



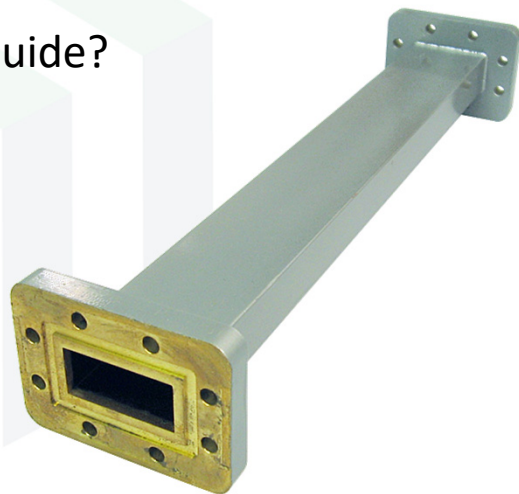
Electromagnetics:
Electromagnetic Field Theory

The Rectangular Waveguide



Lecture Outline

- What is a rectangular waveguide?
- TM Analysis
- TE Analysis
- Visualization of Modes
- Conclusions

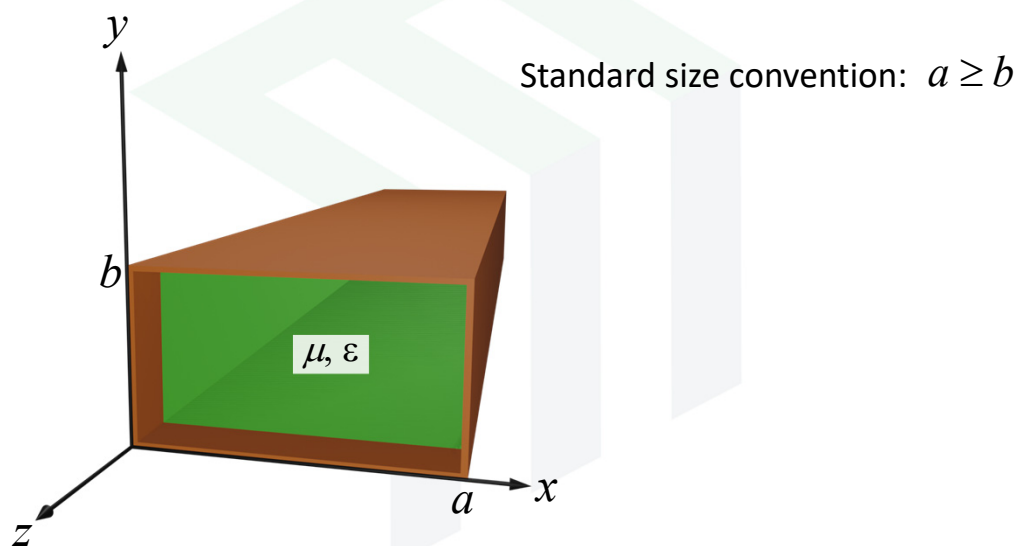


What is a Rectangular Waveguide?



Slide 3

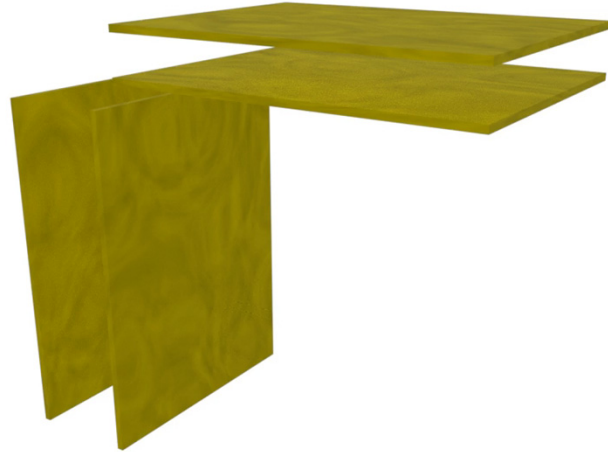
Geometry of Rectangular Waveguide



Slide 4

Analysis of Rectangular Waveguide

Rectangular waveguides are analyzed a bit like each axis were its own parallel plate waveguide.



Notes on the Rectangular Waveguide

- Most classic waveguide example
- Some of the first waveguides used for microwaves
- Not a transmission line because it has only one conductor
- Does not support a TEM mode
- Exhibits a low-frequency cutoff below which no waves will propagate

TE Analysis



Slide 7

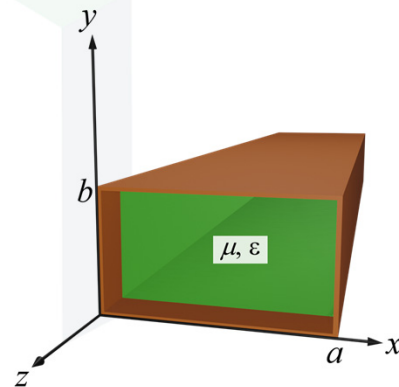
Recall TE Analysis

The governing equation for TE analysis is

$$\frac{\partial^2 H_{0,z}}{\partial x^2} + \frac{\partial^2 H_{0,z}}{\partial y^2} - k_c^2 H_{0,z} = 0 \quad k_c^2 = k^2 - \beta^2$$

After a solution is obtained, the remaining field components are calculated according to

$$\begin{aligned} H_{0,x} &= -\frac{j\beta}{k_c^2} \frac{\partial H_{0,z}}{\partial x} & E_{0,x} &= -\frac{j\omega\mu}{k_c^2} \frac{\partial H_{0,z}}{\partial y} \\ H_{0,y} &= -\frac{j\beta}{k_c^2} \frac{\partial H_{0,z}}{\partial y} & E_{0,y} &= \frac{j\omega\mu}{k_c^2} \frac{\partial H_{0,z}}{\partial x} \\ E_{0,z} &= 0 \end{aligned}$$



Slide 8

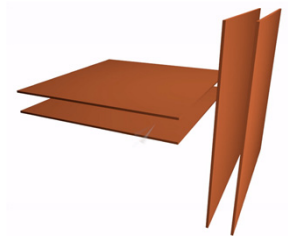
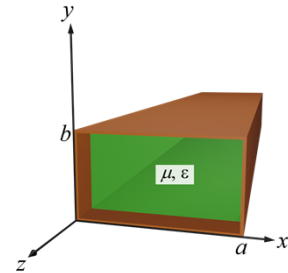
General Form of the Solution

From the geometry of the waveguide, the general form of the solution can be immediately written as

$$H_z(x, y, z) = H_{0,z}(x, y)e^{-j\beta z}$$

Viewing the rectangular waveguide as the combination of two parallel plate waveguides, apply separation of variables to write $H_{0,z}(x, y)$ as the product of two functions.

$$H_{0,z}(x, y) = X(x)Y(y)$$



Separation of Variables (1 of 3)

The solution is written as the product of two 1D functions, $X(x)$ and $Y(y)$. Substitute this solution back into the differential equation.

$$\frac{\partial^2 H_{0,z}}{\partial x^2} + \frac{\partial^2 H_{0,z}}{\partial y^2} - k_c^2 H_{0,z} = 0$$

$H_{0,z}(x, y) = X(x)Y(y)$



$$\frac{\partial^2 XY}{\partial x^2} + \frac{\partial^2 XY}{\partial y^2} - k_c^2 XY = 0$$

To be compact, drop the (x) and (y) notation.

$$\frac{\partial^2 X}{\partial x^2} Y + X \frac{\partial^2 Y}{\partial y^2} - k_c^2 XY = 0$$

Move $X(x)$ out of the $\partial/\partial x$ operation and $Y(y)$ out of the $\partial/\partial y$ operation.

$$\frac{1}{X} \frac{d^2 X}{dx^2} + \frac{1}{Y} \frac{d^2 Y}{dy^2} - k_c^2 = 0$$

The derivatives become ordinary because $X(x)$ and $Y(y)$ have only one independent variable each.

