Advanced Electromagnetics:
21st Century Electromagnetics

# Transmission Lines Embedded in Anisotropic Media

## Lecture Outline

- RF transmission lines
- Transmission lines embedded in anisotropic media
- Finite-difference analysis of transmission lines embedded in arbitrary anisotropic media
  - Formulation
  - Implementation
  - Example
- Generalization of the method

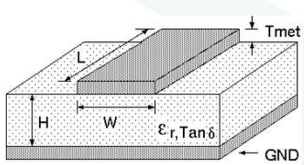
# RF Transmission Lines



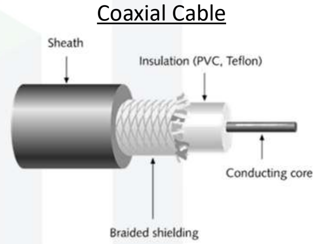
## What are Transmission Lines?

Transmission lines are metallic structures that guide electromagnetic waves from DC to very high frequencies.

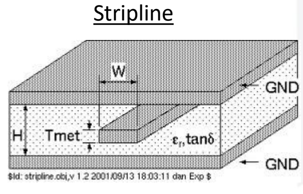
Microstrip




Coaxial Cable




Stripline




Slotline

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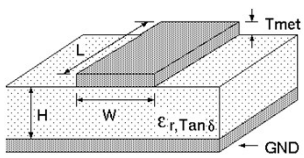
# Characteristic Impedance $Z_0$

The ratio of the voltage to current as well as the electric to magnetic field is a constant along the length of the transmission line. This ratio is the *characteristic impedance*.

$$Z_0 = \sqrt{\frac{L}{C}}$$

The value of the characteristic impedance alone has little meaning. Reflections are caused whenever the impedance changes.




For small H/W,

$$Z_0 \approx \frac{377H}{\sqrt{\varepsilon_r}W}\left(1-\frac{2H}{\pi\varepsilon_r W}\left[(1+\varepsilon_r)\ln\left(\frac{H}{W}\right)+2.230+4.454\varepsilon_r+(4.464+3.89\varepsilon_r)\frac{H}{W}\right]\right)^{-1/2}$$

For large H/W,

$$Z_0 \approx \frac{377H}{\pi\sqrt{2(1+\varepsilon_r)}}\sqrt{\left[\ln\left(8\frac{H}{W}\right)+\frac{1}{32}\left(\frac{W}{H}\right)^2\right] \cdot\left[\ln\left(8\frac{H}{W}\right)+\left(\frac{W}{H}\right)^2\frac{1}{16(1+\varepsilon_r)}+\frac{\varepsilon_r-1}{\varepsilon_r}\left(0.041\left(\frac{W}{H}\right)^2-0.454\right)\right]}$$

S. Y. Poh, W. C. Chew, J. A. Kong, "Approximate Formulas for Line Capacitance and Characteristic Impedance of Microstrip Line", *IEEE Trans. Microwave Theory Tech.*, vol. MTT-29, no. 2, pp. 135-142, May 1981.



# Approximate Equations

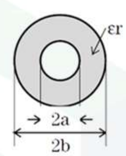
Coaxial $$Z_0 = \frac{60}{\sqrt{\varepsilon_r}}\ln\left(\frac{b}{a}\right)$$

Twin-Lead $$Z_0 = \frac{120}{\sqrt{\varepsilon_r}}\cosh^{-1}\left(\frac{d}{2\sqrt{ab}}\right)$$

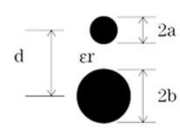
Parallel Plate $$Z_0 = \frac{120\pi d}{w\sqrt{\varepsilon_r}}$$

Microstrip $$Z_0 = \begin{cases} \dfrac{60}{D}\ln\left(\dfrac{2}{4d}+\dfrac{8d}{w}\right) & \dfrac{w}{d}\le 1 \\ \dfrac{120}{D}\dfrac{\pi}{(2/3)\ln(1.444+w/d)+1.393+w/d} & \dfrac{w}{d}\ge 1 \end{cases}$$

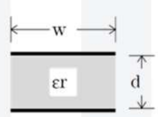
$$D = \sqrt{\frac{\varepsilon_r+1}{2}+\frac{\varepsilon_r-1}{2\sqrt{1+12d/2}}}$$



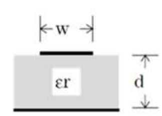
Coaxial



Twin-lead



Parallel Plate



Microstrip

# Analysis: Electrostatic Approximation

Transmission lines are waveguides. To be rigorous, they should be modeled as such. This can be rather computationally intensive. An alternative is to analyze transmission lines using Laplace's equation.

The dimensions of transmission lines are typically much smaller than the operating wavelength so the wave nature of electromagnetics is less important to consider. Therefore, Maxwell's equations are essentially be solved in the limit as $\omega \rightarrow 0$. Setting $\omega = 0$ is called the ***electrostatic approximation*** and the solution approximates the TEM solution of a waveguide.

$$\nabla \bullet \vec{B} = 0 \qquad\qquad \nabla \bullet \vec{B} = 0$$
$$\nabla \bullet \vec{D} = 0 \qquad\qquad \nabla \bullet \vec{D} = 0$$
$$\nabla \times \vec{H} = \vec{J} + \partial \vec{D} / \partial t \quad \Rightarrow \quad \nabla \times \vec{H} = \vec{J}$$
$$\nabla \times \vec{E} = -\partial \vec{B} / \partial t \qquad\qquad \nabla \times \vec{E} = 0$$

All of the field components are separable.
The transmission line is TEM.

Rigorously, the transmission line is quasi-TEM.

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# Analysis: Inhomogeneous Isotropic Laplace's Equation

The divergence condition for the electric flux is

$$\nabla \bullet \vec{D} = 0 \qquad \text{Eq. (1)}$$

But, it is known that $\vec{D} = \varepsilon_0 \varepsilon_{\mathrm{r}} \vec{E}$, so this equation becomes

$$\nabla \bullet \left( \varepsilon_{\mathrm{r}} \vec{E} \right) = 0 \qquad \text{Eq. (2)}$$

The electric field intensity $\vec{E}$ is related to the scalar potential $V$ as follows.

$$\vec{E} = -\nabla V \qquad \text{Eq. (3)}$$

The inhomogeneous Laplace's equation is derived by substituting Eq. (3) into Eq. (2).

$$\boxed{\nabla^2 V + \left( \frac{\nabla \varepsilon_{\mathrm{r}}}{\varepsilon_{\mathrm{r}}} \right) \bullet \left( \nabla V \right) = 0}$$

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# Analysis: Calculating Distributed Capacitance

In the electrostatic approximation, the transmission line is a capacitor. The total energy $U$ stored in a capacitor is

$$U = \tfrac{1}{2}\iint_A \left(\vec{D}\bullet\vec{E}\right) dA$$

The integral is performed over the entire cross section of device. For open devices like microstrips, this is infinite area. In practice, we integrate over a large enough area to incorporate as much of of the electric field as possible.

This equation is valid in anisotropic media.

From circuit theory, the capacitance $C$ is related to the total stored energy $U$ through

$$U = \frac{CV_0^2}{2}$$

$V_0$ is the voltage across the capacitor.

If the above equations are set equal and substitute $\vec{E} = -\nabla V$ into the expression, the equation for the distributed capacitance $C$ is derived.

$$\boxed{C = \frac{1}{V_0^2}\iint_A \left(\vec{D}\bullet\vec{E}\right) dA}$$



# Analysis: Calculating Distributed Inductance

The voltage along the transmission line travels at the same velocity as the electric field so…

$$v_V = v_E \quad\rightarrow\quad \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{\mu\varepsilon}} \quad\rightarrow\quad LC = \frac{\mu_r\varepsilon_r}{c_0^2}$$

Assuming the surrounding medium is homogeneous, the distributed inductance $L$ can be calculated from the above equation as

$$L = \frac{\mu_r\varepsilon_r}{c_0^2 C}$$

This means that for the dielectric only case, the distributed inductance $L$ is calculated directly from the distributed capacitance $C$.

