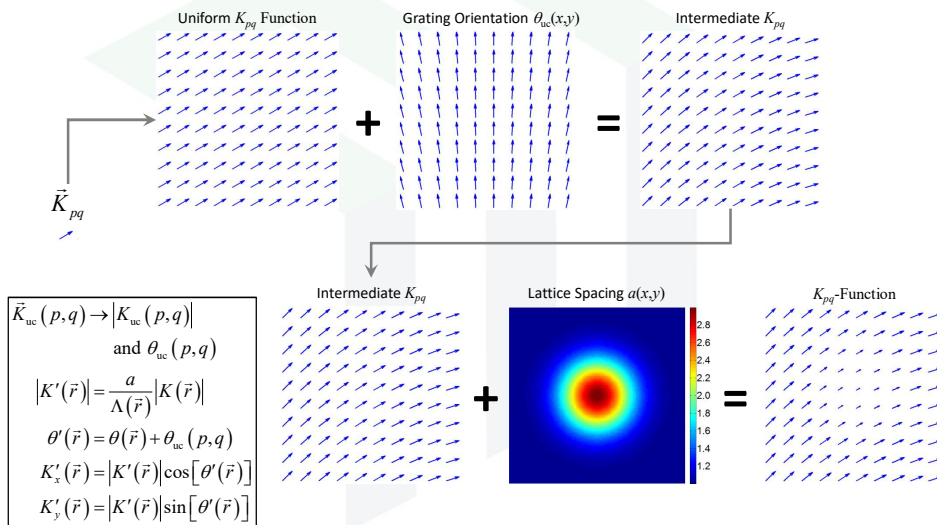
% TRUNCATE SPATIAL HARMONICS
p0 = 1 + floor(Nx/2); %p position of zero-order harmonic
q0 = 1 + floor(Ny/2); %q position of zero-order harmonic
p1 = p0 - floor(P/2); %start of p range
p2 = p0 + floor(P/2); %end of p range
q1 = q0 - floor(Q/2); %start of q range
q2 = q0 + floor(Q/2); %end of q range
ERF = ERF(p1:p2, q1:q2); %truncate harmonics

```

Step 5 – Spatially Vary Each Planar Grating Individually and Sum the Results



Step 5 – For Each Grating a) Construct *K*-Function



Step 5 – For Each Grating

b) Solve for the Grating Phase

Construct Matrix Equation

$$\mathbf{A} = \begin{bmatrix} \mathbf{D}_x \\ \mathbf{D}_y \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} \mathbf{k}_{x,pq} \\ \mathbf{k}_{y,pq} \end{bmatrix}$$

Solve Using Least Squares

$$\mathbf{A}' = \mathbf{A}^T \mathbf{A} \quad \mathbf{b}' = \mathbf{A}^T \mathbf{b}$$

$$\Phi_{pq} = (\mathbf{A}')^{-1} \mathbf{b}'$$

Reshape Back to a 2D Grid

$$\Phi_{pq} \rightarrow \Phi_{pq}(x, y)$$

Interpolate to a Higher Resolution Grid Using `interp2()`

$$\Phi_{pq}(x, y) \rightarrow \Phi_{2,pq}(x_2, y_2)$$

Step 5 – For Each Grating

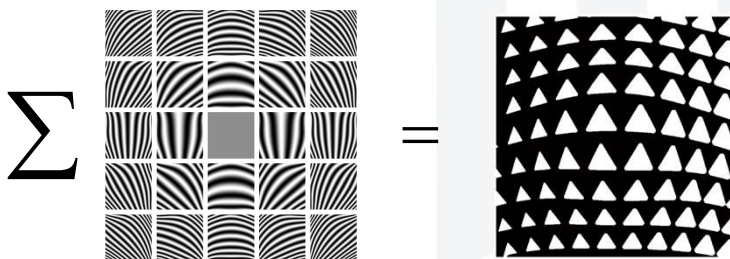
c) Add All of the Spatially-Variant Gratings

Calculate the pq^{th} spatially variant grating

$$\varepsilon_{pq}(\vec{r}) = a_{pq} \exp[j\Phi_{2,pq}(\vec{r})]$$

Add this complex planar grating to the overall analog lattice

$$\varepsilon_{\text{analog}}(\vec{r}) = \varepsilon_{\text{analog}}(\vec{r}) + \varepsilon_{pq}(\vec{r})$$



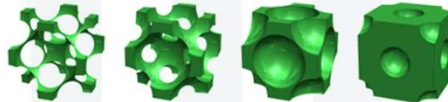
Step 6 – Incorporate Spatially-Variant Fill Fraction

At this point, the analog lattice is obtained. Set any imaginary components to zero as this is just numerical noise.