Separation of Variables (2 of 3)

First, attention is focused on the x -dependence in the differential equation.

$$\frac{1}{X} \frac{d^2 X}{dx^2} + \frac{1}{Y} \frac{d^2 Y}{dy^2} - k_c^2 = 0$$

$\underbrace{\hspace{10em}}_{-k_x^2}$

This definition of k_x lets the differential equation be written as a wave equation.

$$\frac{d^2 X}{dx^2} - k_x^2 X = 0$$

Second, attention is focused on the y -dependence in the differential equation.

$$\frac{1}{X} \frac{d^2 X}{dx^2} - k_c^2 + \frac{1}{Y} \frac{d^2 Y}{dy^2} = 0$$

$\underbrace{\hspace{10em}}_{-k_y^2}$

This definition of k_y lets the differential equation be written as a wave equation.

$$\frac{d^2 Y}{dy^2} - k_y^2 Y = 0$$

Separation of Variables (3 of 3)

If all of this is correct, then it should be possible to add the two new differential equations together to get the original differential equation.

$$\begin{array}{r} \frac{d^2 X}{dx^2} - k_x^2 X = 0 \\ \frac{d^2 Y}{dy^2} - k_y^2 Y = 0 \end{array} \quad \Longrightarrow \quad \begin{array}{r} \frac{1}{X} \frac{d^2 X}{dx^2} - k_x^2 = 0 \\ + \frac{1}{Y} \frac{d^2 Y}{dy^2} - k_y^2 = 0 \\ \hline \frac{1}{X} \frac{d^2 X}{dx^2} + \frac{1}{Y} \frac{d^2 Y}{dy^2} - k_x^2 - k_y^2 = 0 \end{array}$$

The original differential equation is obtained if

$$k_c^2 = k_x^2 + k_y^2$$

Original differential equation

$$\frac{1}{X} \frac{d^2 X}{dx^2} + \frac{1}{Y} \frac{d^2 Y}{dy^2} - k_c^2 = 0$$

General Solution

There are now two differential equations to solve.

$$\frac{d^2 X}{dx^2} - k_x^2 X = 0$$

$$\frac{d^2 Y}{dy^2} - k_y^2 Y = 0$$

These are essentially the same differential equation so their solution has the same general form.

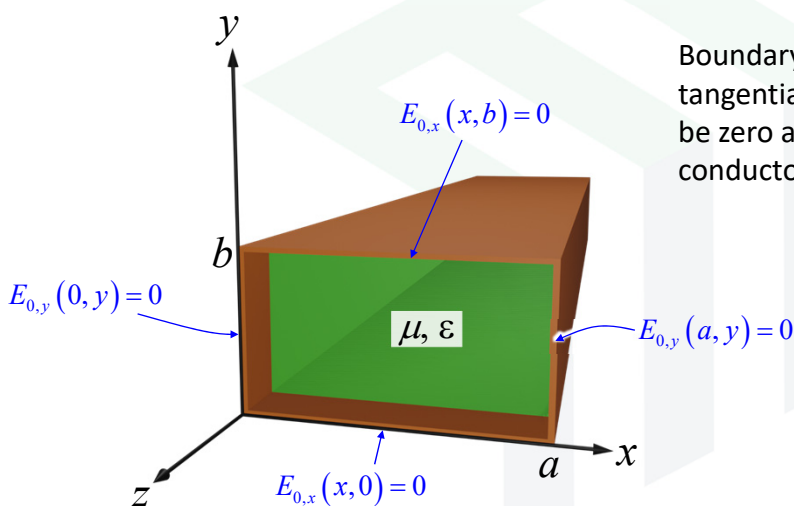
$$\frac{d^2 X}{dx^2} - k_x^2 X = 0 \rightarrow X(x) = A \cos(k_x x) + B \sin(k_x x) \quad \text{PP waveguide along } x$$

$$\frac{d^2 Y}{dy^2} - k_y^2 Y = 0 \rightarrow Y(y) = C \cos(k_y y) + D \sin(k_y y) \quad \text{PP waveguide along } y$$

The overall solution is the product of $X(x)$ and $Y(y)$.

$$H_{0,z}(x, y) = X(x)Y(y) = [A \cos(k_x x) + B \sin(k_x x)][C \cos(k_y y) + D \sin(k_y y)]$$

Electromagnetic Boundary Conditions



Boundary conditions require that the tangential component of the electric field be zero at the boundary with a perfect conductor.

$E_{0,x}$ and $E_{0,y}$

In order to apply the boundary conditions, the electric field components $E_{0,x}$ and $E_{0,y}$ must be derived from the expression for $H_{0,z}$.

$$\begin{aligned} E_{0,x}(x,y) &= -\frac{j\omega\mu}{k_c^2} \frac{\partial H_{0,z}}{\partial y} \\ &= -\frac{j\omega\mu}{k_c^2} \frac{\partial}{\partial y} [A \cos(k_x x) + B \sin(k_x x)] [C \cos(k_y y) + D \sin(k_y y)] \\ &= -\frac{j\omega\mu}{k_c^2} k_y [A \cos(k_x x) + B \sin(k_x x)] [-C \sin(k_y y) + D \cos(k_y y)] \end{aligned}$$

$$\begin{aligned} E_{0,y}(x,y) &= \frac{j\omega\mu}{k_c^2} \frac{\partial H_{0,z}}{\partial x} \\ &= \frac{j\omega\mu}{k_c^2} \frac{\partial}{\partial x} [A \cos(k_x x) + B \sin(k_x x)] [C \cos(k_y y) + D \sin(k_y y)] \\ &= \frac{j\omega\mu}{k_c^2} k_x [-A \sin(k_x x) + B \cos(k_x x)] [C \cos(k_y y) + D \sin(k_y y)] \end{aligned}$$

Apply Boundary Conditions (1 of 2)

At the $x = 0$ boundary,

$$\begin{aligned} 0 &= E_{0,y}(0,y) \\ &= \frac{j\omega\mu}{k_c^2} k_x [-A \sin(0) + B \cos(0)] [C \cos(k_y y) + D \sin(k_y y)] \\ &= \frac{j\omega\mu}{k_c^2} k_x [B] [C \cos(k_y y) + D \sin(k_y y)] \longrightarrow B = 0 \end{aligned}$$

At the $x = a$ boundary,

$$\begin{aligned} 0 &= E_{0,y}(a,y) \\ &= \frac{j\omega\mu}{k_c^2} k_x [-A \sin(k_x a)] [C \cos(k_y y) + D \sin(k_y y)] \end{aligned}$$

$A = 0$ leads to a trivial solution. It must be the $\sin(k_x a)$ term that enforces the BC.

$$0 = \sin(k_x a) \rightarrow k_x a = m\pi \quad m = 0, 1, 2, \dots \quad \text{The specific values } m \text{ can be considered later.}$$

Apply Boundary Conditions (2 of 2)

At the $y = 0$ boundary,

$$\begin{aligned}
 0 &= E_{0,x}(x,0) \\
 &= -\frac{j\omega\mu}{k_c^2} k_y [A \cos(k_x x) + B \sin(k_x x)] [-\cancel{C \sin(0)} + D \cos(0)] \\
 &= -\frac{j\omega\mu}{k_c^2} k_y [A \cos(k_x x) + B \sin(k_x x)] [D] \longrightarrow D = 0
 \end{aligned}$$

At the $y = b$ boundary,

$$\begin{aligned}
 0 &= E_{0,x}(x,b) \\
 &= -\frac{j\omega\mu}{k_c^2} k_y [A \cos(k_x x) + B \sin(k_x x)] [-C \sin(k_y b)] \\
 C = 0 &\text{ leads to a trivial solution. It must be the } \sin(k_y b) \text{ term that enforces the BC.}
 \end{aligned}$$

$$0 = \sin(k_y b) \rightarrow k_y b = n\pi \quad n = 0, 1, 2, \dots$$

The specific values n can be will be considered later.