Dielectric materials should not alter the inductance. However if the value of $C$ calculated previously is used, it will. This is incorrect. The solution is to calculate capacitance with a homogeneous dielectric $C_h$ and then calculate inductance from this. It is simplest to use just air.

$$\boxed{L = \frac{1}{c_0^2 C_h}}$$

# Analysis: Calculating TL Parameters

The characteristic impedance $Z_0$ is calculated from the distributed inductance and capacitance through

$$Z_0 = \sqrt{\frac{L}{C}}$$

The velocity $v$ of a signal travelling along the transmission line is

$$v = \frac{1}{\sqrt{LC}}$$

The effective refractive index $n_{\text{eff}}$ is therefore

$$n_{\text{eff}} = \frac{c_0}{v} = c_0\sqrt{LC}$$

Both $Z_0$ and $n_{\text{eff}}$ are needed to analyze transmission line circuits.

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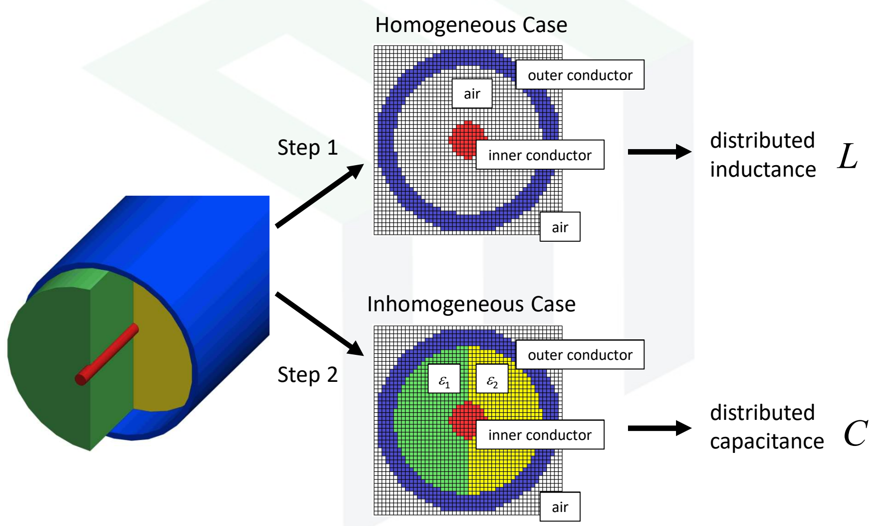
# Two Step Modeling Approach

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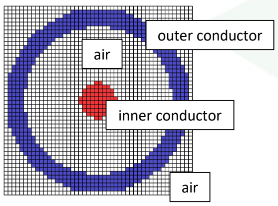
# How We Will Analyze Transmission Lines

**Step 1**
Construct homogeneous TL



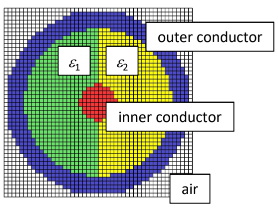

**Step 2**
Construct and solve matrix equation

$$\nabla^2 V_\mathrm{h} = 0$$
$$\Downarrow$$
$$\mathbf{L}_\mathrm{h}\mathbf{v}_\mathrm{h} = \mathbf{v}_\mathrm{src}$$
$$\Downarrow$$
$$\mathbf{v}_\mathrm{h} = \mathbf{L}_\mathrm{h}^{-1}\mathbf{v}_\mathrm{src}$$

**Step 3**
Calculate distributed inductance

$$C_\mathrm{h} = \frac{\varepsilon_0}{V_0^2}\iint_A \left(\vec{D}_\mathrm{h} \bullet \vec{E}_\mathrm{h}\right) dA$$
$$L = \frac{1}{c_0^2 C_\mathrm{h}}$$

**Step 4**
Construct inhomogeneous TL




**Step 5**
Construct and solve matrix equation

$$\nabla^2 V + \left(\frac{\nabla \varepsilon_\mathrm{r}}{\varepsilon_\mathrm{r}}\right) \bullet \left(\nabla V\right) = 0$$
$$\Downarrow$$
$$\mathbf{L}\mathbf{v} = \mathbf{v}_\mathrm{src}$$
$$\Downarrow$$
$$\mathbf{v} = \mathbf{L}^{-1}\mathbf{v}_\mathrm{src}$$

**Step 6**
Calculate distributed capacitance

$$C = \frac{\varepsilon_0}{V_0^2}\iint_A \left(\vec{D} \bullet \vec{E}\right) dA$$

**Step 7**
Calculate TL parameters

$$Z_0 = \sqrt{\frac{L}{C}} \qquad n_\mathrm{eff} = c_0\sqrt{LC}$$

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# Transmission Lines Embedded in Anisotropic Media

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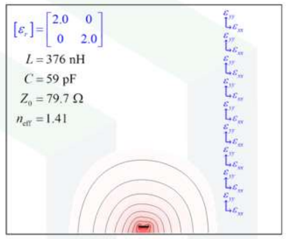
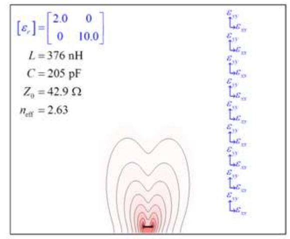
# Effect of Strength of Anisotropy

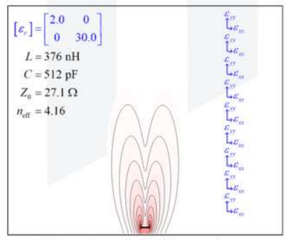
- Dielectric anisotropy has no effect on the distributed inductance.
- Capacitance increases and impedance lowers with increasing $\varepsilon_{yy}$.
- Field develops most strongly in the direction with highest permittivity.
- The degree to which field follows highest $\varepsilon$ is proportional to the strength of the anisotropy.



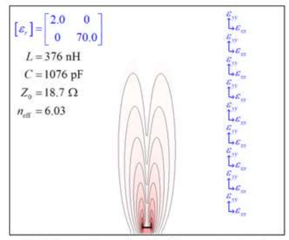
(a) Microstrip embedded in an isotropic medium.



(b) Microstrip embedded in an anisotropic medium with $\Delta\varepsilon = 8.0$.



(c) Microstrip embedded in an anisotropic medium with $\Delta\varepsilon = 28.0$.



(d) Microstrip embedded in an anisotropic medium with $\Delta\varepsilon = 68.0$.

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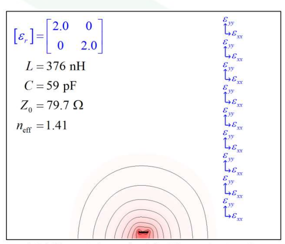
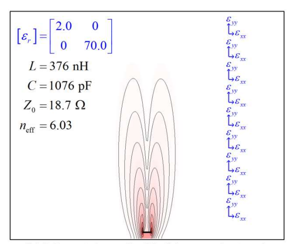
# Effect of Spatially Varying the Tensor Orientation

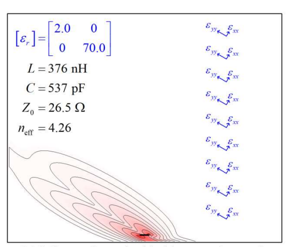
- Impedance changes in the presences of the anisotropic medium.
- Field actually "follows" the anisotropy.
- Impedance changes slightly with tilt.
- Impedance relatively constant regardless of spatial variance.



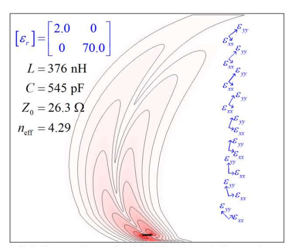
(a) Microstrip embedded in an isotropic medium.



(b) Microstrip embedded in an anisotropic medium.