$$\varepsilon_{\text{analog}}(\vec{r}) = \text{Re}[\varepsilon_{\text{analog}}(\vec{r})]$$

From the analog lattice, calculate the binary lattice with a spatially-variant fill factor $f(\vec{r})$.

$$\varepsilon_{\text{binary}}(\vec{r}) = \begin{cases} \varepsilon_1 & \varepsilon_{\text{analog}}(\vec{r}) \leq \gamma(\vec{r}) \\ \varepsilon_2 & \varepsilon_{\text{analog}}(\vec{r}) > \gamma(\vec{r}) \end{cases}$$



For analog lattices that have a smooth cosine looking profile, the threshold can be estimated as

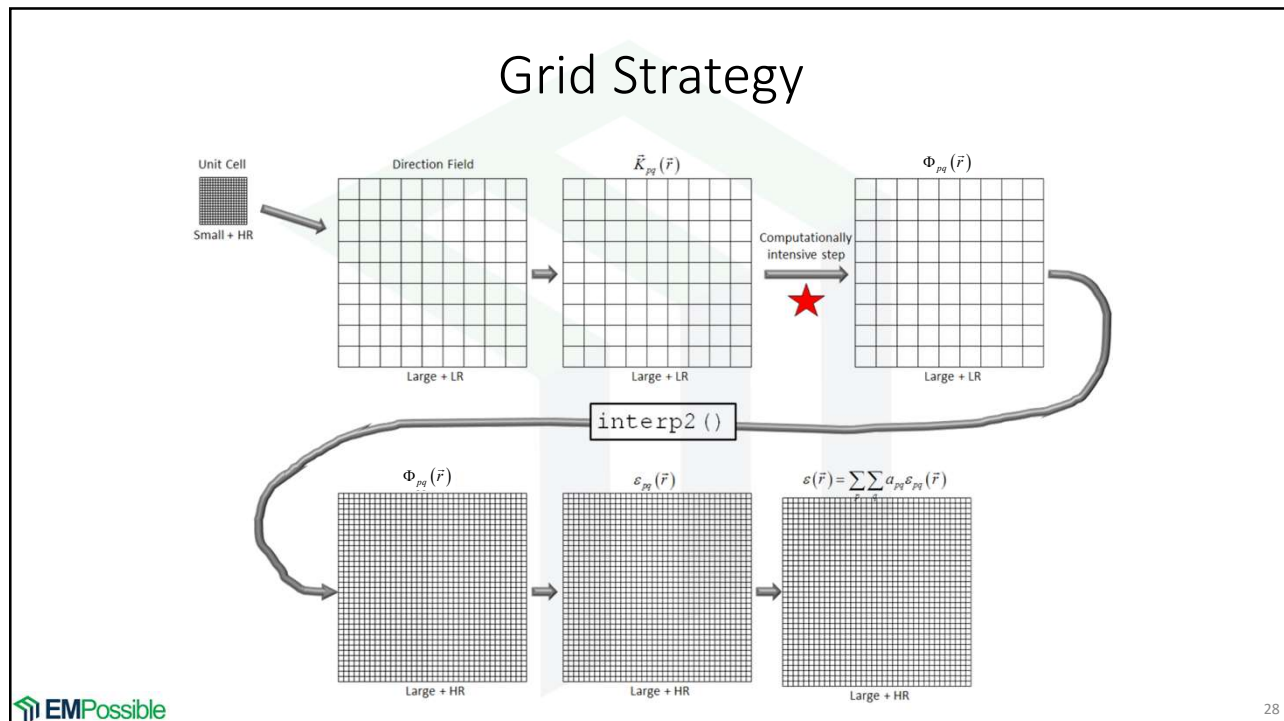
$$\gamma(\vec{r}) \approx \cos[\pi f(\vec{r})]$$

Summary of Spatially Variant Algorithm

1. Define the spatial variance: orientation $\theta(\vec{r})$, lattice spacing $\Lambda(\vec{r})$, fill fraction $f(\vec{r})$,...
