Advanced Electromagnetics:
21^{st} Century Electromagnetics

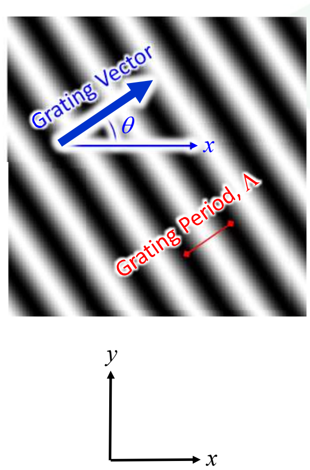
Spatially-Variant Planar Gratings

Lecture Outline

- Preliminary Concepts
  - The grating vector
  - Grating phase
- Generating spatially-variant planar gratings
- Extras
  - More efficient grid strategy
  - Deformation control
  - Spatially-variant planar gratings on curved surfaces

The Grating Vector $\vec{K}$

The Grating Vector in Two Dimensions



- The grating vector $\vec{K}$ is very similar to a wave vector $\vec{k}$.
- The direction of $\vec{K}$ is perpendicular to the grating planes.
- The magnitude of $\vec{K}$ is 2π divided by the period of the grating Λ.

$$\left|\vec{K}\right| = \frac{2\pi}{\Lambda}$$

- Given the slant angle θ, the grating vector is calculated as

$$\vec{K} = \frac{2\pi}{\Lambda}\left(\hat{a}_x \cos\theta + \hat{a}_y \sin\theta\right)$$

- This definition of $\vec{K}$ allows a convenient calculation of the analog grating.

$$\cos\left(\vec{K} \bullet \vec{r}\right)$$

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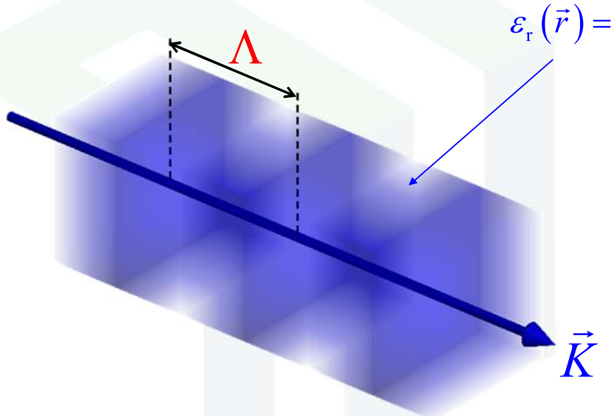
The Grating Vector in Three Dimensions

$$\varepsilon_r(\vec{r}) = \varepsilon_{avg} + \Delta\varepsilon\cos\left(\vec{K}\bullet\vec{r}\right)$$

$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$$

position vector



$$\left|\vec{K}\right| = \frac{2\pi}{\Lambda}$$

$$\vec{K} = K_x\hat{x} + K_y\hat{y} + K_z\hat{z}$$

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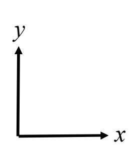
The Analog Grating



For simplicity, calculation of the analog grating will be written as

$$\varepsilon_a(\vec{r}) = \cos\left(\vec{K}\bullet\vec{r}\right)$$

or

$$\varepsilon_a(\vec{r}) = \mathrm{Re}\left[\exp\left(j\vec{K}\bullet\vec{r}\right)\right]$$

If instead it is desired to generate a physical analog grating, it will need to be scaled to convey permittivity values.

$$\varepsilon(\vec{r}) = \varepsilon_{avg} + \Delta\varepsilon\cos\left(\vec{K}\bullet\vec{r}\right)$$

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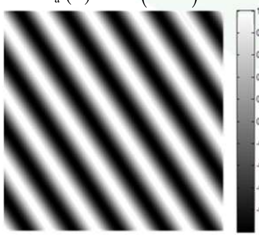
The Binary Grating

Analog Grating

$\varepsilon_a(\vec{r}) = \cos(\vec{K} \bullet \vec{r})$

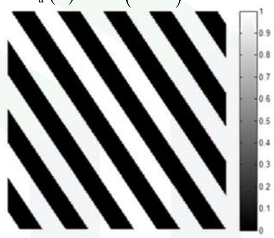


Generation of the grating using a cosine function gives a smooth analog profile.

This requires functionally grading the material properties to realize this. That is hard.

Binary Grating

$\varepsilon_a(\vec{r}) = \cos(\vec{K} \bullet \vec{r}) > 0$

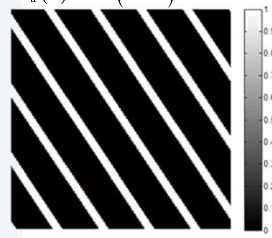


Use a threshold to convert the analog profile to a binary grating.

This is much easier to physically realize.

Controlling the Fill Fraction

$\varepsilon_a(\vec{r}) = \cos(\vec{K} \bullet \vec{r}) > 0.8$



The threshold value can be adjusted to control the duty cycle, or fill fraction, of the grating.

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Grating Phase

$$\Phi(\vec{r})$$

Concept of Grating Phase $\Phi(\vec{r})$

An analogy can be made with a standard wave. A wave propagates in the direction of the wave vector $\vec{k}$.

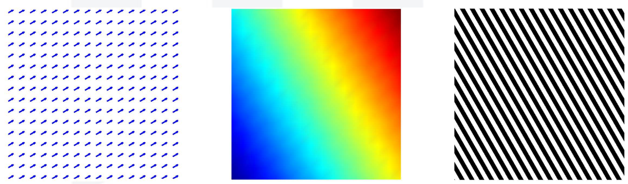
$$\vec{E}(\vec{r}) = \vec{E}_0 \cos\left(\omega t - \vec{k} \bullet \vec{r}\right)$$

As the wave propagates, it accumulates phase ϕ which is a function that increases in the direction of $\vec{k}$. The wave could alternatively be calculated just from the phase $\phi(\vec{r})$.

$$\vec{E}(\vec{r}) = \vec{E}_0 \cos\left[\phi(\vec{r})\right] \qquad \phi(\vec{r}) = \omega t - \vec{k} \bullet \vec{r}$$

For gratings, the grating vector $\vec{K}$ serves a similar purpose as the wave vector $\vec{k}$ does for waves.

$$\vec{k}(\vec{r}) \leftrightarrow \vec{K}(\vec{r})$$
$$\phi(\vec{r}) \leftrightarrow \Phi(\vec{r})$$
$$\vec{E}(\vec{r}) \leftrightarrow \varepsilon_a(\vec{r})$$



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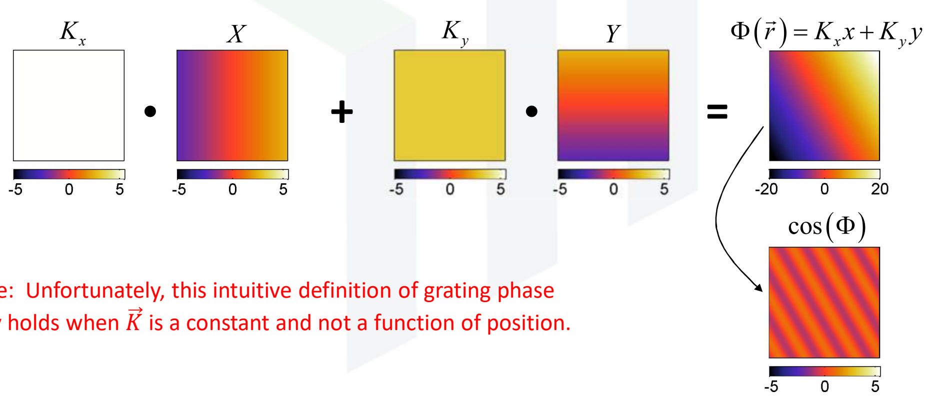
Definition of Grating Phase

Think of a grating as an unmoving wave because it has almost the same mathematical form. As a wave propagates it accumulates phase. The wave can be calculated from just the phase. Think of the grating phase $\Phi(\vec{r})$ this way…

$$\varepsilon_a(\vec{r}) = \cos\left(\vec{K} \bullet \vec{r}\right) = \cos\left[\Phi(\vec{r})\right] \qquad \Phi(\vec{r}) = \vec{K} \bullet \vec{r} = K_x x + K_y y$$



Note: Unfortunately, this intuitive definition of grating phase only holds when $\vec{K}$ is a constant and not a function of position.