Revised Solution for $H_{0,z}$

It was determined that $B = D = 0$ so the expression for $H_{0,z}$ becomes

$$H_{0,z}(x, y) = AC \cos(k_x x) \cos(k_y y)$$

The product AC is written as a single constant A_{mn} .

$$H_{0,z}(x, y) = A_{mn} \cos(k_x x) \cos(k_y y)$$

Also, recall the conditions for k_x and k_y .

$$k_x a = m\pi \rightarrow k_x = \frac{m\pi}{a} \quad k_y b = n\pi \rightarrow k_y = \frac{n\pi}{b}$$

$$H_{0,z}(x, y) = A_{mn} \cos\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right)$$

Entire Solution (1 of 2)

The final expression for $H_{0,z}$ is

$$H_{0,z}(x, y) = A_{mn} \cos\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) \quad E_{0,z}(x, y) = 0$$

From this, the other field components are

$$E_{0,x}(x, y) = \frac{j\omega\mu n\pi}{k_c^2 b} A_{mn} \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right)$$

$$E_{0,y}(x, y) = -\frac{j\omega\mu m\pi}{k_c^2 a} A_{mn} \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right)$$

$$H_{0,x}(x, y) = \frac{j\beta_{mn} m\pi}{k_c^2 a} A_{mn} \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right)$$

$$H_{0,y}(x, y) = \frac{j\beta_{mn} n\pi}{k_c^2 b} A_{mn} \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right)$$

Entire Solution (2 of 2)

The overall electric and magnetic fields at any position are

$$E_x(x, y, z) = \frac{j\omega\mu n\pi}{k_c^2 b} A_{mn} \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) e^{-j\beta_{mn} z}$$

$$E_y(x, y, z) = -\frac{j\omega\mu m\pi}{k_c^2 a} A_{mn} \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) e^{-j\beta_{mn} z}$$

$$E_z(x, y, z) = 0$$

$$H_x(x, y, z) = \frac{j\beta_{mn} m\pi}{k_c^2 a} A_{mn} \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) e^{-j\beta_{mn} z}$$

$$H_y(x, y, z) = \frac{j\beta_{mn} n\pi}{k_c^2 b} A_{mn} \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) e^{-j\beta_{mn} z}$$

$$H_z(x, y, z) = A_{mn} \cos\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) e^{-j\beta_{mn} z}$$

Phase Constant, β

Recall the cutoff wave number

$$k_c^2 = k_x^2 + k_y^2$$

After analyzing the boundary conditions, this expression can be written as

$$k_c^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2$$

The phase constant β is therefore

$$k_c^2 = k^2 - \beta^2$$

$$\beta^2 = k^2 - k_c^2$$

$$\beta_{mn} = \sqrt{k^2 - k_c^2} = \sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}$$

Cutoff Frequency, f_c

Recall the expression for the phase constant

$$\beta_{mn} = \sqrt{k^2 - k_c^2}$$

The phase constant must be a real number for a guided mode. This requires

$$k > k_c$$

Any time $k < k_c$, the mode is cutoff and not supported by the waveguide. From this, the cutoff frequency f_c is derived to be

$$\begin{aligned} k &> k_c \\ \omega\sqrt{\mu\varepsilon} &> k_c \\ 2\pi f_c\sqrt{\mu\varepsilon} &= k_c \end{aligned}$$

$$f_{c,mn} = \frac{k_c}{2\pi\sqrt{\mu\varepsilon}} = \frac{1}{2\pi\sqrt{\mu\varepsilon}} \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$$

Characteristic Impedance, Z_{TE}

The characteristic impedance Z_{TE} of the TE mode is

$$Z_{\text{TE}} = \frac{E_x}{H_y} = \frac{\frac{j\omega\mu n\pi}{k_c^2 b} A_{mn} \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) e^{-j\beta z}}{\frac{j\beta_{mn} n\pi}{k_c^2 b} A_{mn} \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) e^{-j\beta z}} = \frac{\omega\mu}{\beta_{mn}} = \frac{k\eta}{\beta_{mn}}$$

Cutoff for First-Order TE Mode (1 of 2)

The cutoff frequency for the TE_{mn} mode was found to be

$$f_{c,mn} = \frac{1}{2\pi\sqrt{\mu\varepsilon}} \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$$

What about the TE_{00} mode?

$$\text{TE}_{00} \rightarrow m = n = 0$$

$$f_{c,00} = \frac{1}{2\pi\sqrt{\mu\varepsilon}} \sqrt{\left(\frac{0}{a}\right)^2 + \left(\frac{0}{b}\right)^2} = 0$$

The TE_{00} mode does not exist because it is impossible for a mode in this waveguide to have a cutoff frequency of 0 Hz.

Cutoff for First-Order TE Mode (2 of 2)

What about the TE₀₁ mode?

$$\text{TE}_{01} \rightarrow m = 0, n = 1$$

$$f_{c,01} = \frac{1}{2\pi\sqrt{\mu\epsilon}} \sqrt{\left(\frac{0}{a}\right)^2 + \left(\frac{1 \cdot \pi}{b}\right)^2} = \frac{1}{2b\sqrt{\mu\epsilon}}$$

What about the TE₁₀ mode?

$$\text{TE}_{10} \rightarrow m = 1, n = 0$$

$$f_{c,10} = \frac{1}{2\pi\sqrt{\mu\epsilon}} \sqrt{\left(\frac{1 \cdot \pi}{a}\right)^2 + \left(\frac{0}{b}\right)^2} = \frac{1}{2a\sqrt{\mu\epsilon}}$$

Since $a > b$, it is concluded that the first-order TE mode is TE₁₀ because it has the lowest cutoff frequency.

CAUTION: It cannot yet be said that the TE₁₀ is the fundamental mode because the cutoff frequency of the TM modes has not yet been checked.

Single Mode Operation (1 of 2)

Over what range of frequencies does a rectangular waveguide supports only a single TE mode?

$$f_{c1} < f < f_{c2}$$

Low-Frequency Cutoff

The lower frequency cutoff was just found.

$$f_{c1} = \frac{1}{2a\sqrt{\mu\epsilon}}$$

High-Frequency Cutoff

The high-frequency cutoff is the frequency where the second-order TE mode is supported. This could be the TE₀₁, TE₁₁ or TE₂₀ mode. All must be considered.

$$\text{TE}_{01}: f_{c,01} = \frac{1}{2\pi\sqrt{\mu\epsilon}} \sqrt{\left(\frac{0}{a}\right)^2 + \left(\frac{1 \cdot \pi}{b}\right)^2} = \frac{1}{2b\sqrt{\mu\epsilon}}$$

$$\text{TE}_{11}: f_{c,11} = \frac{1}{2\pi\sqrt{\mu\epsilon}} \sqrt{\left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2} = \frac{1}{2b\sqrt{\mu\epsilon}} \sqrt{1 + \left(\frac{b}{a}\right)^2}$$

$$\text{TE}_{20}: f_{c,20} = \frac{1}{2\pi\sqrt{\mu\epsilon}} \sqrt{\left(\frac{2 \cdot \pi}{a}\right)^2 + \left(\frac{0}{b}\right)^2} = \frac{1}{2b\sqrt{\mu\epsilon}} \frac{2b}{a}$$

TE₁₁ will always have a higher cutoff frequency than TE₀₁.