2. Build the baseline unit cell.
3. Decompose unit cell into a set of planar gratings.
4. Truncate the set of planar gratings to a minimal set.
5. Loop over each spatial harmonic
 - a. Construct K -function that is uniform across the grid according to the grating vector of the spatial harmonic.
 - b. Solve for grating phase $\Phi(\vec{r})$ from the K -function.
 - c. Add the planar grating to the overall lattice.
6. Incorporate spatially-variant fill fraction.

Improving Efficiency of the Algorithm

Slide 27




Calculating the Grid Parameters


- High Resolution Unit Cell Grid
 - Need enough points for the FFT to converge.
$$\Delta_{\text{FFT}} < a/(N \cdot P)$$
- Low Resolution Lattice Grid
 - Need enough resolution to resolve the spatial variance. Usually this is $N > 10$ grid cells per unit cell.
$$\Delta_{\text{coarse}} < a/N$$
- High Resolution Lattice Grid
 - Need enough resolution to resolve the shortest period planar grating.


$$\Delta_{\text{fine}} < \frac{a_{\text{min}}}{N} \quad a_{\text{min}} = \frac{2\pi}{\max[\vec{K}_{pq}]}$$

Collinear Spatial Harmonics

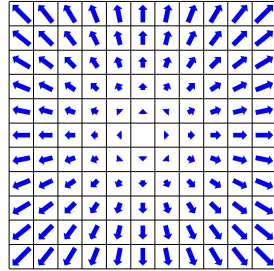
If a spatial harmonic is collinear to another (i.e. their grating vectors are parallel), its grating phase can be calculated simply by scaling the grating phase of the other. Therefore, it is only necessary to solve for one of them.

\vec{K}_1

 $\nabla\Phi_1 = \vec{K}_1$
 Solve this numerically.

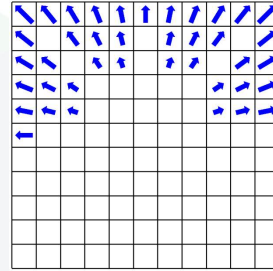
$\vec{K}_3 = a\vec{K}_1$

 $\nabla\Phi_2 = \vec{K}_2 = (a\vec{K}_1)$
 $\nabla\left(\frac{\Phi_2}{a}\right) = \vec{K}_1 \rightarrow \frac{\Phi_2}{a} = \Phi_1$
 $\Phi_2 = a\Phi_1$ Just scale the solution obtained from \vec{K}_1 .

$\vec{K}_i = b_i\vec{K}_1$

 $\Phi_i = b_i\Phi_1$
 Keep doing this for all other collinear spatial harmonics.