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Problem of a Chirped Grating

Suppose the following chirped grating with period $\Lambda(z)$ is to be generated.

$$\Lambda(z) = \Lambda_0 z \qquad 1 \le z \le 2$$

This 1D grating is described by the following grating vector function.

$$K(z) = \frac{2\pi}{\Lambda(z)} = \frac{2\pi}{\Lambda_0 z}$$

Now try to calculate the grating using $\cos(\vec{K} \cdot \vec{r})$?

$$\varepsilon_{\mathrm{a}}(z) = \cos\left[K(z) \cdot z\right] = \cos\left[\frac{2\pi}{\Lambda_0 z} \cdot z\right] = \cos\left[\frac{2\pi}{\Lambda_0}\right]$$

The argument in the cosine is a constant so no grating is even generated.

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Conclusion From the Chirped Grating

When the grating vector $\vec{K}(\vec{r})$ is a function of position, the grating can no longer be directly calculated from it.

$$\varepsilon_{\mathrm{a}}(\vec{r}) \ne \cos\left[\vec{K}(\vec{r}) \bullet \vec{r}\right] \qquad \Phi(\vec{r}) \ne \vec{K}(\vec{r}) \bullet \vec{r}$$

Instead, the grating must be calculated through the intermediate parameter of the grating phase $\Phi(\vec{r})$.

$$\varepsilon_{\mathrm{a}}(\vec{r}) = \cos\left[\Phi(\vec{r})\right]$$

However, in order to do this a more rigorous definition of grating phase must be adopted.

$$\boxed{\nabla\Phi(\vec{r}) = \vec{K}(\vec{r})}$$

⬅ Key equation

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Revised Solution for Chirped Gratings

Now construct the chirped grating using the grating phase.

$$\nabla\Phi = \vec{K}(\vec{r})$$

$$\frac{d\Phi}{dz} = \frac{2\pi}{\Lambda_0 z}$$

$$\Phi(z) = \int_{-\infty}^{z} \frac{2\pi}{\Lambda_0 z'} dz' = \cancel{\int_{-\infty}^{1} \frac{2\pi}{\Lambda_0 z'} dz'} + \int_{1}^{z} \frac{2\pi}{\Lambda_0 z'} dz' = \frac{2\pi}{\Lambda_0} \ln z' \Big|_1^z$$

$$\Phi(z) = \frac{2\pi}{\Lambda_0}(\ln z - 1)$$

This term is just a constant. Since it is a phase, we are free to choose whatever is convenient. Here we choose zero.

The analog grating is then

$$\varepsilon_a(z) = \cos[\Phi(z)] = \cos\left[\frac{2\pi}{\Lambda_0}(\ln z - 1)\right]$$

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Generating Spatially-Variant Planar Gratings

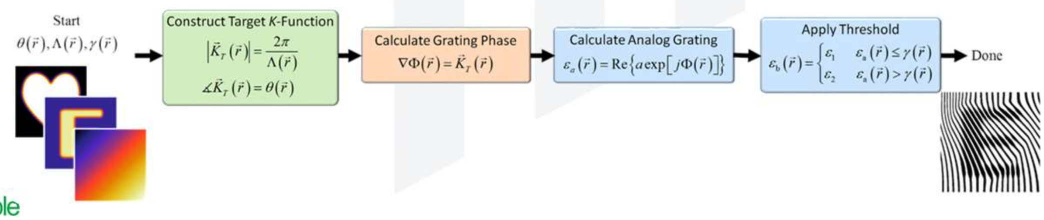
Procedure for Generating Spatially-Variant Planar Gratings

1. Define the grating vector function $\vec{K}(\vec{r})$, or K-function. $$\vec{K}(\vec{r}) = \frac{2\pi}{\Lambda(\vec{r})}\left\{\hat{a}_x \cos\left[\theta(\vec{r})\right] + \hat{a}_y \sin\left[\theta(\vec{r})\right]\right\}$$

2. Calculate the grating phase $\Phi(\vec{r})$ from $\vec{K}(\vec{r})$. $$\nabla\Phi(\vec{r}) = \vec{K}(\vec{r})$$

3. Calculate the analog grating $\varepsilon_a(\vec{r})$ from the grating phase $\Phi(\vec{r})$. $$\varepsilon_a(\vec{r}) = \cos\left[\Phi(\vec{r})\right]$$

4. Calculate the binary grating $\varepsilon_b(\vec{r})$ from the analog grating $\varepsilon_a(\vec{r})$. $$\varepsilon_b(\vec{r}) = \begin{cases} \varepsilon_1 & \varepsilon_a(\vec{r}) \le \gamma(\vec{r}) \\ \varepsilon_2 & \varepsilon_a(\vec{r}) > \gamma(\vec{r}) \end{cases}$$



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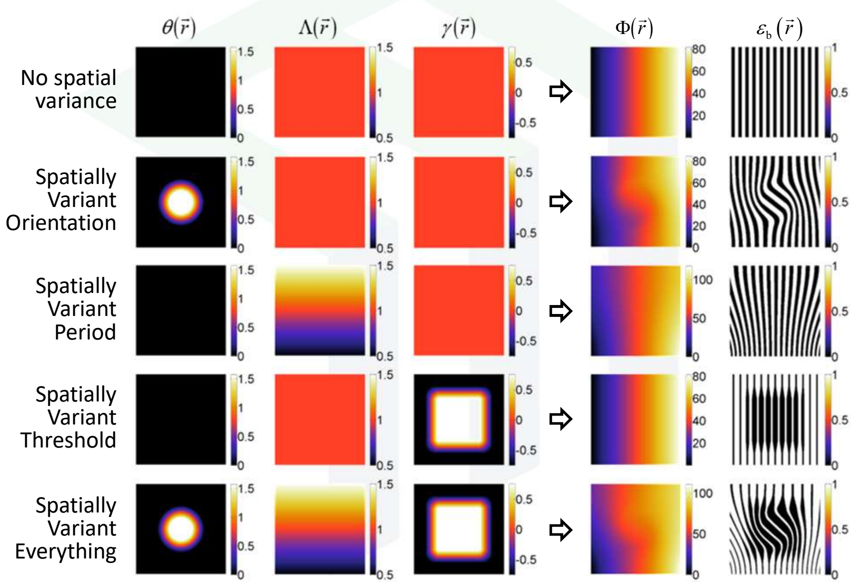
Spatially-Variant Planar Gratings