(c) Microstrip embedded in an anisotropic medium tilted by 60°.



(d) Microstrip embedded in a spatially variant anisotropic medium.

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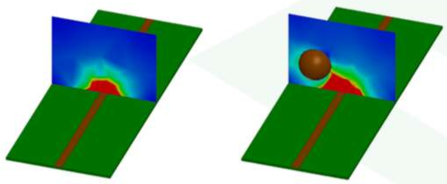
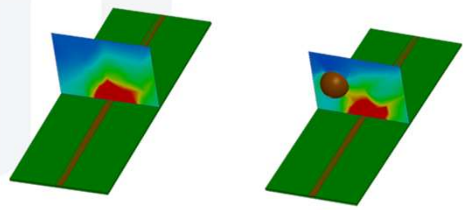
# Isolation of a Microstrip



a) Microstrip line b) Microstrip line with metal object near

Ordinary microstrip with metal ball in proximity. Near-field is perturbed affecting the signal in the line.

Same microstrip embedded in an anisotropic material with dielectric tensor tilted away from ball. Near-field is less perturbed, affecting the line much less.



a) Microstrip line embedded in an SVAM b) Embedded microstrip line with metal object near

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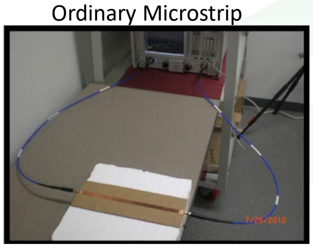
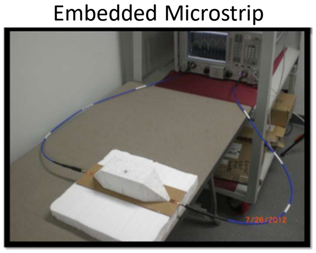
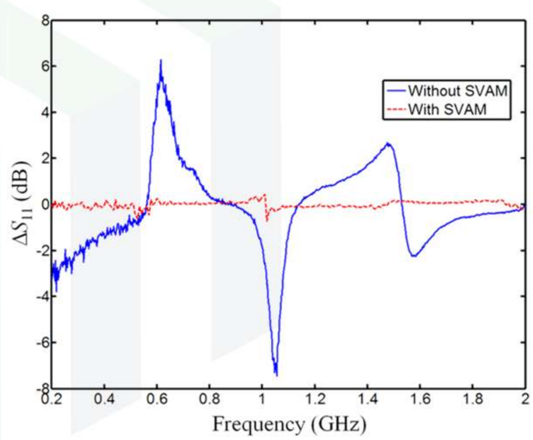
# Experimental Results

Ordinary Microstrip



Embedded Microstrip



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# Finite-Difference Analysis of Transmission Lines Embedded in Arbitrary Anisotropic Media

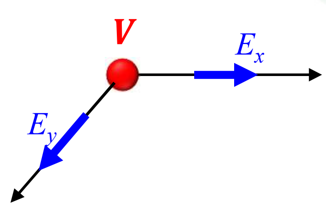
*Formulation*

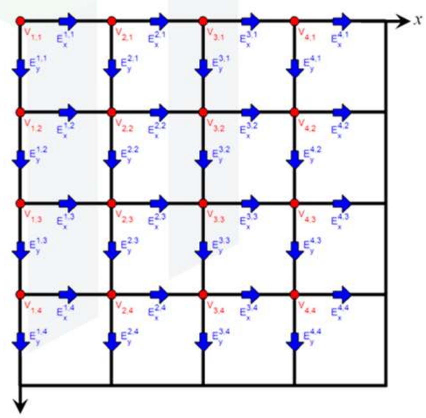


## The Grid

There are two unknowns, $V$ and $\vec{E}$, so a staggered grid is adopted.

It is possible to use the `yeeder()` function to construct the derivative operators.

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# Governing Equations

$$\nabla \bullet \vec{D} = 0 \quad \rightarrow \quad \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{bmatrix} \begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = 0$$

$$\vec{D} = \varepsilon \vec{E} \quad \rightarrow \quad \begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}$$

$$\vec{E} = -\nabla V \quad \rightarrow \quad \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} = -\begin{bmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{bmatrix} V$$

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# Reduction to Two Dimensions

$$\begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \cancel{\frac{\partial}{\partial z}} \end{bmatrix} \begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = 0 \quad \rightarrow \quad D_z = 0 \quad \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \end{bmatrix} \begin{bmatrix} D_x \\ D_y \end{bmatrix} = 0$$

$$\begin{bmatrix} D_x \\ D_y \\ \cancel{D_z} \end{bmatrix} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} \quad \rightarrow \quad E_z = 0 \quad \begin{bmatrix} D_x \\ D_y \end{bmatrix} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} \\ \varepsilon_{yx} & \varepsilon_{yy} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \end{bmatrix}$$

$$\begin{bmatrix} E_x \\ E_y \\ \cancel{E_z} \end{bmatrix} = -\begin{bmatrix} \partial/\partial x \\ \partial/\partial y \\ \cancel{\partial/\partial z} \end{bmatrix} V \quad \rightarrow \quad \begin{bmatrix} E_x \\ E_y \end{bmatrix} = -\begin{bmatrix} \partial/\partial x \\ \partial/\partial y \end{bmatrix} V$$

Transmission line properties are independent of $\varepsilon_{xz}$, $\varepsilon_{yz}$, $\varepsilon_{zx}$, $\varepsilon_{zy}$, and $\varepsilon_{zz}$.

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# Finite-Difference Approximation

$$\begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \end{bmatrix} \begin{bmatrix} D_x \\ D_y \end{bmatrix} = 0 \quad \rightarrow \quad \begin{bmatrix} \mathbf{D}_x^e & \mathbf{D}_y^e \end{bmatrix} \begin{bmatrix} \mathbf{d}_x \\ \mathbf{d}_y \end{bmatrix} = \mathbf{0}$$

$$\begin{bmatrix} D_x \\ D_y \end{bmatrix} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} \\ \varepsilon_{yx} & \varepsilon_{yy} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} \mathbf{d}_x \\ \mathbf{d}_y \end{bmatrix} = \begin{bmatrix} \boldsymbol{\varepsilon}_{xx} & \boldsymbol{\varepsilon}_{xy} \\ \boldsymbol{\varepsilon}_{yx} & \boldsymbol{\varepsilon}_{yy} \end{bmatrix} \begin{bmatrix} \mathbf{e}_x \\ \mathbf{e}_y \end{bmatrix} \quad ???$$

$$\begin{bmatrix} E_x \\ E_y \end{bmatrix} = -\begin{bmatrix} \partial/\partial x \\ \partial/\partial y \end{bmatrix} V \quad \rightarrow \quad \begin{bmatrix} \mathbf{e}_x \\ \mathbf{e}_y \end{bmatrix} = -\begin{bmatrix} \mathbf{D}_x^v \\ \mathbf{D}_y^v \end{bmatrix} \mathbf{v}$$

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# The Problem with the Tensor

$$\begin{bmatrix} D_x \\ D_y \end{bmatrix} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} \\ \varepsilon_{yx} & \varepsilon_{yy} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \end{bmatrix} \quad \rightarrow \quad \begin{aligned} D_x &= \varepsilon_{xx} E_x + \varepsilon_{xy} E_y \\ D_y &= \varepsilon_{yx} E_x + \varepsilon_{yy} E_y \end{aligned}$$

$$\rightarrow \quad \begin{aligned} D_x^{i,j} &\overset{???}{=} \varepsilon_{xx}^{i,j} E_x^{i,j} + \varepsilon_{xy}^{i,j} E_y^{i,j} \\ D_y^{i,j} &\overset{???}{=} \varepsilon_{yx}^{i,j} E_x^{i,j} + \varepsilon_{yy}^{i,j} E_y^{i,j} \end{aligned}$$

In the first finite-difference equation, $E_y$ is in a physically different position than $E_x$ and $D_x$.