The second-order mode depends on choice of a and b .

$$f_{c2} = \begin{cases} f_{c,01} & a \leq 2b \\ f_{c,20} & a > 2b \quad (\text{typical}) \end{cases}$$

Single Mode Operation (2 of 2)

Bandwidth

Typical rectangular waveguides will have $a > 2b$, so

$$f_{c1} = \frac{1}{2a\sqrt{\mu\epsilon}} \quad f_{c2} = \frac{1}{a\sqrt{\mu\epsilon}}$$

$$\Delta f = f_{c2} - f_{c1} = \frac{1}{a\sqrt{\mu\epsilon}} - \frac{1}{2a\sqrt{\mu\epsilon}} = \frac{1}{2a\sqrt{\mu\epsilon}}$$

Fractional Bandwidth

Continuing the assumption that $a > 2b$, the fractional bandwidth can be calculated from f_{c1} and f_{c2} above as follows

$$\text{FBW} = \frac{\Delta f}{f_c} = \frac{f_{c2} - f_{c1}}{(f_{c2} + f_{c1})/2} = 2 \frac{f_{c2} - f_{c1}}{f_{c2} + f_{c1}} = 2 \frac{\frac{1}{a\sqrt{\mu\epsilon}} - \frac{1}{2a\sqrt{\mu\epsilon}}}{\frac{1}{a\sqrt{\mu\epsilon}} + \frac{1}{2a\sqrt{\mu\epsilon}}} = \frac{2}{3} = 66.7\%$$

Example #1 – TE Mode Analysis (1 of 4)

Suppose there exists an air-filled rectangular waveguide with $a = 3$ cm and $b = 2$ cm.

What is the cutoff frequency of the waveguide?

$$f_{c1} = \frac{1}{2a\sqrt{\mu\epsilon}} = \frac{c_0}{2a\sqrt{\mu_r\epsilon_r}} = \frac{299792458 \text{ m/s}}{2(0.03 \text{ m})\sqrt{(1.0)(1.0)}} = \boxed{5.0 \text{ GHz}}$$

Over what range of frequencies is the waveguide single mode?

Observing that $a < 2b$, so the second-order mode is TE_{01} .

$$f_{c2} = \frac{1}{2b\sqrt{\mu\epsilon}} = \frac{c_0}{2b\sqrt{\mu_r\epsilon_r}} = \frac{299792458 \text{ m/s}}{2(0.02 \text{ m})\sqrt{(1.0)(1.0)}} = 7.5 \text{ GHz}$$

$$\boxed{5.0 \text{ GHz} < f < 7.5 \text{ GHz}}$$

Example #1 – TE Mode Analysis (2 of 4)

What is the fractional bandwidth of the waveguide?

$$\text{FBW} = (100\%) \frac{\Delta f}{f} = (200\%) \frac{f_2 - f_1}{f_2 + f_1} = (200\%) \frac{7.5 - 5.0}{7.5 + 5.0} = \boxed{40\%}$$

Plot the phase constant and effective refractive index for the first-order and second-order modes from DC up to 15 GHz.

The phase constant is calculated as:

$$\beta_{mn} = \sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2} \rightarrow \begin{aligned} \text{TE}_{10}: \beta_1 &= \sqrt{\left(\frac{2\pi f}{c_0}\right)^2 - \left(\frac{\pi}{a}\right)^2} \\ \text{TE}_{01}: \beta_2 &= \sqrt{\left(\frac{2\pi f}{c_0}\right)^2 - \left(\frac{\pi}{b}\right)^2} \end{aligned}$$

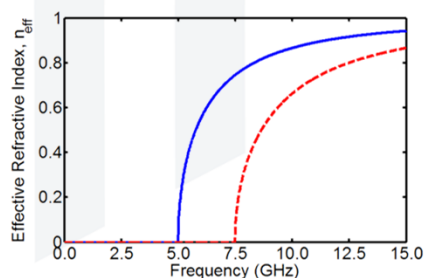
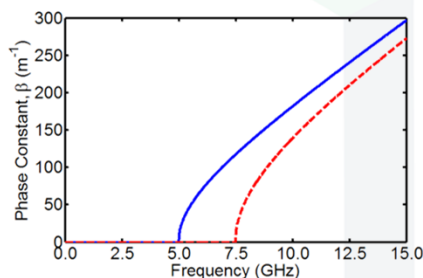
The effective refractive index is calculated as:

$$\beta = k_0 n_{\text{eff}} \rightarrow n_{\text{eff}} = \frac{\beta}{2\pi f} = \beta \frac{c_0}{2\pi f}$$

Example #1 – TE Mode Analysis (3 of 4)

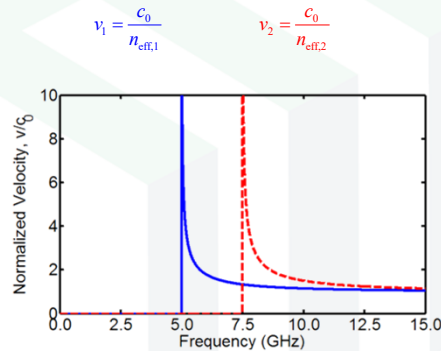
Plot the phase constant and effective refractive index for the first-order and second-order modes from DC up to 15 GHz.

$$\beta_1 = \sqrt{\left(\frac{2\pi f}{c_0}\right)^2 - \left(\frac{\pi}{a}\right)^2} \quad \beta_2 = \sqrt{\left(\frac{2\pi f}{c_0}\right)^2 - \left(\frac{\pi}{b}\right)^2} \quad n_{\text{eff},1} = \beta_1 \frac{c_0}{2\pi f} \quad n_{\text{eff},2} = \beta_2 \frac{c_0}{2\pi f}$$



Example #1 – TE Mode Analysis (4 of 4)

Plot the velocity of the modes as a function of frequency.



Are the modes travelling faster than the speed of light?

Summary of TE Analysis

Field Solution

$$E_x(x, y, z) = \frac{j\omega\mu n\pi}{k_c^2 b} A_{mn} \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) e^{-j\beta_{mn}z}$$

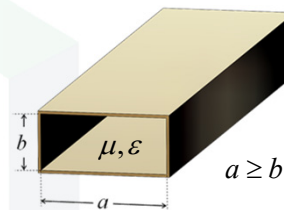
$$E_y(x, y, z) = -\frac{j\omega\mu m\pi}{k_c^2 a} A_{mn} \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) e^{-j\beta_{mn}z}$$

$$E_z(x, y, z) = 0$$

$$H_x(x, y, z) = \frac{j\beta_{mn} m\pi}{k_c^2 a} A_{mn} \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) e^{-j\beta_{mn}z}$$

$$H_y(x, y, z) = \frac{j\beta_{mn} n\pi}{k_c^2 b} A_{mn} \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) e^{-j\beta_{mn}z}$$

$$H_z(x, y, z) = A_{mn} \cos\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) e^{-j\beta_{mn}z}$$



- TE₀₀ mode does not exist
- TE₁₀ is the lowest order TE mode

Phase Constant

$$\beta_{mn} = \sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}$$

Same equation as for TM

Cutoff Frequency

$$f_{c,mn} = \frac{1}{2\pi\sqrt{\mu\epsilon}} \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$$

Same equation as for TM

Characteristic Impedance

$$Z_{TE,mn} = \frac{k\eta}{\beta_{mn}}$$

TM Analysis



Slide 33

Recall TM Analysis

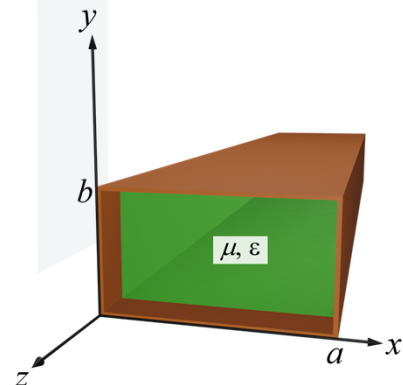
The governing equation for TM analysis is

$$\frac{\partial^2 E_{0,z}}{\partial x^2} + \frac{\partial^2 E_{0,z}}{\partial y^2} - k_c^2 E_{0,z} = 0 \quad H_{0,z} = 0 \quad k_c^2 = k^2 - \beta^2$$