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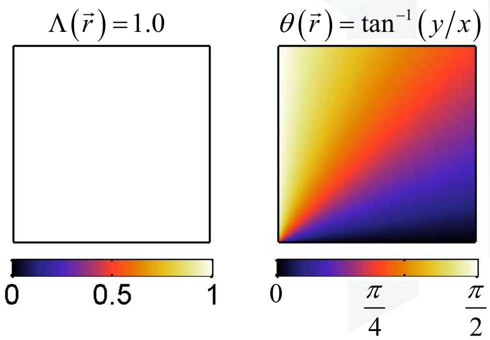
Build Input Functions

Two things are needed to build the K-function:

1. Orientation of the grating as a function of position, $\theta(\vec{r})$.
2. Period of the grating as a function of position, $\Lambda(\vec{r})$.




```
% GRATING PARAMETERS
a   = 1;
er1 = 2.5;
er2 = 1.0;
gth = 0;

% GRID PARAMETERS
Sx = 5*a;
Sy = Sx;
Nx = 100;
Ny = round(Nx*Sy/Sx);

% CALCULATE GRID
dx = Sx/Nx;
dy = Sy/Ny;
xa = [1:Nx]*dx;
ya = [1:Ny]*dy;
[Y,X] = meshgrid(ya,xa);

% SPATIALLY-VARIANT PARAMETERS
PER   = a*ones(Nx,Ny);
THETA = atan2(Y,X);
```

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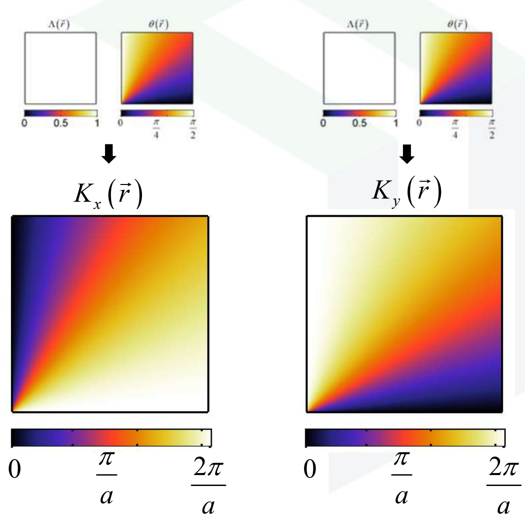
Calculate the K-Function

The K-function is calculated directly from the input functions $\Lambda(\vec{r})$ and $\theta(\vec{r})$.



$$K_x(\vec{r}) = \frac{2\pi}{\Lambda(\vec{r})}\cos\left[\theta(\vec{r})\right]$$

$$K_y(\vec{r}) = \frac{2\pi}{\Lambda(\vec{r})}\sin\left[\theta(\vec{r})\right]$$

```
% CALCULATE K-FUNCTION
Kx = 2*pi./PER.*cos(THETA);
Ky = 2*pi./PER.*sin(THETA);
```

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Finite-Difference Solution to $\nabla\Phi(\vec{r}) = \vec{K}(\vec{r})$

Begin by expanding the governing equation.

$$\nabla\Phi(\vec{r}) = \vec{K}(\vec{r}) \quad \rightarrow \quad \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{bmatrix} \Phi(\vec{r}) = \begin{bmatrix} K_x(\vec{r}) \\ K_y(\vec{r}) \\ K_z(\vec{r}) \end{bmatrix}$$

Approximate this equation using finite-differences.

$$\begin{bmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{bmatrix} \Phi(\vec{r}) = \begin{bmatrix} K_x(\vec{r}) \\ K_y(\vec{r}) \\ K_z(\vec{r}) \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} \mathbf{D}_x \\ \mathbf{D}_y \\ \mathbf{D}_z \end{bmatrix} \mathbf{\Phi} = \begin{bmatrix} \mathbf{k}_x \\ \mathbf{k}_y \\ \mathbf{k}_z \end{bmatrix}$$

Neumann boundary conditions can be used in the derivative operators, but there may exist better boundary conditions to use.

$\mathbf{\Phi} \equiv$ column vector containing grating phase throughout grid

$\mathbf{k}_i \equiv$ column vector containing K_i throughout grid

$\mathbf{D}_i \equiv$ banded matrix that calculates a partial derivative along ith axis

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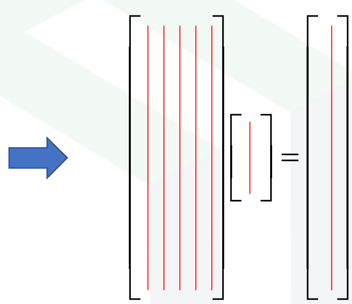
More Equations Than Unknowns

The resulting linear algebra problem has more equations than it has unknowns.

$$\begin{bmatrix} \mathbf{D}_x \\ \mathbf{D}_y \\ \mathbf{D}_z \end{bmatrix} \mathbf{\Phi} = \begin{bmatrix} \mathbf{k}_x \\ \mathbf{k}_y \\ \mathbf{k}_z \end{bmatrix}$$

$$\mathbf{D}_x\mathbf{\Phi} = \mathbf{k}_x$$
$$\mathbf{D}_y\mathbf{\Phi} = \mathbf{k}_y$$
$$\mathbf{D}_z\mathbf{\Phi} = \mathbf{k}_z$$



It is usually not possible for the solution to simultaneously satisfy all of the equations.

For this reason, the solution must be a “best fit.” One way to do this is to solve it in the sense of least squares.

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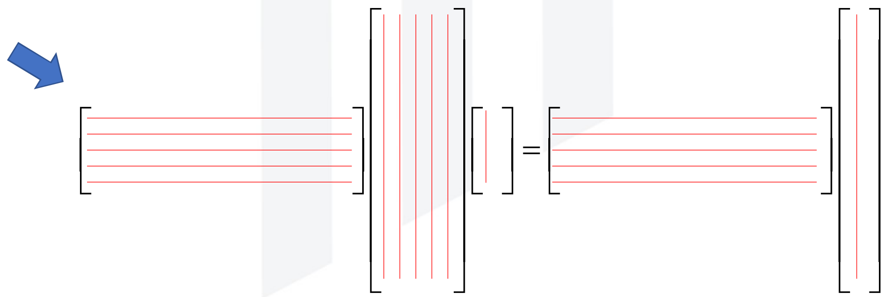
Solution in the Sense of Least Squares (1 of 2)

First, the problem is cast into the standard form $\mathbf{Ax} = \mathbf{b}$.

$$\mathbf{Ax} = \mathbf{b} \qquad \mathbf{A} = \begin{bmatrix} \mathbf{D}_x \\ \mathbf{D}_y \\ \mathbf{D}_z \end{bmatrix}, \quad \mathbf{x} = \mathbf{\Phi}, \quad \mathbf{b} = \begin{bmatrix} \mathbf{k}_x \\ \mathbf{k}_x \\ \mathbf{k}_x \end{bmatrix}$$

Next, premultiply both sides by $\mathbf{A}^T$.

$$\mathbf{A}^T \mathbf{Ax} = \mathbf{A}^T \mathbf{b}$$



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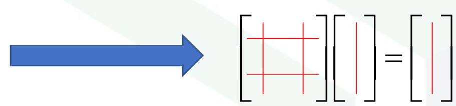
Solution in the Sense of Least Squares (2 of 2)

This is now a new system of equations with the same number of equations as there are unknowns.

$$\mathbf{A}'\mathbf{x} = \mathbf{b}'$$



$$\mathbf{A}' = \mathbf{A}^T \mathbf{A}$$
$$\mathbf{b}' = \mathbf{A}^T \mathbf{b}$$