Similarly, in the second finite-difference equation, $E_x$ is in a physically different position than $E_y$ and $D_y$.

The finite-difference equations above are incorrect because not all of the terms exist at the same position in space.

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# The Correction

Recall the interpolation matrices $\mathbf{R}_x^+$, $\mathbf{R}_y^+$, $\mathbf{R}_z^+$, $\mathbf{R}_x^-$, $\mathbf{R}_y^-$, and $\mathbf{R}_z^-$ from *Computational Electromagnetics*.

$$D_x^{i,j} = \varepsilon_{xx}^{i,j} E_x^{i,j} + \frac{\varepsilon_{xy}^{i,j} E_y^{i,j} + \varepsilon_{xy}^{i+1,j} E_y^{i+1,j} + \varepsilon_{xy}^{i,j-1} E_y^{i,j-1} + \varepsilon_{xy}^{i+1,j-1} E_y^{i+1,j-1}}{4}$$

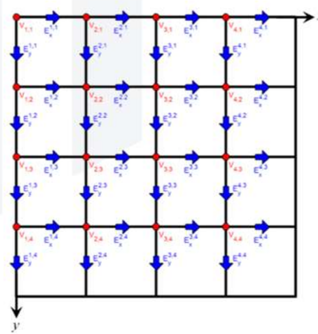
$$D_y^{i,j} = \frac{\varepsilon_{yx}^{i,j} E_x^{i,j} + \varepsilon_{yx}^{i-1,j} E_x^{i-1,j} + \varepsilon_{yx}^{i,j+1} E_x^{i,j+1} + \varepsilon_{yx}^{i-1,j+1} E_x^{i-1,j+1}}{4} + \varepsilon_{yy}^{i,j} E_y^{i,j}$$

$$\downarrow$$

$$\mathbf{d}_x = \boldsymbol{\varepsilon}_{xx}\mathbf{e}_x + \mathbf{R}_x^+\mathbf{R}_y^-\boldsymbol{\varepsilon}_{xy}\mathbf{e}_y$$

$$\mathbf{d}_y = \mathbf{R}_x^-\mathbf{R}_y^+\boldsymbol{\varepsilon}_{yx}\mathbf{e}_x + \boldsymbol{\varepsilon}_{xy}\mathbf{e}_y$$

$\mathbf{R}_a^\pm \equiv$ interpolation matrix
$a \equiv$ axis along which average is taken
$\pm \equiv$ direction of adjacent cell






# Revised Finite-Difference Approximation

The dielectric tensor is modified by incorporating the interpolation matrices.

$$\begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \end{bmatrix} \begin{bmatrix} D_x \\ D_y \end{bmatrix} = 0 \quad \rightarrow \quad \begin{bmatrix} \mathbf{D}_x^e & \mathbf{D}_y^e \end{bmatrix} \begin{bmatrix} \mathbf{d}_x \\ \mathbf{d}_y \end{bmatrix} = \mathbf{0}$$

$$\begin{bmatrix} D_x \\ D_y \end{bmatrix} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} \\ \varepsilon_{yx} & \varepsilon_{yy} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} \mathbf{d}_x \\ \mathbf{d}_y \end{bmatrix} = \begin{bmatrix} \boldsymbol{\varepsilon}_{xx} & \mathbf{R}_x^+\mathbf{R}_y^-\boldsymbol{\varepsilon}_{xy} \\ \mathbf{R}_x^-\mathbf{R}_y^+\boldsymbol{\varepsilon}_{yx} & \boldsymbol{\varepsilon}_{yy} \end{bmatrix} \begin{bmatrix} \mathbf{e}_x \\ \mathbf{e}_y \end{bmatrix}$$

$$\begin{bmatrix} E_x \\ E_y \end{bmatrix} = -\begin{bmatrix} \partial/\partial x \\ \partial/\partial y \end{bmatrix} V \quad \rightarrow \quad \begin{bmatrix} \mathbf{e}_x \\ \mathbf{e}_y \end{bmatrix} = -\begin{bmatrix} \mathbf{D}_x^v \\ \mathbf{D}_y^v \end{bmatrix} \mathbf{v}$$

## Matrix Form of the Inhomogeneous Laplace's Equation

Now, the matrix form of the inhomogeneous Laplaces' equation is derived.

Eq. (1) $\begin{bmatrix} \mathbf{D}_x^e & \mathbf{D}_y^e \end{bmatrix} \begin{bmatrix} \mathbf{d}_x \\ \mathbf{d}_y \end{bmatrix} = \mathbf{0}$

Eq. (2) $\begin{bmatrix} \mathbf{d}_x \\ \mathbf{d}_y \end{bmatrix} = \begin{bmatrix} \boldsymbol{\varepsilon}_{xx} & \mathbf{R}_x^+ \mathbf{R}_y^- \boldsymbol{\varepsilon}_{xy} \\ \mathbf{R}_x^- \mathbf{R}_y^+ \boldsymbol{\varepsilon}_{yx} & \boldsymbol{\varepsilon}_{yy} \end{bmatrix} \begin{bmatrix} \mathbf{e}_x \\ \mathbf{e}_y \end{bmatrix}$

Eq. (3) $\begin{bmatrix} \mathbf{e}_x \\ \mathbf{e}_y \end{bmatrix} = - \begin{bmatrix} \mathbf{D}_x^v \\ \mathbf{D}_y^v \end{bmatrix} \mathbf{v}$

Step 1: Substitute Eq. (2) into Eq. (1)

$$\begin{bmatrix} \mathbf{D}_x^e & \mathbf{D}_y^e \end{bmatrix} \begin{bmatrix} \boldsymbol{\varepsilon}_{xx} & \mathbf{R}_x^+ \mathbf{R}_y^- \boldsymbol{\varepsilon}_{xy} \\ \mathbf{R}_x^- \mathbf{R}_y^+ \boldsymbol{\varepsilon}_{yx} & \boldsymbol{\varepsilon}_{yy} \end{bmatrix} \begin{bmatrix} \mathbf{e}_x \\ \mathbf{e}_y \end{bmatrix} = \mathbf{0}$$

Step 2: Substitute Eq. (3) into the above

$$-\begin{bmatrix} \mathbf{D}_x^e & \mathbf{D}_y^e \end{bmatrix} \begin{bmatrix} \boldsymbol{\varepsilon}_{xx} & \mathbf{R}_x^+ \mathbf{R}_y^- \boldsymbol{\varepsilon}_{xy} \\ \mathbf{R}_x^- \mathbf{R}_y^+ \boldsymbol{\varepsilon}_{yx} & \boldsymbol{\varepsilon}_{yy} \end{bmatrix} \begin{bmatrix} \mathbf{D}_x^v \\ \mathbf{D}_y^v \end{bmatrix} \mathbf{v} = \mathbf{0}$$

Step 3: Drop the negative sign

$$\boxed{\begin{bmatrix} \mathbf{D}_x^e & \mathbf{D}_y^e \end{bmatrix} \begin{bmatrix} \boldsymbol{\varepsilon}_{xx} & \mathbf{R}_x^+ \mathbf{R}_y^- \boldsymbol{\varepsilon}_{xy} \\ \mathbf{R}_x^- \mathbf{R}_y^+ \boldsymbol{\varepsilon}_{yx} & \boldsymbol{\varepsilon}_{yy} \end{bmatrix} \begin{bmatrix} \mathbf{D}_x^v \\ \mathbf{D}_y^v \end{bmatrix} \mathbf{v} = \mathbf{0}}$$



## Matrix Form of the Homogeneous Laplace's Equation

For the homogeneous case, the dielectric tensor reduces to

$$\begin{bmatrix} \boldsymbol{\varepsilon}_{xx} & \mathbf{R}_x^+ \mathbf{R}_y^- \boldsymbol{\varepsilon}_{xy} \\ \mathbf{R}_x^- \mathbf{R}_y^+ \boldsymbol{\varepsilon}_{yx} & \boldsymbol{\varepsilon}_{yy} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}$$