After a solution is obtained, the remaining field components are calculated according to

$$H_{0,x} = \frac{j\omega\epsilon}{k_c^2} \frac{\partial E_{0,z}}{\partial y} \quad E_{0,x} = -\frac{j\beta}{k_c^2} \frac{\partial E_{0,z}}{\partial x}$$

$$H_{0,y} = -\frac{j\omega\epsilon}{k_c^2} \frac{\partial E_{0,z}}{\partial x} \quad E_{0,y} = -\frac{j\beta}{k_c^2} \frac{\partial E_{0,z}}{\partial y}$$



Slide 34

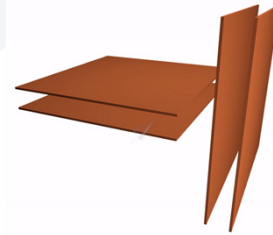
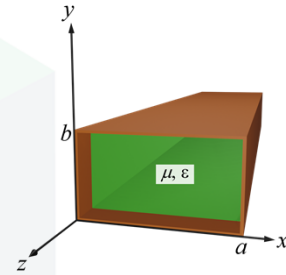
General Form of the Solution

From the geometry of the waveguide, the general form of the solution can be immediately written as

$$E_z(x, y, z) = E_{0,z}(x, y)e^{-j\beta z}$$

Viewing the rectangular waveguide as the combination of two parallel plate waveguides, apply separation of variables to write $E_{0,z}(x, y)$ as the product of two functions.

$$E_{0,z}(x, y) = X(x)Y(y)$$



Separation of Variables (1 of 3)

The solution is written as the product of two 1D functions, $X(x)$ and $Y(y)$. Substitute this solution back into the differential equation.

$$\frac{\partial^2 E_{0,z}}{\partial x^2} + \frac{\partial^2 E_{0,z}}{\partial y^2} - k_c^2 E_{0,z} = 0$$

$E_{0,z}(x, y) = X(x)Y(y)$



$$\frac{\partial^2 XY}{\partial x^2} + \frac{\partial^2 XY}{\partial y^2} - k_c^2 XY = 0$$

To be compact, drop the (x) and (y) notation.

$$\frac{\partial^2 X}{\partial x^2} Y + X \frac{\partial^2 Y}{\partial y^2} - k_c^2 XY = 0$$

Move $X(x)$ out of the $\partial/\partial x$ operation and $Y(y)$ out of the $\partial/\partial y$ operation.

$$\frac{1}{X} \frac{d^2 X}{dx^2} + \frac{1}{Y} \frac{d^2 Y}{dy^2} - k_c^2 = 0$$

The derivatives become ordinary because $X(x)$ and $Y(y)$ have only one independent variable each.

Separation of Variables (2 of 3)

First, attention is focused on the x -dependence in the differential equation.

$$\frac{1}{X} \frac{d^2 X}{dx^2} + \frac{1}{Y} \frac{d^2 Y}{dy^2} - k_c^2 = 0$$

$\underbrace{\hspace{10em}}_{-k_x^2}$

This definition of k_x lets the differential equation be written as a wave equation.

$$\frac{d^2 X}{dx^2} - k_x^2 X = 0$$

Second, attention is focused on the y -dependence in the differential equation.

$$\frac{1}{X} \frac{d^2 X}{dx^2} - k_c^2 + \frac{1}{Y} \frac{d^2 Y}{dy^2} = 0$$

$\underbrace{\hspace{10em}}_{-k_y^2}$

This definition of k_y lets the differential equation be written as a wave equation.

$$\frac{d^2 Y}{dy^2} - k_y^2 Y = 0$$

Separation of Variables (3 of 3)

It should be possible to add the two new differential equations together to get the original differential equation.

$$\begin{array}{r} \frac{d^2 X}{dx^2} - k_x^2 X = 0 \\ \frac{d^2 Y}{dy^2} - k_y^2 Y = 0 \end{array} \quad \Longrightarrow \quad \begin{array}{r} \frac{1}{X} \frac{d^2 X}{dx^2} - k_x^2 = 0 \\ + \frac{1}{Y} \frac{d^2 Y}{dy^2} - k_y^2 = 0 \\ \hline \frac{1}{X} \frac{d^2 X}{dx^2} + \frac{1}{Y} \frac{d^2 Y}{dy^2} - k_x^2 - k_y^2 = 0 \end{array}$$

The original differential equation is obtained if

$$k_c^2 = k_x^2 + k_y^2$$

Original differential equation

$$\frac{1}{X} \frac{d^2 X}{dx^2} + \frac{1}{Y} \frac{d^2 Y}{dy^2} - k_c^2 = 0$$

General Solution

There are now two differential equations to solve.

$$\frac{d^2 X}{dx^2} - k_x^2 X = 0 \qquad \frac{d^2 Y}{dy^2} - k_y^2 Y = 0$$

These are essentially the same differential equation so their solution has the same general form.

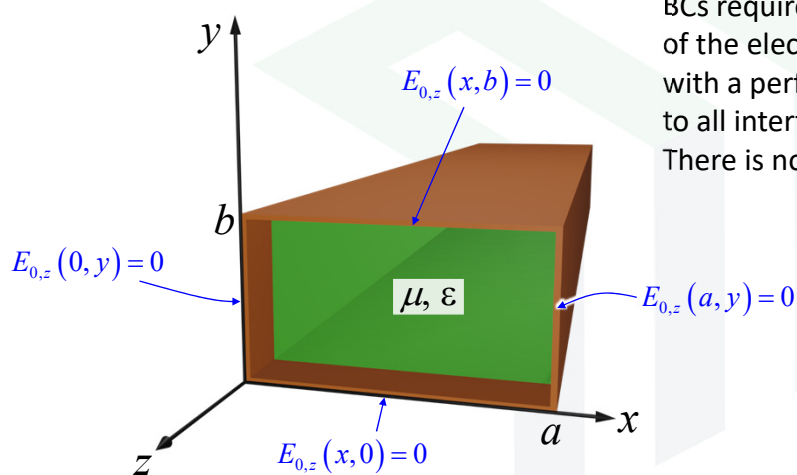
$$\frac{d^2 X}{dx^2} - k_x^2 X = 0 \rightarrow X(x) = A \cos(k_x x) + B \sin(k_x x) \quad \text{PP waveguide along } x$$

$$\frac{d^2 Y}{dy^2} - k_y^2 Y = 0 \rightarrow Y(y) = C \cos(k_y y) + D \sin(k_y y) \quad \text{PP waveguide along } y$$

The overall solution is the product of $X(x)$ and $Y(y)$.

$$E_{0,z}(x, y) = X(x)Y(y) = [A \cos(k_x x) + B \sin(k_x x)][C \cos(k_y y) + D \sin(k_y y)]$$

Electromagnetic Boundary Conditions



BCs require that the tangential component of the electric field be zero at the boundary with a perfect conductor. $E_{0,z}$ is tangential to all interfaces so it is just used directly. There is no need to calculate $E_{0,x}$ or $E_{0,y}$.