This is solved for $\mathbf{x}$ using standard linear algebra methods.

$$\mathbf{x} = \left(\mathbf{A}'\right)^{-1} \mathbf{b}'$$

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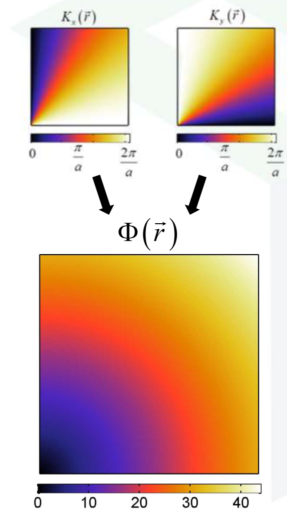
Calculate Grating Phase in 2D

We calculate the grating phase $\Phi(\vec{r})$ from the K-function as a best fit.



$$\nabla\Phi(\vec{r}) = \vec{K}$$

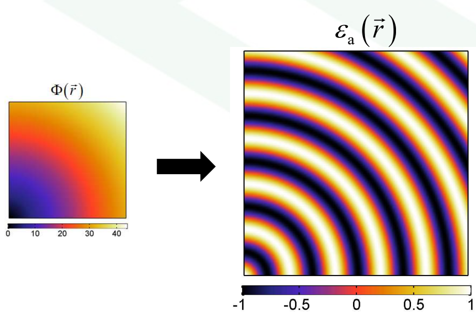
$$\begin{bmatrix} \mathbf{D}_x \\ \mathbf{D}_y \end{bmatrix} \mathbf{\Phi} = \begin{bmatrix} \mathbf{k}_x \\ \mathbf{k}_y \end{bmatrix}$$

```
% CONSTRUCT DERIVATIVE MATRICES
NS  = [Nx Ny];
RES = [dx dy];
BC  = [1 1];
[DX,~,DY,~] = fdder(NS,RES,BC);

% COMPUTE GRATING PHASE
A   = [ DX ; DY ];
b   = [ Kx(:) ; Ky(:) ];
PHI = (A.'*A)\(A.'*b);
PHI = reshape(PHI,Nx,Ny);
```


Calculate Analog Grating

The analog grating $\varepsilon_a(\vec{r})$ is calculated directly from the grating phase $\Phi(\vec{r})$.

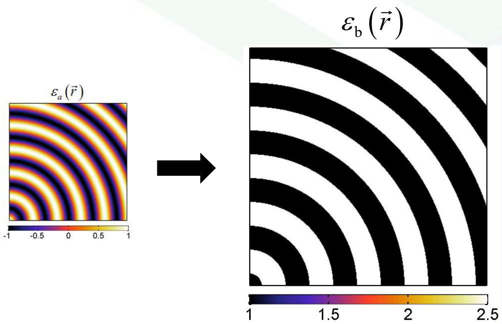


$$\varepsilon_a(\vec{r}) = \cos\left[\Phi(\vec{r})\right]$$

```
% COMPUTE ANALOG GRATING
ERA = cos(PHI);
```

Calculate Binary Grating

The binary grating $\varepsilon_{\text{b}}(\vec{r})$ is calculated directly from the analog grating $\varepsilon_{\text{a}}(\vec{r})$ using the threshold technique and a threshold function $\gamma(\vec{r})$.



$$\varepsilon_{\text{b}}(\vec{r}) = \begin{cases} \varepsilon_1 & \varepsilon_{\text{a}}(\vec{r}) \leq \gamma(\vec{r}) \\ \varepsilon_2 & \varepsilon_{\text{a}}(\vec{r}) > \gamma(\vec{r}) \end{cases}$$

```
% COMPUTE BINARY GRATING
ERB = er1*(ERA <= gth) + er2*(ERA > gth);
```

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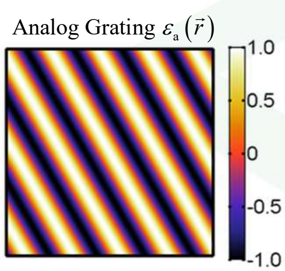
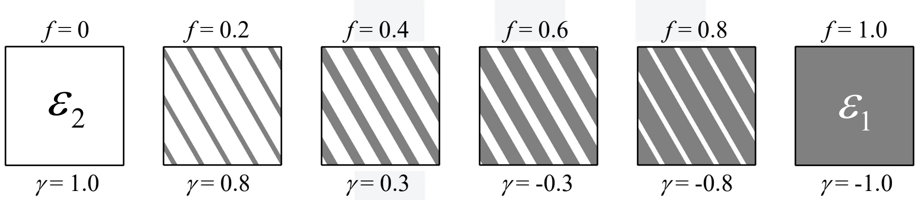
Controlling Fill Fraction Through the Threshold Function



$$\varepsilon_{\text{b}}(\vec{r}) = \begin{cases} \varepsilon_1 & \varepsilon_{\text{a}}(\vec{r}) \leq \gamma(\vec{r}) \\ \varepsilon_2 & \varepsilon_{\text{a}}(\vec{r}) > \gamma(\vec{r}) \end{cases}$$

Calculate the threshold value in order to realize a given fill factor f of ε_1.

$$\gamma(\vec{r}) = \cos\left[\pi f(\vec{r})\right]$$



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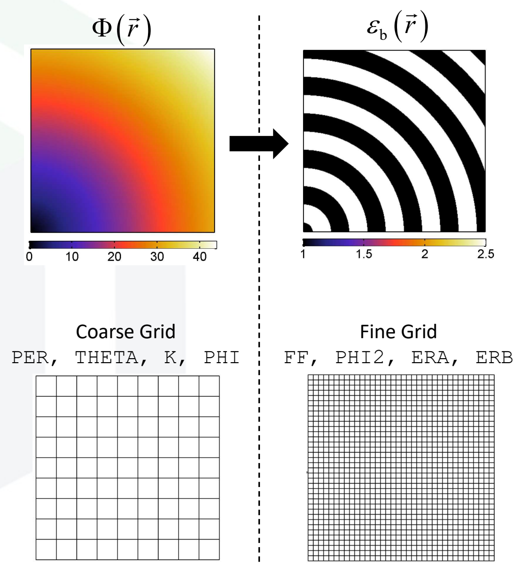
More Efficient Grid Strategy

Low-Res / High-Res Grids

Observe how smooth the grating phase function $\Phi(\vec{r})$ is compared to the final binary grating $\varepsilon_{\mathrm{b}}(\vec{r})$.

It is usually possible get away with a much coarser grid up to the point where the grating phase is calculated.