Identifying Collinear Planar Gratings

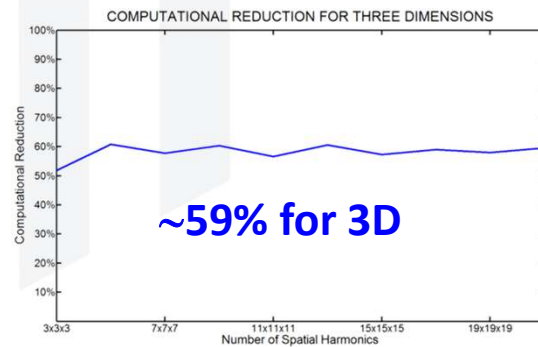
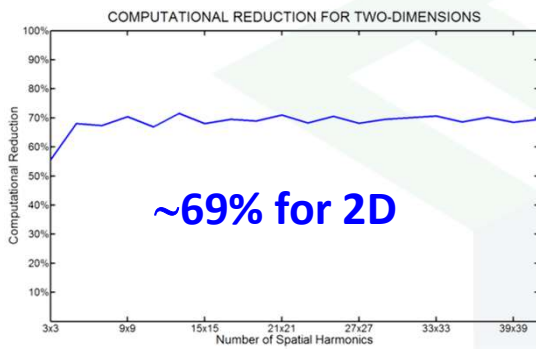


121 spatial harmonics

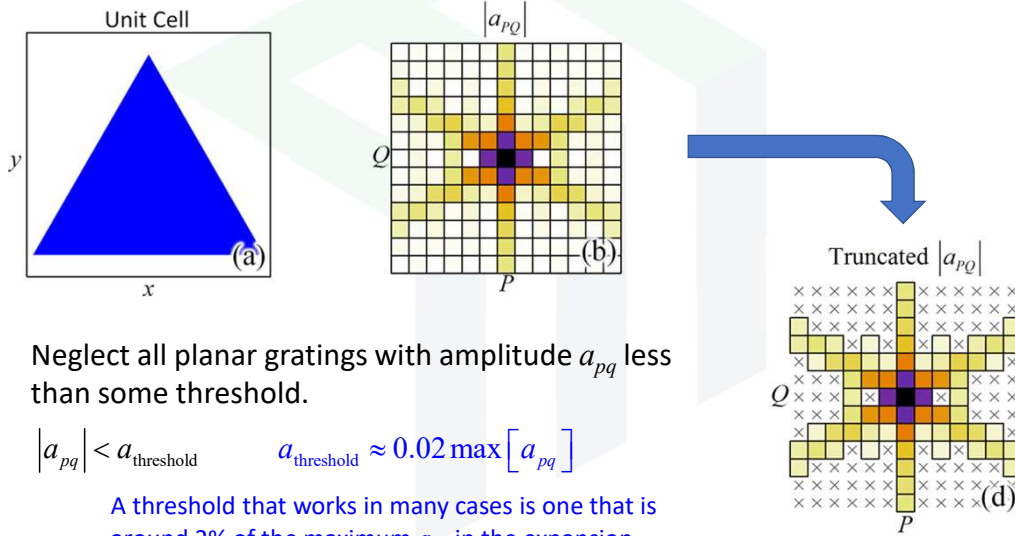


40 "unique" spatial harmonics.
The rest are collinear.

Performance Gain by Eliminating Collinear Gratings



Eliminating Gratings According to Their Amplitude

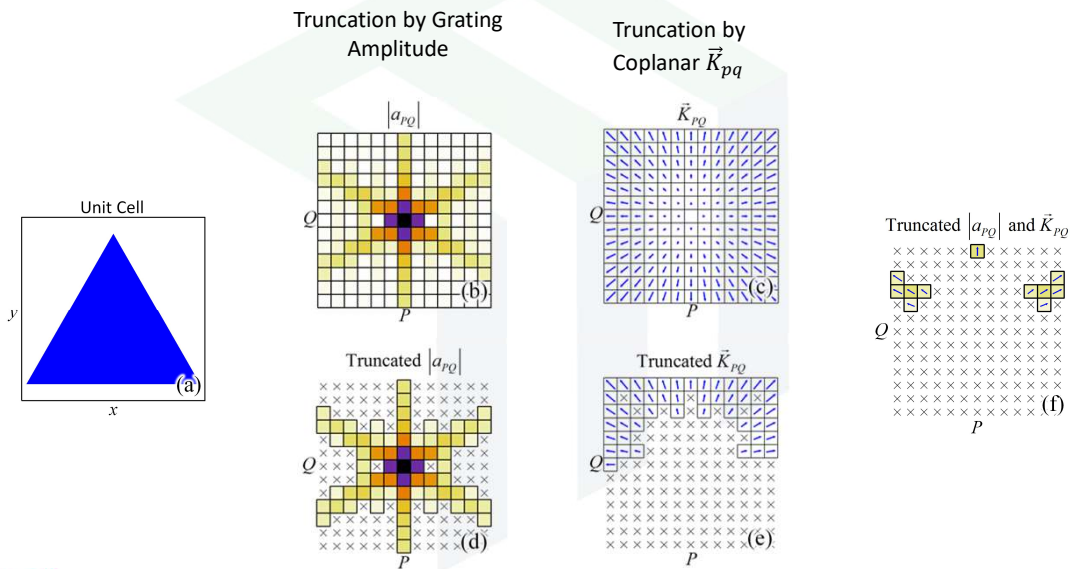


Neglect all planar gratings with amplitude a_{pq} less than some threshold.