Apply Boundary Conditions (1 of 2)

At the $x = 0$ boundary,

$$\begin{aligned} 0 &= E_{0,z}(0, y) \\ &= [A \cos(0) + \cancel{B \sin(0)}] [C \cos(k_y y) + D \sin(k_y y)] \\ &= [A] [C \cos(k_y y) + D \sin(k_y y)] \longrightarrow A = 0 \end{aligned}$$

At the $x = a$ boundary,

$$\begin{aligned} 0 &= E_{0,z}(a, y) \\ &= [B \sin(k_x a)] [C \cos(k_y y) + D \sin(k_y y)] \end{aligned}$$

$B = 0$ leads to a trivial solution. It must be the $\sin(k_x a)$ term that enforces the BC.

$$0 = \sin(k_x a) \rightarrow k_x a = m\pi \quad m = 1, 2, \dots \quad m = 0 \text{ leads to a trivial solution.}$$

Apply Boundary Conditions (2 of 2)

At the $y = 0$ boundary,

$$\begin{aligned} 0 &= E_{0,z}(x, 0) \\ &= [A \cos(k_x x) + B \sin(k_x x)] [C \cos(0) + \cancel{D \sin(0)}] \\ &= [A \cos(k_x x) + B \sin(k_x x)] [C] \longrightarrow C = 0 \end{aligned}$$

At the $y = b$ boundary,

$$\begin{aligned} 0 &= E_{0,z}(x, b) \\ &= [A \cos(k_x x) + B \sin(k_x x)] [D \sin(k_y b)] \end{aligned}$$

$D = 0$ leads to a trivial solution. It must be the $\sin(k_y b)$ term that enforces the BC.

$$0 = \sin(k_y b) \rightarrow k_y b = n\pi \quad n = 1, 2, \dots \quad n = 0 \text{ leads to a trivial solution.}$$

Revised Solution for $E_{0,z}$

It was determined that $A = C = 0$ so the expression for $E_{0,z}$ becomes

$$E_{0,z}(x, y) = \boxed{BD} \sin(k_x x) \sin(k_y y)$$

The product BD is written as a single constant B_{mn} .

$$E_{0,z}(x, y) = B_{mn} \sin(k_x x) \sin(k_y y)$$

Also, recall the conditions for k_x and k_y .

$$k_x a = m\pi \rightarrow k_x = \frac{m\pi}{a} \quad k_y b = n\pi \rightarrow k_y = \frac{n\pi}{b}$$

$$E_{0,z}(x, y) = B_{mn} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right)$$

* Note: Neither m nor n can be zero or the entire solution will be zero.

Entire Solution (1 of 2)

The final expression for $E_{0,z}$ is

$$E_{0,z}(x, y) = B_{mn} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \quad H_{0,z}(x, y) = 0$$

From this, the other field components are

$$E_{0,x}(x, y) = -\frac{j\beta_{mn} m\pi}{k_c^2 a} B_{mn} \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right)$$

$$E_{0,y}(x, y) = -\frac{j\beta_{mn} n\pi}{k_c^2 b} B_{mn} \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right)$$

$$H_{0,x}(x, y) = \frac{j\omega\epsilon n\pi}{k_c^2 b} B_{mn} \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right)$$

$$H_{0,y}(x, y) = -\frac{j\omega\epsilon m\pi}{k_c^2 a} B_{mn} \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right)$$

Entire Solution (2 of 2)

The overall electric and magnetic fields at any position are

$$E_x(x, y, z) = -\frac{j\beta_{mn}m\pi}{k_c^2 a} B_{mn} \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) e^{-j\beta_{mn}z}$$

$$E_y(x, y, z) = -\frac{j\beta_{mn}n\pi}{k_c^2 b} B_{mn} \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) e^{-j\beta_{mn}z}$$

$$E_z(x, y, z) = B_{mn} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) e^{-j\beta_{mn}z}$$

$$H_x(x, y, z) = \frac{j\omega\epsilon n\pi}{k_c^2 b} B_{mn} \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) e^{-j\beta_{mn}z}$$

$$H_y(x, y, z) = -\frac{j\omega\epsilon m\pi}{k_c^2 a} B_{mn} \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) e^{-j\beta_{mn}z}$$

$$H_z(x, y, z) = 0$$

Phase Constant, β

Recall the cutoff wave number

$$k_c^2 = k_x^2 + k_y^2$$

After analyzing the boundary conditions, this expression can be written as

$$k_c^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2$$

The phase constant β is therefore

$$k_c^2 = k^2 - \beta^2$$

$$\beta^2 = k^2 - k_c^2$$

$$\beta_{mn} = \sqrt{k^2 - k_c^2} = \sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}$$

Cutoff Frequency, f_c

Recall the expression for the phase constant

$$\beta_{mn} = \sqrt{k^2 - k_c^2}$$

The phase constant must be a real number for a guided mode. This requires

$$k > k_c$$

Any time $k < k_c$, the mode is cutoff and not supported by the waveguide. From this, the cutoff frequency f_c can be derived as

$$\begin{aligned}
 k &> k_c \\
 \omega\sqrt{\mu\varepsilon} &> k_c \\
 2\pi f_c\sqrt{\mu\varepsilon} &= k_c
 \end{aligned}
 \quad
 \boxed{f_{c,mn} = \frac{k_c}{2\pi\sqrt{\mu\varepsilon}} = \frac{1}{2\pi\sqrt{\mu\varepsilon}} \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}}$$

This is the same equation as for the TE modes.

Characteristic Impedance, Z_{TM}

The characteristic impedance Z_{TM} for the TM mode is

$$Z_{TM} = \frac{E_x}{H_y} = \frac{-\frac{j\beta_{mn}m\pi}{k_c^2 a} B_{mn} \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) e^{-j\beta z}}{-\frac{j\omega\varepsilon m\pi}{k_c^2 a} B_{mn} \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) e^{-j\beta z}} = \frac{\beta_{mn}}{\omega\varepsilon} = \frac{\beta_{mn}\eta}{k}$$

Cutoff for First-Order TM Mode (1 of 2)

The cutoff frequency for the TM_{mn} mode was found to be

$$f_{c,mn} = \frac{1}{2\pi\sqrt{\mu\varepsilon}} \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$$

Note, it is not possible to have $n = 0$ or $m = 0$ for the TM mode. So...

The TM_{00} mode does not exist.
 The TM_{01} mode does not exist.
 The TM_{10} mode does not exist.
 The TM_{02} mode does not exist.
 The TM_{20} mode does not exist.
 The TM_{03} mode does not exist.
 The TM_{30} mode does not exist.

etc.

Cutoff for First-Order TM Mode (2 of 2)

What combination of m and n minimizes f_c ?

$$f_{c,mn} = \frac{1}{2\pi\sqrt{\mu\varepsilon}} \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$$

The TM_{11} mode will have the lowest cutoff frequency.

$$m = 1, n = 1$$

$$f_{c1} = \frac{1}{2\pi\sqrt{\mu\varepsilon}} \sqrt{\left(\frac{1\cdot\pi}{a}\right)^2 + \left(\frac{1\cdot\pi}{b}\right)^2} = \frac{1}{2\sqrt{\mu\varepsilon}} \sqrt{\left(\frac{1}{a}\right)^2 + \left(\frac{1}{b}\right)^2}$$

CAUTION: It cannot yet be said that the TM_{11} is the fundamental mode because the TE modes have not been checked.

Example #2 – TM Mode Analysis (1 of 3)

Given an air-filled rectangular waveguide with $a = 3$ cm and $b = 2$ cm.