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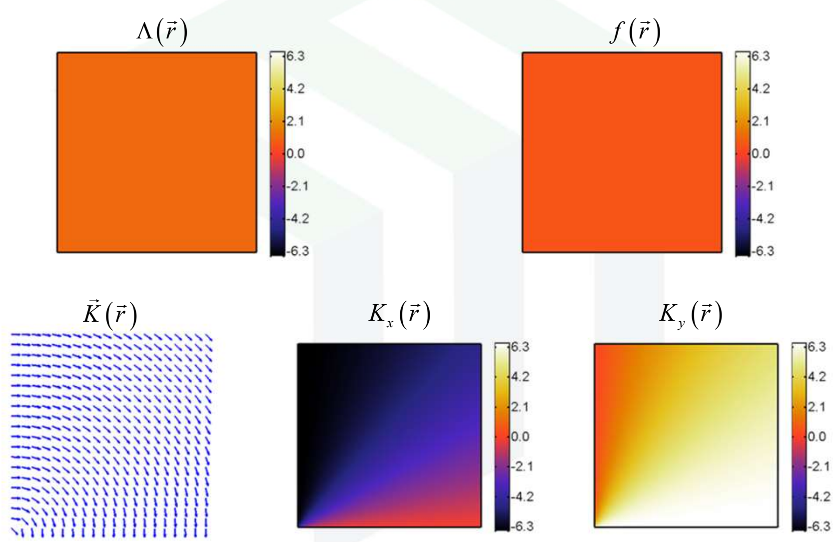
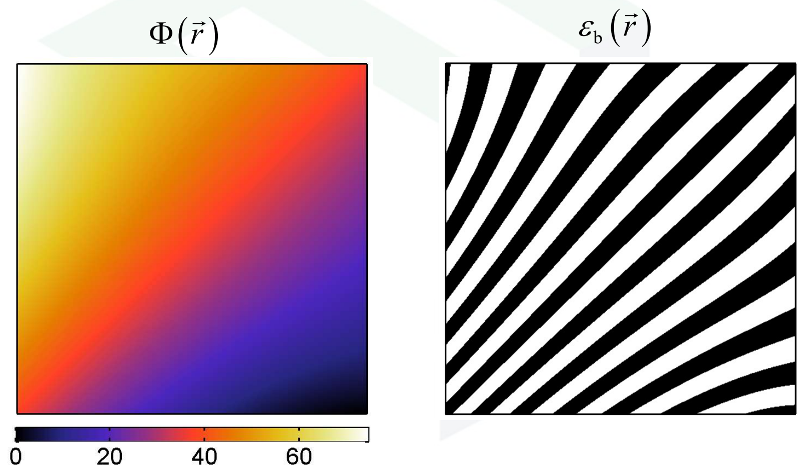
Deformation Control

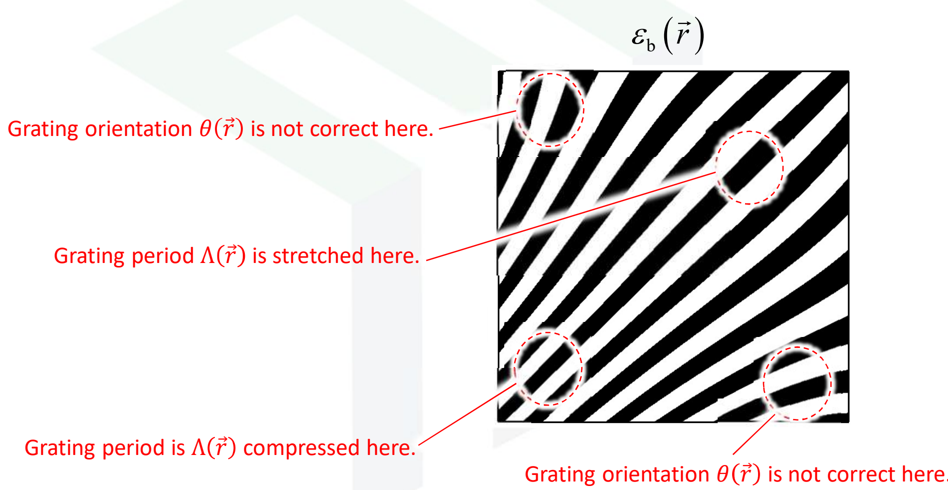
Typical Input Functions



Typical Results

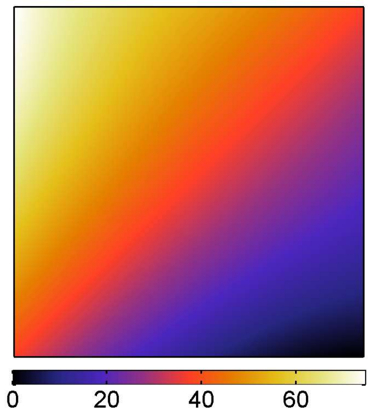


Typical Problems



What Grating Was Actually Generated?

$\Phi(\vec{r})$



The process was started with the target K-function, $\vec{K}_{\mathrm{T}}(\vec{r})$.

$$\vec{K}_T(\vec{r}) = \frac{2\pi}{\Lambda(\vec{r})}\left\{\hat{a}_x \cos\left[\theta(\vec{r})\right] + \hat{a}_y \sin\left[\theta(\vec{r})\right]\right\}$$

From this, the grating phase $\Phi(\vec{r})$ was calculated as a best fit.

$$\nabla\Phi(\vec{r}) \cong \vec{K}_T(\vec{r})$$

Given the grating phase $\Phi(\vec{r})$, the actual K-function $\vec{K}_{\mathrm{a}}(\vec{r})$ that was calculated is.

$$\vec{K}_{\mathrm{a}}(\vec{r}) = \nabla\Phi(\vec{r})$$

```
% CALCULATE ACTUAL K-FUNCTION
KA  = [DX;DY]*PHI(:);
M   = Nx*Ny;
Kxa = KA(1:M);
Kya = KA(M+1:2*M);
Kxa = reshape(Kxa,Nx,Ny);
Kya = reshape(Kya,Nx,Ny);
```

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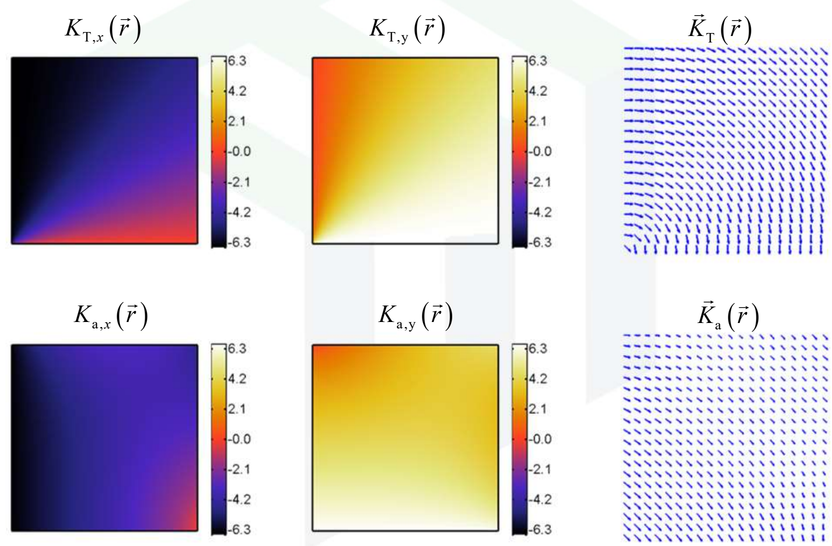
Comparison of K-Functions



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Quantifying the Problems (1 of 2)

The two parameters quantified by the K-function $\vec{K}(\vec{r})$ are the grating period $\Lambda(\vec{r})$ and the grating orientation $\theta(\vec{r})$.