Substituting this into the inhomogeneous Laplace's equation gives

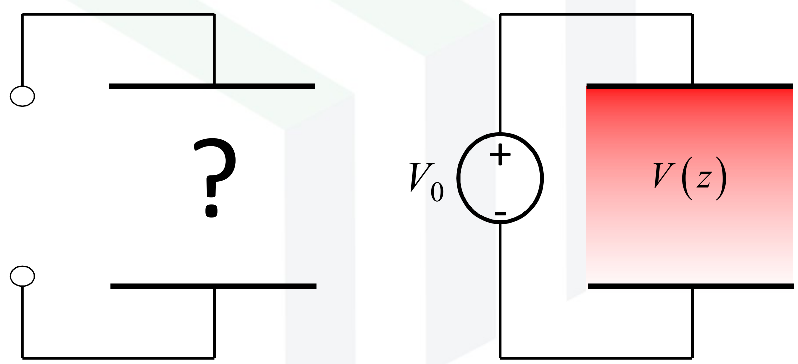
$$\begin{bmatrix} \mathbf{D}_x^e & \mathbf{D}_y^e \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{D}_x^v \\ \mathbf{D}_y^v \end{bmatrix} \mathbf{v} = \mathbf{0}$$

$$\downarrow$$

$$\boxed{\begin{bmatrix} \mathbf{D}_x^e & \mathbf{D}_y^e \end{bmatrix} \begin{bmatrix} \mathbf{D}_x^v \\ \mathbf{D}_y^v \end{bmatrix} \mathbf{v} = \mathbf{0}}$$

## The Excitation Vector $\mathbf{b}$

The matrix equation at this point is $\mathbf{Lv} = \mathbf{0}$. This is not yet solvable because no applied potentials has been incorporated. When this is done, the equation becomes $\mathbf{Lv} = \mathbf{b}$.




$\mathbf{Lv} = \mathbf{0}$ $\qquad$ $\mathbf{Lv} = \mathbf{b}$



## Enforcing the Known Potentials

It turns out $\mathbf{L}$ must be modified in addition to constructing $\mathbf{b}$.

Each point on the grid containing a metal must be forced to a known value of potential $V$. This is done by modifying the corresponding row in the matrix equation as follows:

1. Replace entire $m$th row in $\mathbf{L}$ with all zeroes.
2. Place a 1 in the diagonal element of the $m$th row of $\mathbf{L}$.
3. Place the value of the applied potential $V_m^{\text{applied}}$ in the $m$th row of $\mathbf{b}$.

$$V_m^{\text{applied}} \rightarrow \underbrace{\begin{bmatrix} (\#) & (\#) & (\#) & \cdots & (\#) & (\#) & (\#) \\ (\#) & (\#) & (\#) & \cdots & (\#) & (\#) & (\#) \\ & & \ddots & & \iddots & & \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ & & \iddots & & \ddots & & \\ (\#) & (\#) & (\#) & \cdots & (\#) & (\#) & (\#) \\ (\#) & (\#) & (\#) & \cdots & (\#) & (\#) & (\#) \end{bmatrix}}_{\mathbf{L}'} \underbrace{\begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_m^{\text{metal}} \\ \vdots \\ V_{N_x N_y - 1} \\ V_{N_x N_y} \end{bmatrix}}_{\mathbf{v}} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ V_m^{\text{applied}} \\ \vdots \\ 0 \\ 0 \end{bmatrix}}_{\mathbf{b}}$$

# Fancy Way of Forcing Potentials

Construct two terms:

$\mathbf{F} \equiv$ Force matrix — Diagonal matrix containing 1's in the positions corresponding to values we wish to force.

$\mathbf{v}_f \equiv$ forced potentials — Column vector containing the forced potentials. Numbers in positions not being forced are ignored.

1. Replace forced rows in **L** with all zeros.

$$\mathbf{L}' = (\mathbf{I} - \mathbf{F})\mathbf{L}$$

2. Place a 1 in the diagonal elements.

$$\boxed{\mathbf{L}' = \mathbf{F} + (\mathbf{I} - \mathbf{F})\mathbf{L}}$$

2. Place the forced potentials in **b**.

$$\boxed{\mathbf{b} = \mathbf{F}\mathbf{v}_f}$$

Use same **b** for both homogeneous and inhomogeneous cases.

$$V_m^{\text{applied}} \rightarrow \underbrace{\begin{bmatrix} (\#) & (\#) & (\#) & \cdots & (\#) & (\#) & (\#) \\ (\#) & (\#) & (\#) & \cdots & (\#) & (\#) & (\#) \\ & & \ddots & & \unicode{x22F0} & & \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ & & \unicode{x22F0} & & \ddots & & \\ (\#) & (\#) & (\#) & \cdots & (\#) & (\#) & (\#) \\ (\#) & (\#) & (\#) & \cdots & (\#) & (\#) & (\#) \end{bmatrix}}_{\mathbf{L}'} \underbrace{\begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_m^{\text{metal}} \\ \vdots \\ V_{N_xN_y-1} \\ V_{N_xN_y} \end{bmatrix}}_{\mathbf{v}} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ V_m^{\text{applied}} \\ \vdots \\ 0 \\ 0 \end{bmatrix}}_{\mathbf{b}}$$

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# Calculating Transmission Line Parameters

Calculate the field quantities

$$\begin{bmatrix} \mathbf{e}_x \\ \mathbf{e}_y \end{bmatrix} = -\begin{bmatrix} \mathbf{D}_x^v \\ \mathbf{D}_y^v \end{bmatrix} \mathbf{v} \qquad \begin{bmatrix} \mathbf{e}_{x,h} \\ \mathbf{e}_{y,h} \end{bmatrix} = -\begin{bmatrix} \mathbf{D}_x^v \\ \mathbf{D}_y^v \end{bmatrix} \mathbf{v}_h$$

Note: the constant $\varepsilon_0$ has been moved to to the capacitance equation for convenience.

$$\begin{bmatrix} \mathbf{d}_x \\ \mathbf{d}_y \end{bmatrix} = \begin{bmatrix} \boldsymbol{\varepsilon}_{xx} & \mathbf{R}_x^+\mathbf{R}_y^-\boldsymbol{\varepsilon}_{xy} \\ \mathbf{R}_x^-\mathbf{R}_y^+\boldsymbol{\varepsilon}_{yx} & \boldsymbol{\varepsilon}_{yy} \end{bmatrix} \begin{bmatrix} \mathbf{e}_x \\ \mathbf{e}_y \end{bmatrix} \qquad \begin{bmatrix} \mathbf{d}_{x,h} \\ \mathbf{d}_{y,h} \end{bmatrix} = \begin{bmatrix} \mathbf{e}_{x,h} \\ \mathbf{e}_{y,h} \end{bmatrix}$$

Calculate the distributed inductance $L$ and capacitance $C$ using numerical integration

$$C \cong \varepsilon_0 \Delta x \Delta y \begin{bmatrix} \mathbf{d}_x \\ \mathbf{d}_y \end{bmatrix}^T \begin{bmatrix} \mathbf{e}_x \\ \mathbf{e}_y \end{bmatrix} \qquad C_h \cong \varepsilon_0 \Delta x \Delta y \begin{bmatrix} \mathbf{d}_{x,h} \\ \mathbf{d}_{y,h} \end{bmatrix}^T \begin{bmatrix} \mathbf{e}_{x,h} \\ \mathbf{e}_{y,h} \end{bmatrix} \qquad L = \frac{1}{c_0^2 C_h}$$

Calculate the transmission line parameters

$$Z_0 = \sqrt{\frac{L}{C}} \qquad n_{\text{eff}} = c_0\sqrt{LC} \qquad \varepsilon_{\text{r,eff}} = n_{\text{eff}}^2$$

CAUTION:
Be careful to be consistent with the units. For example, it is incorrect to set `centimeters=1` and then set `e0=8.854e-12` because $\varepsilon_0$ has units of F/m.