What is the cutoff frequency of the waveguide?

$$\begin{aligned}
 f_{c1} &= \frac{1}{2\pi\sqrt{\mu\epsilon}} \sqrt{\left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2} \\
 &= \frac{1}{2\pi\sqrt{\mu_0\mu_r\epsilon_0\epsilon_r}} \sqrt{\left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2} \\
 &= \frac{1}{2\pi\sqrt{\mu_0\epsilon_0}\sqrt{\mu_r\epsilon_r}} \sqrt{\left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2} \\
 &= \frac{c_0}{2\sqrt{\mu_r\epsilon_r}} \sqrt{\left(\frac{1}{a}\right)^2 + \left(\frac{1}{b}\right)^2} \\
 &= \frac{299792458 \text{ m/s}}{2\sqrt{(1.0)(1.0)}} \sqrt{\left(\frac{1}{0.03 \text{ m}}\right)^2 + \left(\frac{1}{0.02 \text{ m}}\right)^2} = \boxed{9.0 \text{ GHz}}
 \end{aligned}$$

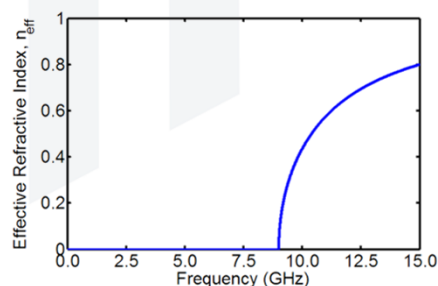
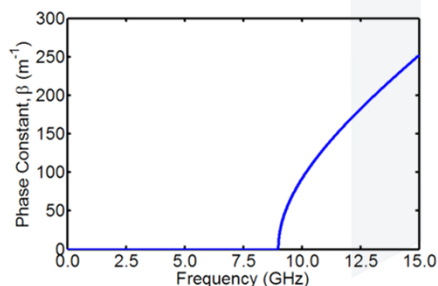
Recall that $c_0 = \frac{1}{\sqrt{\mu_0\epsilon_0}}$

Example #2 – TM Mode Analysis (2 of 3)

Plot the phase constant and effective refractive index for the first-order mode from DC up to 15 GHz.

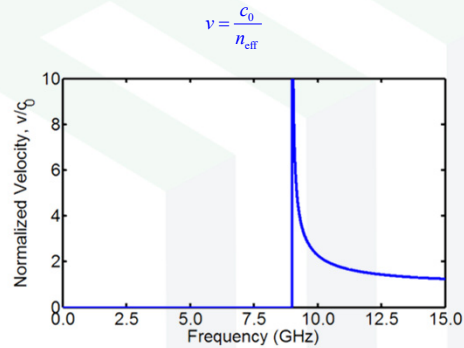
The phase constant is calculated as: $\beta_{mn} = \sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2} \rightarrow \text{TM}_{11}: \beta = \sqrt{\left(\frac{2\pi f}{c_0}\right)^2 - \left(\frac{\pi}{a}\right)^2}$

The effective refractive index is calculated as: $\beta = k_0 n_{\text{eff}} \rightarrow n_{\text{eff}} = \frac{\beta}{2\pi f} = \beta \frac{c_0}{2\pi f}$



Example #2 – TM Mode Analysis (3 of 3)

Plot the velocity of the modes as a function of frequency.



Are our modes travelling faster than the speed of light?

Summary of TM Analysis

Field Solution

$$E_x(x, y, z) = -\frac{j\beta_{mn}m\pi}{k_c^2 a} B_{mn} \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) e^{-j\beta_{mn}z}$$

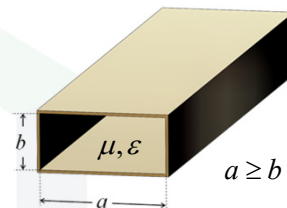
$$E_y(x, y, z) = -\frac{j\beta_{mn}n\pi}{k_c^2 b} B_{mn} \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) e^{-j\beta_{mn}z}$$

$$E_z(x, y, z) = B_{mn} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) e^{-j\beta_{mn}z}$$

$$H_x(x, y, z) = \frac{j\omega\epsilon n\pi}{k_c^2 b} B_{mn} \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) e^{-j\beta_{mn}z}$$

$$H_y(x, y, z) = -\frac{j\omega\epsilon m\pi}{k_c^2 a} B_{mn} \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) e^{-j\beta_{mn}z}$$

$$H_z(x, y, z) = 0$$



- $m \neq 0$ and $n \neq 0$, so TM_{00} , TM_{01} , TM_{02} , TM_{10} , TM_{20} , etc. are not supported modes.
- TM_{11} is the lowest order TM mode

Phase Constant

$$\beta_{mn} = \sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}$$

Same equation as for TE

Cutoff Frequency

$$f_{c,mn} = \frac{1}{2\pi\sqrt{\mu\epsilon}} \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$$

Same equation as for TE

Characteristic Impedance

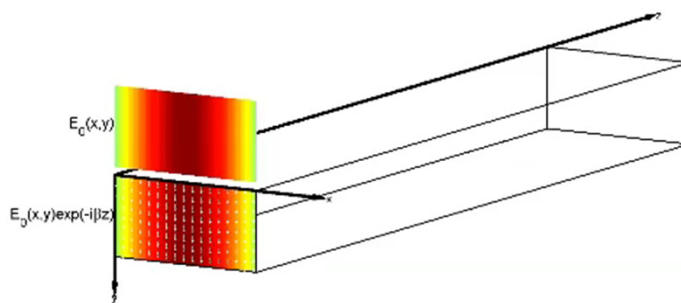
$$Z_{TM,mn} = \frac{\eta\beta_{mn}}{k}$$

Visualization of Modes



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Animation of TE_{10}

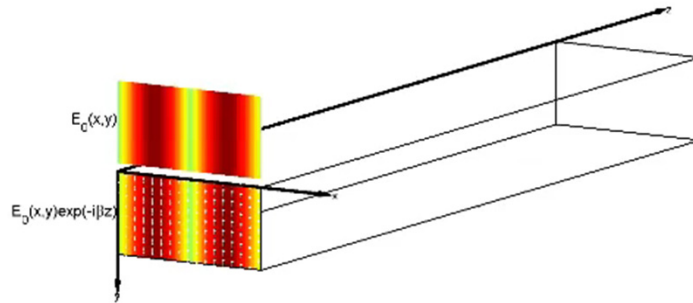


Notice one bright spot along x and zero along y . ($m = 1, n = 0$)



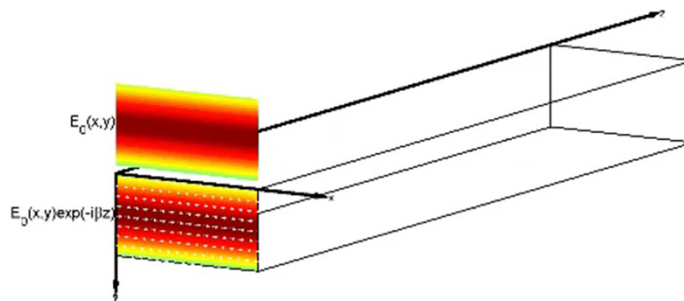
Slide 55

Animation of TE_{20}



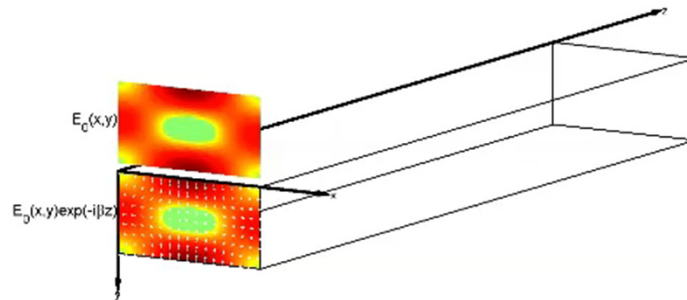
Notice two bright spots along x and zero along y . ($m = 2, n = 0$)

Animation of TE_{01}



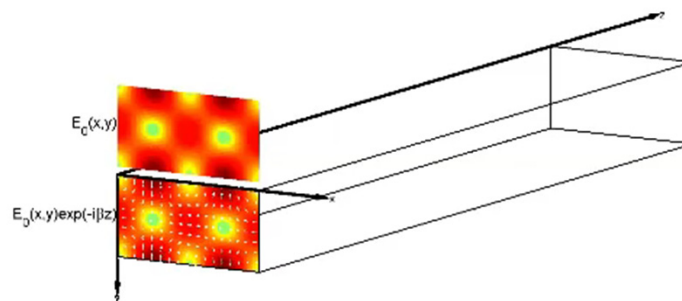
Notice zero bright spots along x and one along y . ($m = 0, n = 1$)