The actual grating period $\Lambda_a(\vec{r})$ and actual grating orientation $\theta_a(\vec{r})$ from the actual K-function $\vec{K}_a(\vec{r})$.

$$\Lambda_a(\vec{r}) = \frac{2\pi}{\left|\vec{K}_a(\vec{r})\right|}$$

$$\theta_a(\vec{r}) = \tan^{-1}\left[\frac{K_y(\vec{r})}{K_x(\vec{r})}\right]$$

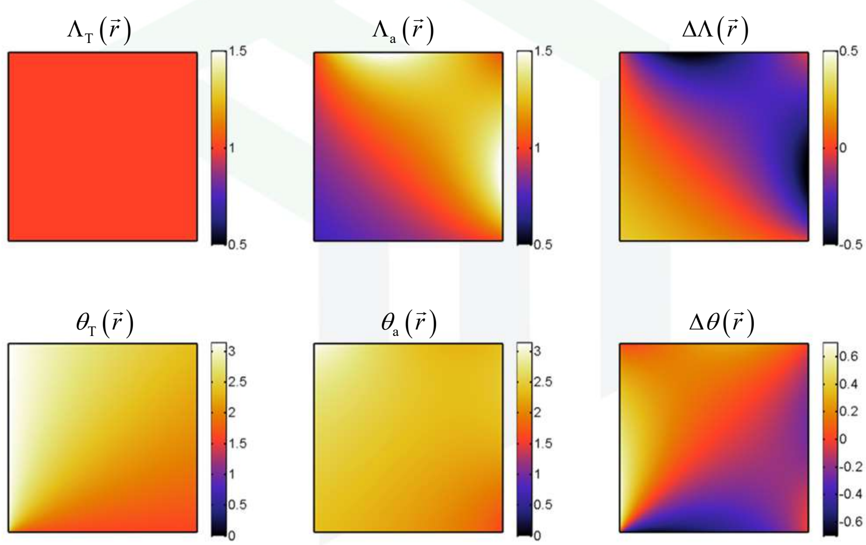
$$\Delta\Lambda(\vec{r}) = \Lambda(\vec{r}) - \Lambda_a(\vec{r})$$

$$\Delta\theta(\vec{r}) = \theta(\vec{r}) - \theta_a(\vec{r})$$

```
% CALCULATE PROBLEMS
KMa    = sqrt(abs(Kxa).^2 + abs(Kya).^2);
PERA   = 2*pi./KMa;
THETAA = atan2(Kya,Kxa);

DPER   = PER - PERA;
DTHETA = THETA - THETAA;
```


Quantifying the Problems (2 of 2)



Four Approaches to Improve the Lattice

1. Improve the grating period $\Lambda(\vec{r})$ at the cost of the grating orientation $\theta(\vec{r})$.
2. Improve the grating orientation $\theta(\vec{r})$ at the cost of the grating period $\Lambda(\vec{r})$.
3. Improve the grating in some parts at the cost of the other parts.
4. Hybrids and weighted combinations of the above approaches.

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Improve Grating Period at the Cost of Grating Orientation (1 of 2)

Step 1 – Compute the target K-function

$$\vec{K}_{\mathrm{T}}(\vec{r}) = \frac{2\pi}{\Lambda_{\mathrm{T}}(\vec{r})}\left\{\hat{a}_x \cos\left[\theta_{\mathrm{T}}(\vec{r})\right] + \hat{a}_y \sin\left[\theta_{\mathrm{T}}(\vec{r})\right]\right\}$$

Step 2 – Solve for the grating phase $\Phi(\vec{r})$

$$\nabla\Phi(\vec{r}) = \vec{K}_{\mathrm{T}}(\vec{r})$$

Step 3 – Compute the actual functions

$$\vec{K}_{\mathrm{a}}(\vec{r}) = \nabla\Phi(\vec{r}) \qquad \Lambda_{\mathrm{a}}(\vec{r}) = \frac{2\pi}{\left|\vec{K}_{\mathrm{a}}(\vec{r})\right|} \qquad \theta_{\mathrm{a}}(\vec{r}) = \tan^{-1}\left[\frac{K_y(\vec{r})}{K_x(\vec{r})}\right]$$

Step 4 – Enforce the grating period $\Lambda(\vec{r})$ in a new K-function

$$\vec{K}_{\mathrm{new}}(\vec{r}) = \frac{2\pi}{\Lambda_{\mathrm{T}}(\vec{r})}\left\{\hat{a}_x \cos\left[\theta_{\mathrm{a}}(\vec{r})\right] + \hat{a}_y \sin\left[\theta_{\mathrm{a}}(\vec{r})\right]\right\}$$

Step 5 – Compute the grating phase $\Phi(\vec{r})$ from the new K-function

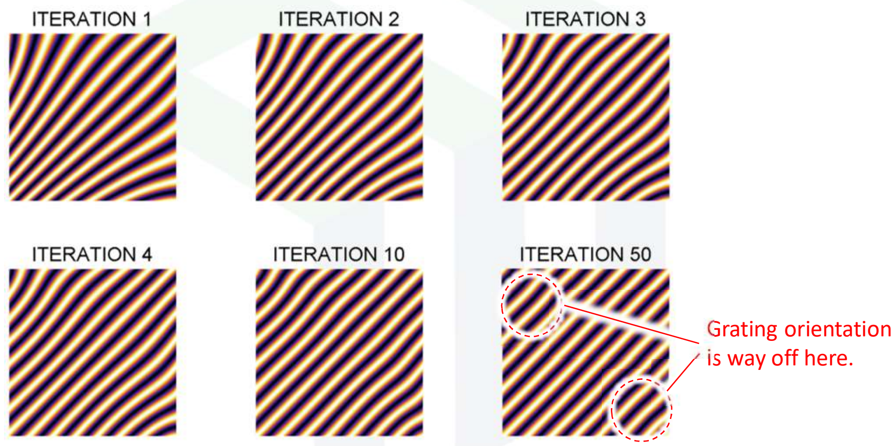
$$\nabla\Phi(\vec{r}) = \vec{K}_{\mathrm{new}}(\vec{r})$$

Step 6 – If grating has not converged, go back to Step 3.

```
for iter = 1 : NITER
    % CALCULATE K-FUNCTION
    Kx = 2*pi./PER.*cos(THETA);
    Ky = 2*pi./PER.*sin(THETA);

    % SOLVE FOR GRATING PHASE
    A   = [ DX ; DY ];
    b   = [ Kx(:) ; Ky(:) ];
    b   = A.'*b;
    A   = A.'*A;
    PHI = A\b;

    % COMPUTE ACTUAL K-FUNCTION
    KA  = [DX;DY]*PHI(:);
    Kxa = KA(1:M);
    Kya = KA(M+1:2*M);

    % COMPUTE PERIOD AND ANGLE
    THETA = atan2(Kya,Kxa);
    THETA = reshape(THETA,Nx,Ny);
end
```

EMPossible

Improve Grating Period at the Cost of Grating Orientation (2 of 2)



Here, convergence is observed after about 10 iterations.
Notice that orientation has been sacrificed.