$$|a_{pq}| < a_{\text{threshold}} \quad a_{\text{threshold}} \approx 0.02 \max[a_{pq}]$$

A threshold that works in many cases is one that is around 2% of the maximum a_{pq} in the expansion.

Overall Efficiency Improvement

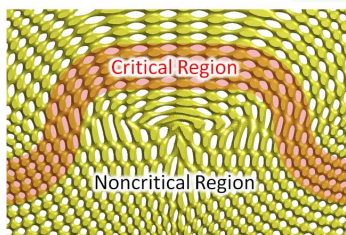


Extras

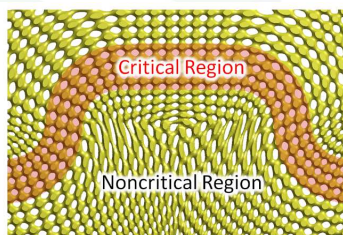
Slide 35

Deformation Control (1 of 2)

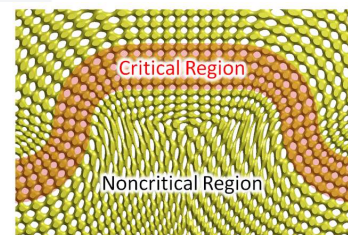
To control deformations of a lattice, simply control the deformations of the constituent planar gratings in the desired manner.



(a) basic algorithm



(b) 10 iterations of modified algorithm

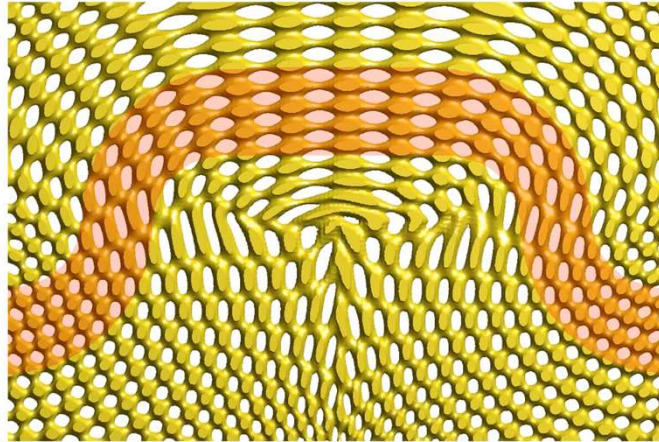


(c) 40 iterations of modified algorithm

R. C. Rumpf, et al, "Spatially-variant periodic structures in electromagnetics," accepted for publication in Phil. Trans. A, Dec. 2014.