Animation of TE_{11}



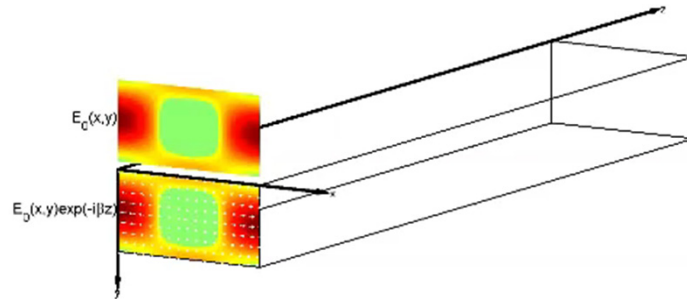
Notice one bright spot along x and one along y . ($m = 1, n = 1$)

Animation of TE_{21}



Notice two bright spots along x and one along y . ($m = 2, n = 1$)

Animation of TM_{11}



Conclusions

The Fundamental Mode

The fundamental mode is the mode which has the lowest cutoff frequency. This is either the TE₁₀ or the TM₁₁ mode.

$$f_{c,TE} = \frac{1}{2\sqrt{\mu\epsilon}} \sqrt{\left(\frac{1}{a}\right)^2}$$

$$f_{c,TM} = \frac{1}{2\sqrt{\mu\epsilon}} \sqrt{\left(\frac{1}{a}\right)^2 + \left(\frac{1}{b}\right)^2}$$

It can be observed that the TE₁₀ mode will always have the lowest cutoff frequency.

It is concluded that the TE₁₀ mode is the fundamental mode of the waveguide.

This is also called the dominant mode. When multiple modes are excited, usually most of the power ends up in the fundamental mode.

Example #3 – Mode Analysis (1 of 3)

Given an air-filled rectangular waveguide with $a = 4$ cm and $b = 2$ cm,

Over what range of frequencies is this waveguide single mode?

An easy way to do this is to calculate a table using a desktop computer.

m	n	TE Cutoff	TM Cutoff
0	0		
1	0	3.74 GHz	
2	0	7.48 GHz	
3	0	11.22 GHz	
4	0	14.96 GHz	
0	1	7.48 GHz	
1	1	8.37 GHz	8.37 GHz
2	1	10.58 GHz	10.58 GHz
3	1	13.49 GHz	13.49 GHz
4	1	16.73 GHz	16.73 GHz
0	2	14.96 GHz	
1	2	15.43 GHz	15.43 GHz
2	2	16.73 GHz	16.73 GHz
3	2	18.71 GHz	18.71 GHz
4	2	21.16 GHz	21.16 GHz
0	3	22.45 GHz	
1	3	22.76 GHz	22.76 GHz
2	3	23.66 GHz	23.66 GHz
3	3	25.10 GHz	25.10 GHz
4	3	26.98 GHz	26.98 GHz
0	4	29.93 GHz	
1	4	30.16 GHz	30.16 GHz
2	4	30.85 GHz	30.85 GHz
3	4	31.96 GHz	31.96 GHz
4	4	33.46 GHz	33.46 GHz

Example #3 – Mode Analysis (2 of 3)

Given an air-filled rectangular waveguide with $a = 4$ cm and $b = 2$ cm,

Over what range of frequencies is this waveguide single mode?

An easy way to do this is to calculate a table using a desktop computer.

Then sort the table in order of increasing cutoff frequency.

Mode	Cutoff
TE10	3.74 GHz
TE20	7.48 GHz
TE01	7.48 GHz
TE11	8.37 GHz
TM11	8.37 GHz
TE21	10.58 GHz
TM21	10.58 GHz
TE30	11.22 GHz
TE31	13.49 GHz
TM31	13.49 GHz
TE40	14.96 GHz
TE02	14.96 GHz
TE12	15.43 GHz
TM12	15.43 GHz
TE41	16.73 GHz

Example #3 – Mode Analysis (3 of 3)

Given an air-filled rectangular waveguide with $a = 4$ cm and $b = 2$ cm,

Over what range of frequencies is this waveguide single mode?

An easy way to do this is to calculate a table using a desktop computer.

Then sort the table in order of increasing cutoff frequency.

It is immediately seen that the TE_{10} mode is the fundamental mode with the lowest cutoff frequency of 3.74 GHz.

The second-order mode is taken from the table to be either the TE_{20} or TE_{01} mode because both of these have the same cutoff frequency of 7.48 GHz.

The overall range of frequencies for single-mode operation is therefore

$$3.74 \text{ GHz} < f < 7.48 \text{ GHz}$$

Mode	Cutoff
TE10	3.74 GHz
TE20	7.48 GHz
TE01	7.48 GHz
TE11	8.37 GHz
TM11	8.37 GHz
TE21	10.58 GHz
TM21	10.58 GHz
TE30	11.22 GHz
TE31	13.49 GHz
TM31	13.49 GHz
TE40	14.96 GHz
TE02	14.96 GHz
TE12	15.43 GHz
TM12	15.43 GHz
TE41	16.73 GHz

Key Points

- The rectangular waveguide is not a transmission line because it has less than two conductors.
- When filled with a homogeneous dielectric, the rectangular waveguide supports TE and TM modes, but not TEM.
- The cutoff frequencies for TE_{mn} and TM_{mn} modes are the same.
- The TE_{00} mode does not exist.
- For TM_{mn} modes, $m \neq 0$ and $n \neq 0$.
- The TE_{10} is the dominant mode because the TM_{10} mode does not exist.
- Phase velocity of the modes exceeds the vacuum speed of light.

Summary of Rectangular Waveguide

Parameter	TE_{mn} $m = n = 0$ not allowed	TM_{mn} $m \neq 0$ and $n \neq 0$
k	$\omega\sqrt{\mu\epsilon}$	$\omega\sqrt{\mu\epsilon}$
k_c	$\sqrt{(m\pi/a)^2 - (n\pi/b)^2}$	$\sqrt{(m\pi/a)^2 - (n\pi/b)^2}$
β	$\sqrt{k^2 - k_c^2}$	$\sqrt{k^2 - k_c^2}$
λ_c	$2\pi/k_c$	$2\pi/k_c = 2d/n$
λ_g	$2\pi / \beta_{mn}$	$2\pi / \beta_{mn}$
v_p	ω / β_{mn}	ω / β_{mn}
α_d	$k^2 \tan \delta / 2\beta_{mn}$	$k^2 \tan \delta / 2\beta_{mn}$
E_x	$\frac{j\omega\mu\pi}{k_c^2 b} A_m \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) e^{-j\beta_{mn} z}$	$-\frac{j\beta_{mn} m\pi}{k_c^2 a} B_m \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) e^{-j\beta_{mn} z}$
E_y	$-\frac{j\omega\mu\pi}{k_c^2 a} A_m \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) e^{-j\beta_{mn} z}$	$-\frac{j\beta_{mn} n\pi}{k_c^2 b} B_m \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) e^{-j\beta_{mn} z}$
E_z	0	$B_m \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) e^{-j\beta_{mn} z}$
H_x	$-\frac{j\beta_{mn} m\pi}{k_c^2 a} A_m \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) e^{-j\beta_{mn} z}$	$\frac{j\omega\epsilon n\pi}{k_c^2 b} B_m \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) e^{-j\beta_{mn} z}$
H_y	$\frac{j\beta_{mn} n\pi}{k_c^2 b} A_m \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) e^{-j\beta_{mn} z}$	$-\frac{j\omega\epsilon m\pi}{k_c^2 a} B_m \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) e^{-j\beta_{mn} z}$
H_z	$A_m \cos\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) e^{-j\beta_{mn} z}$	0
Z	$k\eta / \beta_{mn}$	$\beta_{mn} \eta / k$