Improve Grating Orientation at the Cost of Grating Period (1 of 2)

Step 1 – Compute the target K-function

$$\vec{K}_{\mathrm{T}}(\vec{r}) = \frac{2\pi}{\Lambda_{\mathrm{T}}(\vec{r})}\left\{\hat{a}_x \cos\left[\theta_{\mathrm{T}}(\vec{r})\right] + \hat{a}_y \sin\left[\theta_{\mathrm{T}}(\vec{r})\right]\right\}$$

Step 2 – Solve for the grating phase $\Phi(\vec{r})$

$$\nabla\Phi(\vec{r}) = \vec{K}_{\mathrm{T}}(\vec{r})$$

Step 3 – Compute the actual functions

$$\vec{K}_{\mathrm{a}}(\vec{r}) = \nabla\Phi(\vec{r}) \qquad \Lambda_{\mathrm{a}}(\vec{r}) = \frac{2\pi}{\left|\vec{K}_{\mathrm{a}}(\vec{r})\right|} \qquad \theta_{\mathrm{a}}(\vec{r}) = \tan^{-1}\left[\frac{K_y(\vec{r})}{K_x(\vec{r})}\right]$$

Step 4 – Enforce the grating orientation $\theta(\vec{r})$ in a new K-function

$$\vec{K}_{\mathrm{new}}(\vec{r}) = \frac{2\pi}{\Lambda_{\mathrm{a}}(\vec{r})}\left\{\hat{a}_x \cos\left[\theta_{\mathrm{T}}(\vec{r})\right] + \hat{a}_y \sin\left[\theta_{\mathrm{T}}(\vec{r})\right]\right\}$$

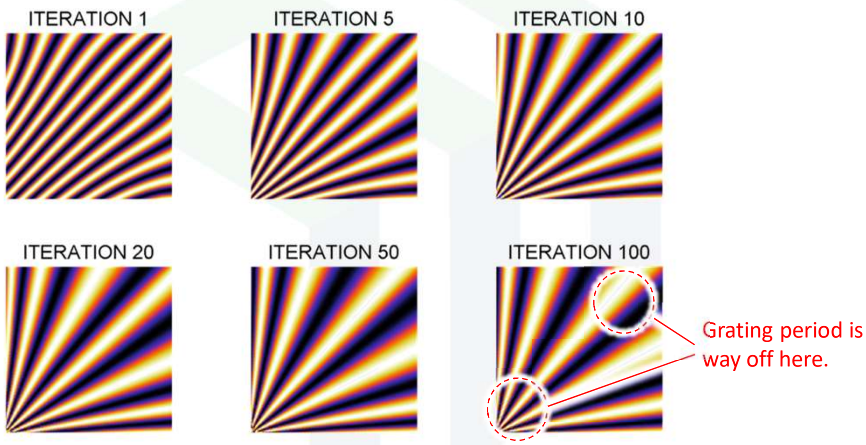
Step 5 – Compute the grating phase $\Phi(\vec{r})$ from the new K-function

$$\nabla\Phi(\vec{r}) = \vec{K}_{\mathrm{new}}(\vec{r})$$

Step 6 – If grating has not converged, go back to Step 3.

```
for iter = 1 : NITER

    % CALCULATE K-FUNCTION
    Kx = 2*pi./PER.*cos(THETA);
    Ky = 2*pi./PER.*sin(THETA);

    % COMPUTE GRATING PHASE
    A   = [ DX ; DY ];
    b   = [ Kx(:) ; Ky(:) ];
    b   = A.'*b;
    A   = A.'*A;
    PHI = A\b;

    % COMPUTE ACTUAL K-FUNCTION
    KA  = [DX;DY]*PHI(:);
    Kxa = KA(1:M);
    Kya = KA(M+1:2*M);

    % COMPUTE PERIOD AND ANGLE
    KMa = sqrt(abs(Kxa).^2 ...
        + abs(Kya).^2);
    PER = 2*pi./KMa;
    PER = reshape(PER,Nx,Ny);
end
```

Improve Grating Orientation at the Cost of Grating Period (2 of 2)



Here, convergence is observed after about 50 iterations.
Notice that period has been sacrificed.

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Improve Grating In Some Regions at the Cost of Other Regions (1 of 2)

Step 1 – Compute the target K-function

$$\vec{K}_{\mathrm{T}}(\vec{r}) = \frac{2\pi}{\Lambda_{\mathrm{T}}(\vec{r})}\left\{\hat{a}_x \cos\left[\theta_{\mathrm{T}}(\vec{r})\right] + \hat{a}_y \sin\left[\theta_{\mathrm{T}}(\vec{r})\right]\right\}$$

Step 2 – Solve for the grating phase $\Phi(\vec{r})$

$$\nabla\Phi(\vec{r}) = \vec{K}_{\mathrm{T}}(\vec{r})$$

Step 3 – Compute the actual functions

$$\vec{K}_{\mathrm{a}}(\vec{r}) = \nabla\Phi(\vec{r}) \qquad \Lambda_{\mathrm{a}}(\vec{r}) = \frac{2\pi}{\left|\vec{K}_{\mathrm{a}}(\vec{r})\right|} \qquad \theta_{\mathrm{a}}(\vec{r}) = \tan^{-1}\left[\frac{K_y(\vec{r})}{K_x(\vec{r})}\right]$$

Step 4 – Enforce the grating orientation $\theta(\vec{r})$ in a new K-function

$$\vec{K}_{\mathrm{new}}(\vec{r}) = \begin{cases} \vec{K}_{\mathrm{T}}(\vec{r}) & \text{inside of regions to optimize} \\ \vec{K}_{\mathrm{a}}(\vec{r}) & \text{outside of regions to optimize} \end{cases}$$

Step 5 – Compute the grating phase $\Phi(\vec{r})$ from the new K-function

$$\nabla\Phi(\vec{r}) = \vec{K}_{\mathrm{new}}(\vec{r})$$

Step 6 – If grating has not converged, go back to Step 3.

```
% CALCULATE TARGET K-FUNCTION
KTx = 2*pi./PER.*cos(THETA);
KTy = 2*pi./PER.*sin(THETA);
Kx  = KTx;
Ky  = KTy;

for iter = 1 : NITER
    % COMPUTE GRATING PHASE
    A   = [ DX ; DY ];
    b   = [ Kx(:) ; Ky(:) ];
    b   = A.'*b;
    A   = A.'*A;
    PHI = A\b;

    % COMPUTE ACTUAL K-FUNCTION
    KA  = [DX;DY]*PHI(:);
    KAx = KA(1:M);
    KAy = KA(M+1:2*M);

    % ENFORCE CRITICAL REGION R
    KAx = reshape(KAx,Nx,Ny);
    KAy = reshape(KAy,Nx,Ny);
    Kx  = KTx.*R + KAx.*(1 - R);
    Ky  = KTy.*R + KAy.*(1 - R);
end
```