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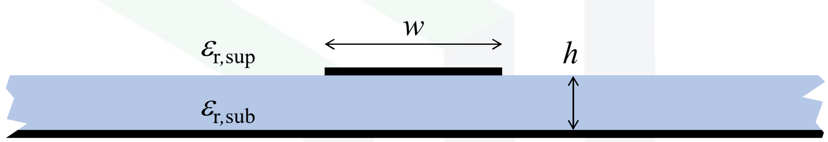
# Finite-Difference Analysis of Transmission Lines Embedded in Arbitrary Anisotropic Media

*Implementation*



## Step 1:
## *Choose a Transmission Line*

Model a microstrip transmission line with the following parameters:




$$\varepsilon_{\mathrm{r,sup}} = \begin{bmatrix} 1.0 & 0 & 0 \\ 0 & 1.0 & 0 \\ 0 & 0 & 1.0 \end{bmatrix} \quad \text{air superstrate}$$

$$\varepsilon_{\mathrm{r,sub}} = \begin{bmatrix} 9.0 & 0 & 0 \\ 0 & 9.0 & 0 \\ 0 & 0 & 9.0 \end{bmatrix} \quad \text{dielectric substrate}$$

$w = 4.0$ mm
$h = 3.0$ mm

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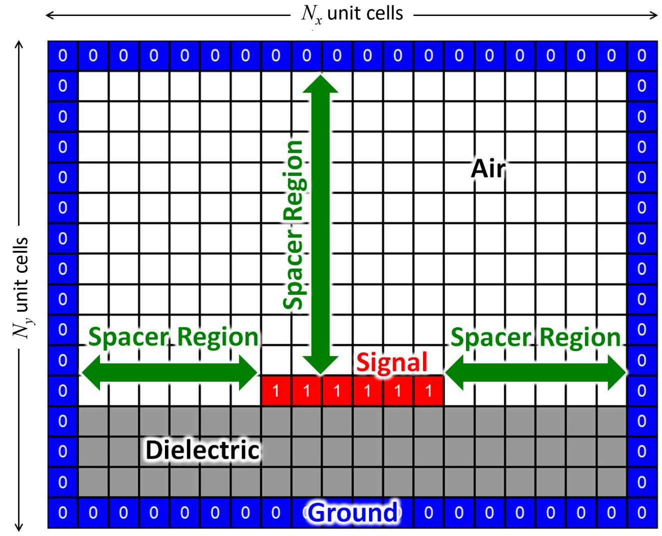
# Step 2:
## *Build Device on Grid (1 of 2)*




Use Dirichlet boundary conditions and ensure there is sufficient space between the microstrip and boundaries.

Rule-of-Thumb:
Spacer regions should be around three times the width of the line.

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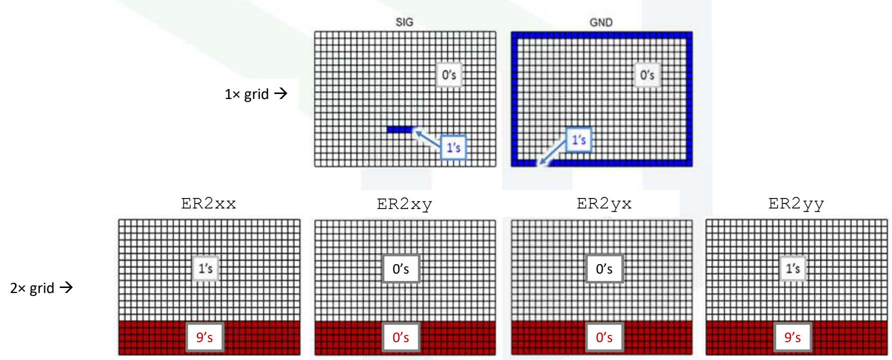
# Step 2:
## *Build Device on Grid (2 of 2)*

Six arrays are used to describe the transmission line.




We do not need to build ER2xz, ER2yz, ER2zx, ER2zy, or ER2zz.

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# Step 3:
## *Construct Matrix Operators*

Use `yeeder()` to calculate the derivative matrices.

```
% CALL FUNCTION TO CONSTRUCT DERIVATIVE OPERATORS
NS  = [Nx Ny];                                   %grid size
RES = [dx dy];                                   %grid resolution
BC  = [0 0];                                     %Dirichlet BCs
[DVX,DVY,DEX,DEY] = yeeder(NS,RES,BC);           %Build matrices
```

Since Dirichlet boundary conditions are being used, the interpolation matrices can be constructed directly from the derivative matrices.

$$\mathbf{R}_x^+ = \frac{\Delta x}{2}\left|\mathbf{D}_x^v\right| \qquad \mathbf{R}_y^- = \frac{\Delta y}{2}\left|\mathbf{D}_y^e\right| \qquad \mathbf{R} = \mathbf{R}_x^+\mathbf{R}_y^- \qquad \frac{\Delta}{2}\left|\frac{f_{i+1} - f_i}{\Delta}\right| = \frac{f_{i+1} + f_i}{2}$$

```
% FORM INTERPOLATION MATRIX FROM DERIVATIVE OPERATORS
RXP = (dx/2)*abs(DVX);
RYM = (dy/2)*abs(DEY);
R   = RXP*RYM;
```

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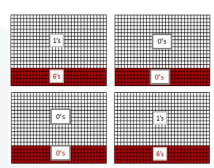
# Step 4:
## *Construct Diagonalized Tensor*

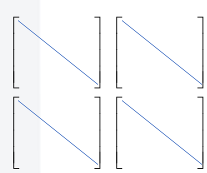
Step 1: Parse materials from 2× grid to 1× grid.

```
ERxx = ER2xx(2:2:Nx2,1:2:Ny2);
ERxy = ER2xy(1:2:Nx2,2:2:Ny2);
ERyx = ER2yx(2:2:Nx2,1:2:Ny2);
ERyy = ER2yy(1:2:Nx2,2:2:Ny2);
```



Step 2: Diagonalize all four tensor elements.

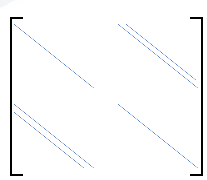
```
ERxx = diag(sparse(ERxx(:)));
ERxy = diag(sparse(ERxy(:)));
ERyx = diag(sparse(ERyx(:)));
ERyy = diag(sparse(ERyy(:)));
```



Step 3: Construct composite tensor

```
ER = [ ERxx , R*ERxy ; R'*ERyx , ERyy ];
```

Note the complex transpose here.



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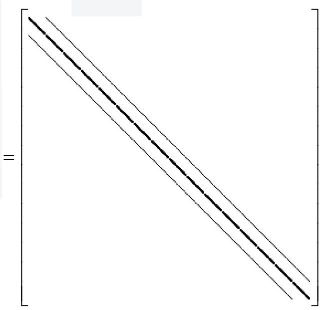
# Step 5:
## *Build* $\mathbf{L}$ *and* $\mathbf{L}_h$

Step 1. Build the inhomogeneous matrix $\mathbf{L}$

```
L = [ DEX DEY ] * ER * [ DVX ; DVY ];
```

Step 2. Build the homogeneous matrix $\mathbf{L}_h$

```
Lh = [ DEX DEY ] * [ DVX ; DVY ];
```

$\mathbf{L} =$ 

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# Step 6:
## *Force Known Potentials*

Step 1. Build the force matrix $\mathbf{F}$ that identifies the locations of the forced potentials.

```
F = SIG | GND;
F = diag(sparse(F(:)));
```

For multiple conductors…

```
F = SIG1 | SIG2 | … | GND;
```

Step 2: Build the column vector containing the values of the forced potentials.