Deformation Control (2 of 2)

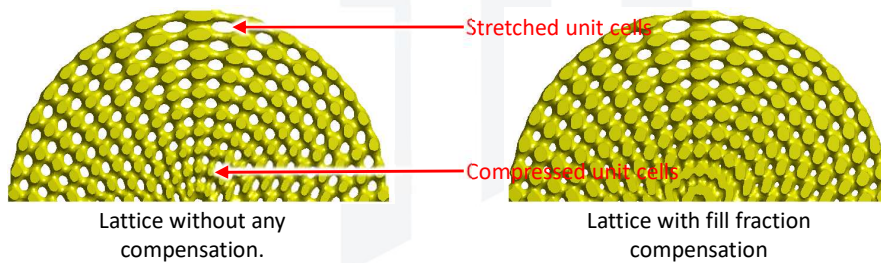
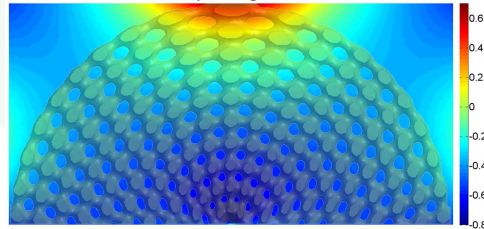
1 ITERATIONS

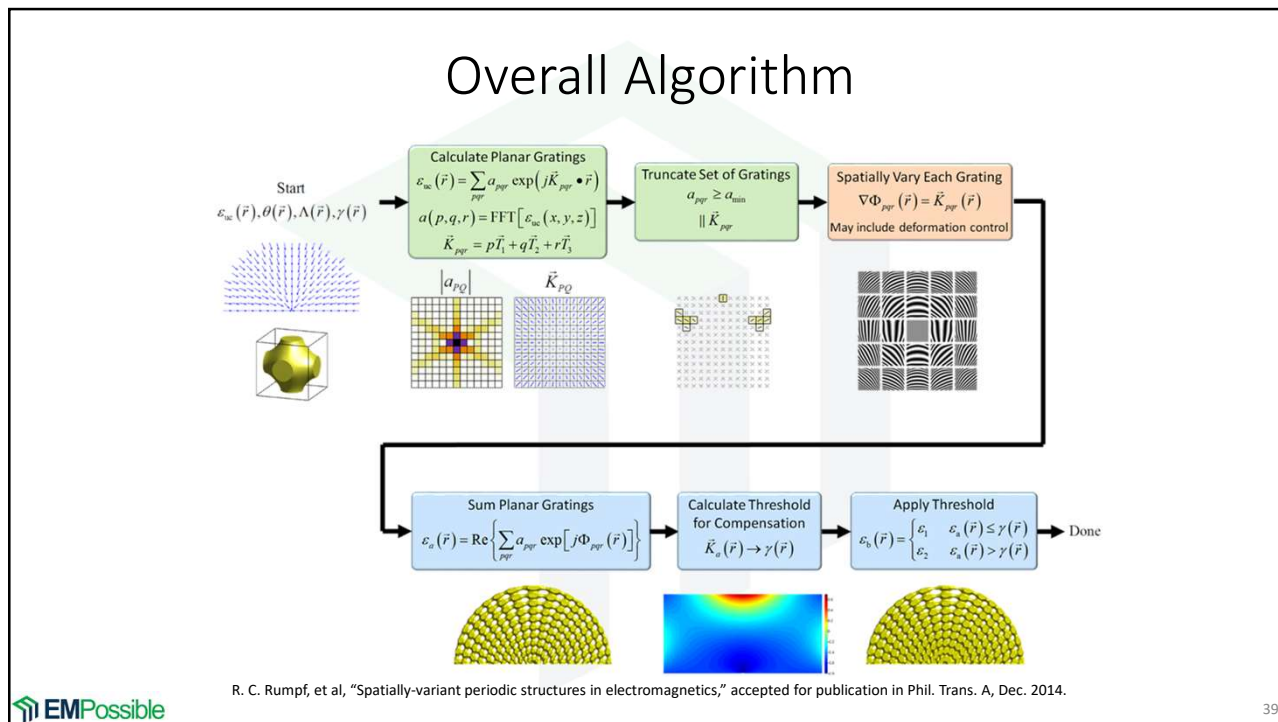


R. C. Rumpf, et al, "Spatially-variant periodic structures in electromagnetics," accepted for publication in Phil. Trans. A, Dec. 2014.