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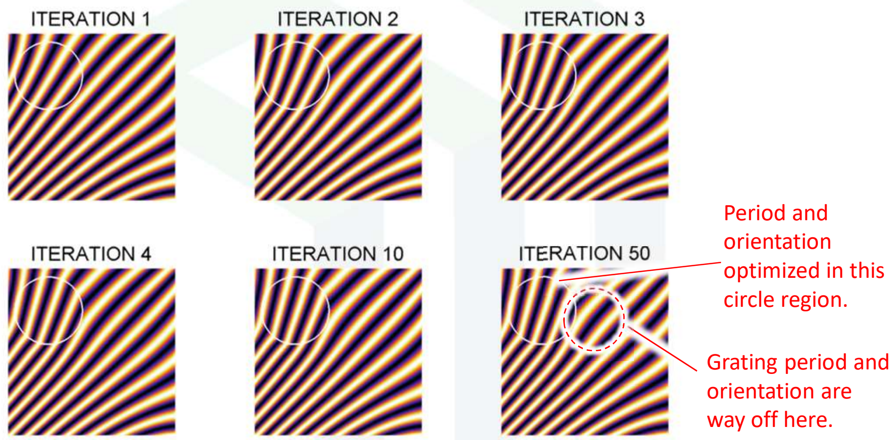
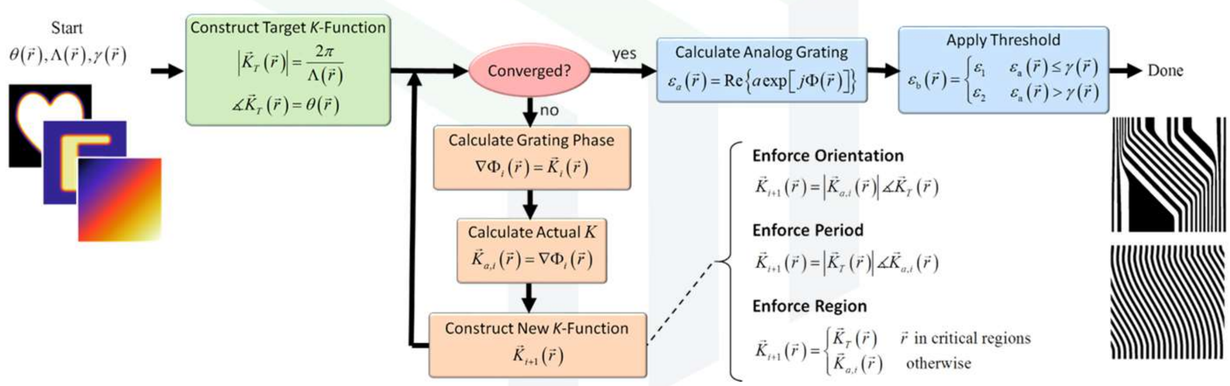
Improve Grating In Some Regions at the Cost of Other Regions (2 of 2)



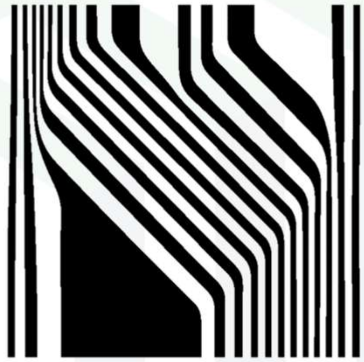
Here, convergence is observed after about 3 iterations.
Notice the lattice is most distorted just outside the critical region.

Block Diagram for Controlling Deformations

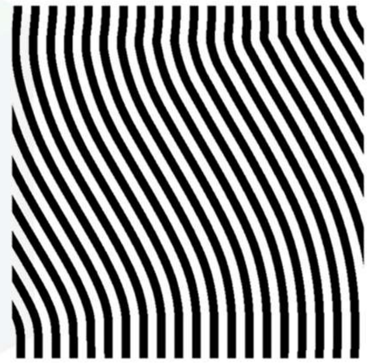


Another Example

Grating generated from standard algorithm.



Grating generated after enforcing the orientation at the expense of period.



Grating generated after enforcing the period at the expense of orientation.

Spatially-Variant Planar Gratings on Curved Surfaces

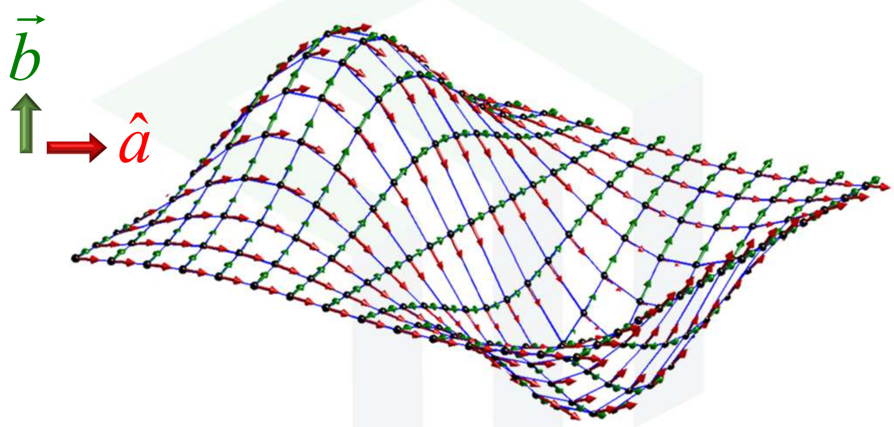
Raymond C. Rumpf, Javier J. Pazos, Jennefir L. Digaum, Stephen M. Kuebler, “Spatially-Variant Periodic Structures in Electromagnetics,” accepted for publication in Phil. Trans. A, December 2014.

The Problem

Suppose it is desired to put a planar grating onto a curved surface without deforming the orientation or period of the grating.

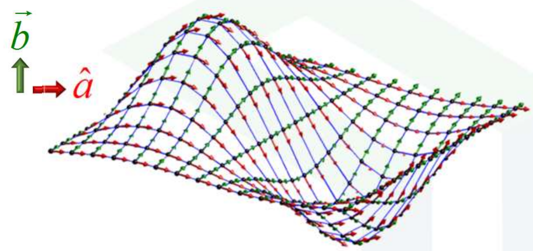
In this case, a grating must be spatially varied in order to not spatially vary it!

Curved Surface Mesh



On a curved surface, the sense of x and y directions must be modified to conform to the curvature. Call these directions $\hat{a}$ and $\hat{b}$. They are defined to always be tangential to the surface.

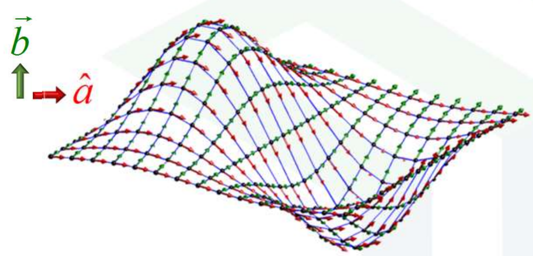
Modified Grating Phase Equation



The grating phase equation must be modified to operate on the curved mesh.

$$\nabla' \Phi\left(\vec{r}\right) = \vec{K}\left(\vec{r}\right) \qquad \nabla' = \frac{\partial}{\partial a}\hat{a} + \frac{\partial}{\partial b}\hat{b}$$

Modified Finite-Difference Approximation

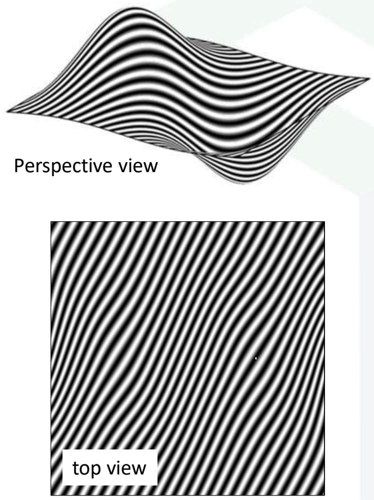


$$\nabla' \Phi\left(\vec{r}\right) = \frac{\partial \Phi\left(\vec{r}\right)}{\partial a}\hat{a} + \frac{\partial \Phi\left(\vec{r}\right)}{\partial b}\hat{b}$$

$$\nabla' \Phi\left(\vec{r}_{p,q}\right) = \frac{\Phi\left(\vec{r}_{p+1,q}\right) - \Phi\left(\vec{r}_{p-1,q}\right)}{\left|\vec{r}_{p+1,q} - \vec{r}_{p-1,q}\right|}\hat{a} + \frac{\Phi\left(\vec{r}_{p,q+1}\right) - \Phi\left(\vec{r}_{p,q-1}\right)}{\left|\vec{r}_{p,q+1} - \vec{r}_{p,q-1}\right|}\hat{b}$$

Example Planar Gratings

Grating #1



Grating #2