1 volt applied to `SIG`

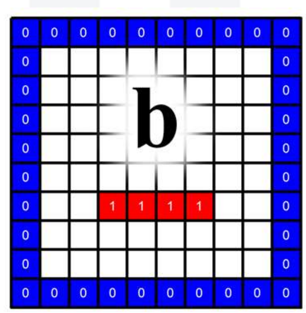
```
vf = 1*SIG + 0*GND;
```

For two conductors…

```
vf = 0.5*SIG1 - 0.5*SIG2 + 0*GND;
```

Step 3: Force the known potentials.

```
L  = (I - F)*L + F;
Lh = (I - F)*Lh + F;
b  = F*vf(:);
```

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# Step 7:
## *Compute the Fields*

Step 1. Compute the potentials. SLOWEST COMPUTATIONAL STEP!!!!

$$\mathbf{v} = \mathbf{L}^{-1}\mathbf{b} \qquad \mathbf{v}_\mathrm{h} = \mathbf{L}_\mathrm{h}^{-1}\mathbf{b}$$

Step 2: Calculate $\vec{E}$.

$$\vec{\mathbf{e}} = \begin{bmatrix} \mathbf{e}_x \\ \mathbf{e}_y \end{bmatrix} = -\begin{bmatrix} \mathbf{D}_x^v \\ \mathbf{D}_y^v \end{bmatrix}\mathbf{v} \qquad \vec{\mathbf{e}}_\mathrm{h} = \begin{bmatrix} \mathbf{e}_{x,\mathrm{h}} \\ \mathbf{e}_{y,\mathrm{h}} \end{bmatrix} = -\begin{bmatrix} \mathbf{D}_x^v \\ \mathbf{D}_y^v \end{bmatrix}\mathbf{v}_\mathrm{h}$$

Step 3: Compute $\vec{D}$.

$$\vec{\mathbf{d}} = \begin{bmatrix} \mathbf{d}_x \\ \mathbf{d}_y \end{bmatrix} = \begin{bmatrix} \boldsymbol{\varepsilon}_{xx} & \mathbf{R}\boldsymbol{\varepsilon}_{xy} \\ \mathbf{R}^H\boldsymbol{\varepsilon}_{yx} & \boldsymbol{\varepsilon}_{yy} \end{bmatrix}\begin{bmatrix} \mathbf{e}_x \\ \mathbf{e}_y \end{bmatrix} \qquad \vec{\mathbf{d}}_\mathrm{h} = \begin{bmatrix} \mathbf{d}_{x,\mathrm{h}} \\ \mathbf{d}_{y,\mathrm{h}} \end{bmatrix} = \begin{bmatrix} \mathbf{e}_{x,\mathrm{h}} \\ \mathbf{e}_{y,\mathrm{h}} \end{bmatrix}$$

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# Step 8:
## *Compute TL Parameters*

Step 1. Compute the distributed capacitance $C$.

$$C = \frac{1}{V_0^2}\iint_A \left(\vec{D}\bullet\vec{E}\right)dA \quad \rightarrow$$

```
C = (e0*dx*dy)*dxy'*exy;
```

Recall that the $\varepsilon_0$ term was moved for convenience.

Step 2: Calculate the distributed inductance $L$.

$$C_h = \frac{1}{V_0^2}\iint_A \left(\vec{D}_h\bullet\vec{E}_h\right)dA$$

$$L = 1\big/\left(c_0^2 C_h\right) \quad \rightarrow$$

```
Ch = (e0*dx*dy)*dxyh'*exyh;
L  = 1/(c0^2*Ch);
```

Step 3: Calculate characteristic impedance $Z_0$.

$$Z_0 = \sqrt{\frac{L}{C}}$$

Be sure to be consistent with the units!

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# Step 9: *Visualize the Fields*

Step 1. Extract the $x$ and $y$ components of $\vec{E}$.

$$\vec{\mathbf{e}} = \begin{bmatrix} \mathbf{e}_x \\ \mathbf{e}_y \end{bmatrix} \rightarrow$$

```
Ex = exy(1:M);
Ey = exy(M+1:2*M);
```

Step 2: Reshape from column vectors to 2D arrays.

```
v  = reshape(v,Nx,Ny);
Ex = reshape(Ex,Nx,Ny);
Ey = reshape(Ey,Nx,Ny);
```

Step 3: Visualize in MATLAB using `imagesc()`, `pcolor()`, and/or `quiver()`.

For day-to-day data analysis, I use `imagesc()`. For generating pictures for publications and presentations, I use `pcolor()`.

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# Finite-Difference Analysis of Transmission Lines Embedded in Arbitrary Anisotropic Media

*Examples*

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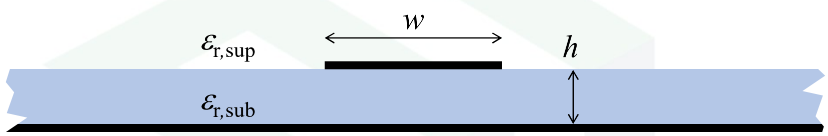
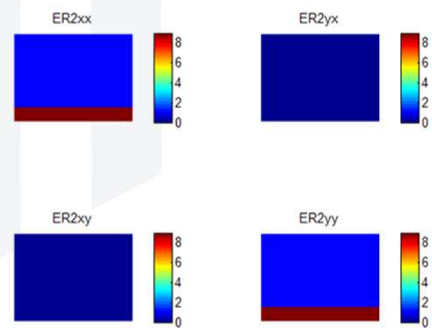
# Example #1: *Ordinary Microstrip (1 of 2)*




$w = 4.0$ mm
$h = 3.0$ mm

$$\varepsilon_{\mathrm{r,sup}} = \begin{bmatrix} 1.0 & 0 & 0 \\ 0 & 1.0 & 0 \\ 0 & 0 & 1.0 \end{bmatrix} \text{ air}$$

$$\varepsilon_{\mathrm{r,sub}} = \begin{bmatrix} 9.0 & 0 & 0 \\ 0 & 9.0 & 0 \\ 0 & 0 & 9.0 \end{bmatrix} \text{ dielectric}$$

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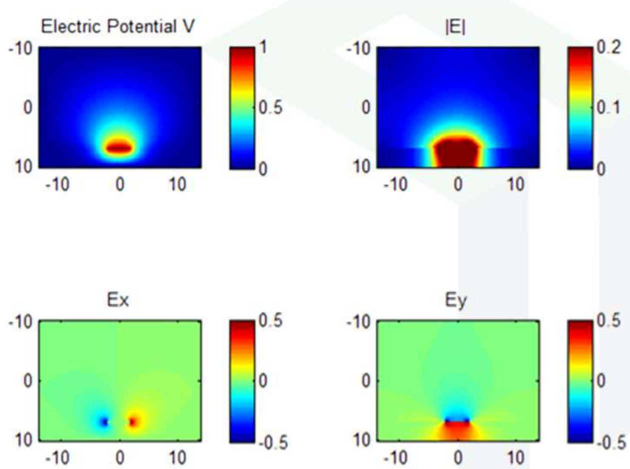
# Example #1: *Ordinary Microstrip (2 of 2)*




```
Nx = 200
Ny = 150
dx = 0.13793
dy = 0.13636
Z0 = 44.3436 ohms
```

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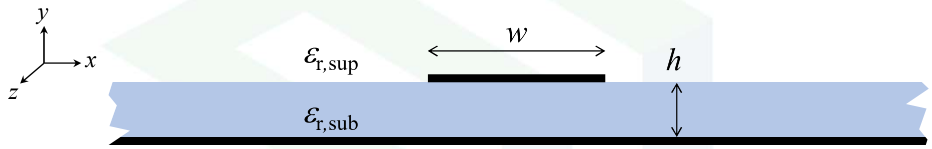
# Example #2:
## *Anisotropic Microstrip (1 of 3)*