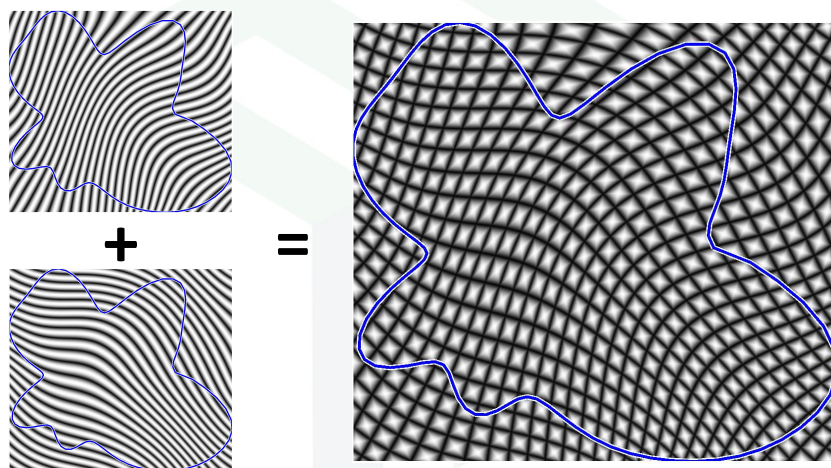
Compensating for Deformations

Lattice Spacing Deviation



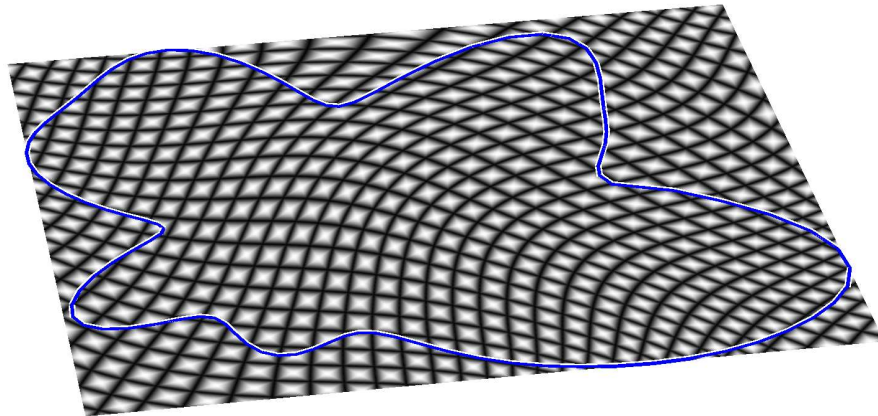


Arrays of Discontinuous Metallic Elements (1 of 2)



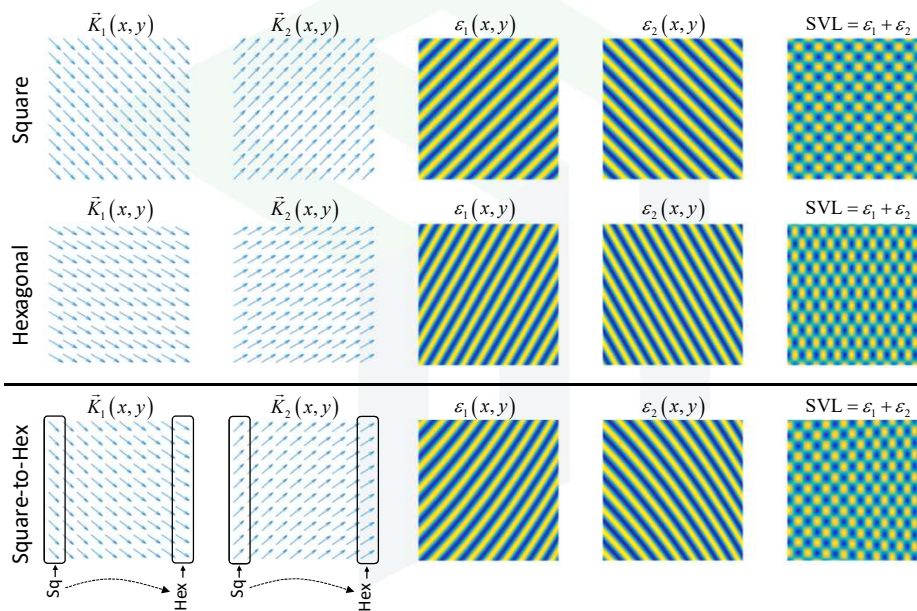
Spatially vary two planar gratings.

Arrays of Discontinuous Metallic Elements (2 of 2)



Place metallic elements at the intersections.

Spatially Varying Lattice Symmetry



Arrays of Metallic Elements Over Curved Surfaces