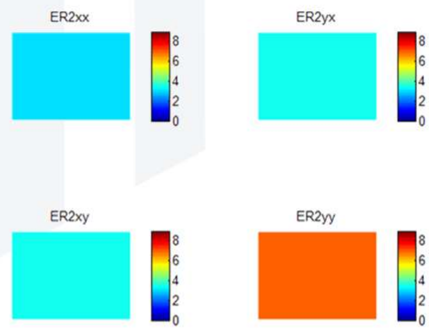



$w = 4.0$ mm

$h = 3.0$ mm

$$\varepsilon_{\text{r,sup}} = \begin{bmatrix} 3.0000 & 3.4641 \\ 3.4641 & 7.0000 \end{bmatrix}$$

$$\varepsilon_{\text{r,sub}} = \begin{bmatrix} 3.0000 & 3.4641 \\ 3.4641 & 7.0000 \end{bmatrix}$$

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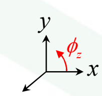
# Example #2:
## *Anisotropic Microstrip (2 of 3)*

To generate correct tensors, start with the diagonal tensor and the angle that it should be rotated around the $z$-axis.

$$[\varepsilon_{\text{r}}] = \begin{bmatrix} 1.0 & 0 & 0 \\ 0 & 9.0 & 0 \\ 0 & 0 & 1.0 \end{bmatrix} \qquad \phi_z = -30°$$




The rotation matrix is

$$[R] = [R_z(-30°)] = \begin{bmatrix} \cos(-30°) & -\sin(-30°) & 0 \\ \sin(-30°) & \cos(-30°) & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.8660 & 0.5000 & 0 \\ -0.5000 & 0.8660 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The rotated tensor is then

$$[R][\varepsilon_{\text{r}}][R]^{-1} = \begin{bmatrix} 0.8660 & 0.5000 & 0 \\ -0.5000 & 0.8660 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1.0 & 0 & 0 \\ 0 & 9.0 & 0 \\ 0 & 0 & 1.0 \end{bmatrix} \begin{bmatrix} 0.8660 & 0.5000 & 0 \\ -0.5000 & 0.8660 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 3.0000 & 3.4641 & 0 \\ 3.4641 & 7.0000 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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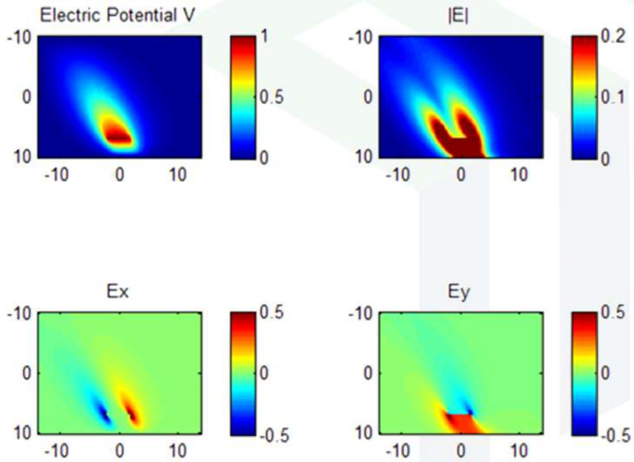
# Example #2:
# *Anisotropic Microstrip (3 of 3)*




```
Nx = 200
Ny = 150
dx = 0.13793
dy = 0.13636
Z0 = 49.9482 ohms
```

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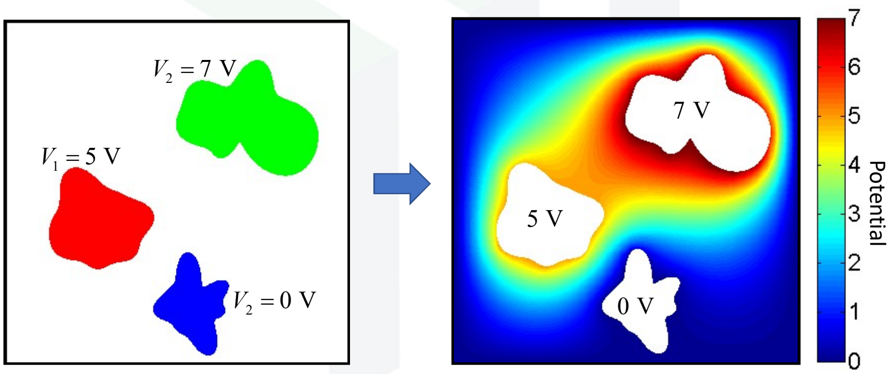
# Generalization of the Method

# Method Can Do More than Transmission Lines!

The model just covered simply calculates the potential in the vicinity of multiple forced potentials. It was used it to model a transmission line, but the method has other applications.






# Think of Laplace's Equation as a Linear "Number Filler Inner"

Given known values at certain points (boundary conditions), Laplace's equation calculates the numbers everywhere else so they vary linearly.

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | | | | |
| | | | | | | | | | | 9.0 | 9.0 | |
| | | | | | | | | | | 9.0 | 9.0 | |
| | | | | | | | | | | 9.0 | 9.0 | |
| | | | | | | | | | | 9.0 | 9.0 | |
| | | | | | | | | | | 9.0 | 9.0 | |
| | | | | | | | | | | 9.0 | 9.0 | |
| | | | | | | | | | | 9.0 | 9.0 | |
| | | | | | | | | | | | | |
| | -9.0 | -9.0 | -9.0 | | | | | | | | | |
| | -9.0 | -9.0 | -9.0 | | | | | | | | | |
| | -9.0 | -9.0 | -9.0 | | | | | | | | | |
| | | | | | | | | | | | | |

Map of known values (boundary values)

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.1 | 0.4 | 0.8 | 1.3 | 1.8 | 2.5 | 3.2 | 4.1 | 5.0 | 6.0 | 6.9 | 7.0 | 6.4 |
| -0.2 | 0.1 | 0.5 | 1.0 | 1.7 | 2.4 | 3.3 | 4.3 | 5.5 | 6.9 | 9.0 | 9.0 | 7.4 |
| -0.6 | -0.3 | 0.1 | 0.7 | 1.4 | 2.2 | 3.2 | 4.4 | 5.7 | 7.2 | 9.0 | 9.0 | 7.8 |
| -1.1 | -0.8 | -0.4 | 0.2 | 1.0 | 1.9 | 3.0 | 4.3 | 5.7 | 7.3 | 9.0 | 9.0 | 7.9 |
| -1.7 | -1.5 | -1.0 | -0.4 | 0.4 | 1.4 | 2.6 | 4.0 | 5.5 | 7.2 | 9.0 | 9.0 | 7.9 |
| -2.4 | -2.3 | -1.9 | -1.3 | -0.4 | 0.8 | 2.1 | 3.5 | 5.2 | 7.0 | 9.0 | 9.0 | 7.8 |
| -3.3 | -3.3 | -3.0 | -2.4 | -1.3 | -0.1 | 1.3 | 2.9 | 4.7 | 6.7 | 9.0 | 9.0 | 7.6 |
| -4.4 | -4.7 | -4.5 | -3.8 | -2.6 | -1.1 | 0.5 | 2.1 | 3.9 | 6.0 | 9.0 | 9.0 | 7.1 |
| -5.6 | -6.4 | -6.4 | -5.8 | -4.1 | -2.2 | -0.5 | 1.2 | 2.9 | 4.5 | 6.0 | 6.4 | 5.8 |
| -6.8 | -9.0 | -9.0 | -9.0 | -5.6 | -3.3 | -1.3 | 0.3 | 1.8 | 3.1 | 4.2 | 4.6 | 4.6 |
| -7.1 | -9.0 | -9.0 | -9.0 | -6.1 | -3.8 | -2.0 | -0.4 | 0.9 | 2.0 | 2.9 | 3.4 | 3.5 |
| -7.0 | -9.0 | -9.0 | -9.0 | -6.1 | -4.0 | -2.3 | -0.9 | 0.2 | 1.2 | 2.0 | 2.4 | 2.7 |
| -6.1 | -6.9 | -7.0 | -6.6 | -5.2 | -3.7 | -2.4 | -1.2 | -0.2 | 0.6 | 1.3 | 1.8 | 2.1 |

Laplace's equation is solved to fill in the values away from the boundaries.