Electromagnetics:

Electromagnetic Field Theory

# Waveguide Analysis Setup

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## Lecture Outline

- Governing Equations for Waveguide Analysis
- Reduced Set of Equations for Waveguide Analysis
- Analysis Setup
  - for hybrid modes
  - for TEM analysis
  - for TE and TM analysis
  - for analyzing slab waveugides

Note, this lecture only covers setting up Maxwell's equations for analysis of waveguides. This lecture does not attempt to obtain solutions.

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# Governing Equations for Waveguide Analysis



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## Steps for Waveguide Analysis

1. Draw the waveguide.
2. Assume a form of the solution. Outer regions must decay exponentially or be equal to zero.
3. Substitute solution into Maxwell's equations.
4. Simplify equations based on the geometry of the waveguide.
5. Manipulate equations into a differential equation to solve. This is called the *governing equation*.
6. Solve the governing equation in each homogeneous region of the waveguide.
7. "Connect" the solutions in each region using boundary conditions.
8. Calculate the overall field solution.
9. Use the field solution to calculate the waveguide parameters such as $\beta$, $Z_0$, and the profile of the fields.

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# Various Wave Equations

**1. Maxwell's Curl Equations**

$$\nabla \times \vec{E} = -j\omega\mu\vec{H}$$
$$\nabla \times \vec{H} = j\omega\varepsilon\vec{E}$$

**2. Wave Equation in General Media**

$$\nabla \times \vec{E} = -j\omega\mu\vec{H} \quad \rightarrow \quad \vec{H} = -\frac{\nabla \times \vec{E}}{j\omega\mu}$$
$$\nabla \times \vec{H} = j\omega\varepsilon\vec{E}$$
$$\nabla \times \left( -\frac{\nabla \times \vec{E}}{j\omega\mu} \right) = j\omega\varepsilon\vec{E}$$
$$\nabla \times \left( \mu^{-1} \nabla \times \vec{E} \right) = \omega^2 \varepsilon \vec{E}$$

**3. Wave Equation in LHI Media**

$$\nabla \times \left( \mu^{-1} \nabla \times \vec{E} \right) = \omega^2 \varepsilon \vec{E}$$
$$\nabla \times \nabla \times \vec{E} = \omega^2 \mu\varepsilon \vec{E}$$
$$\nabla \left( \cancel{\nabla \bullet \vec{E}} \right) - \nabla^2 \vec{E} = k^2 \vec{E}$$
$$\nabla^2 \vec{E} + k^2 \vec{E} = 0$$
$$k = \omega\sqrt{\mu\varepsilon} \equiv \text{wave number}$$

**4. Wave Equation Decouples**

$$\nabla^2 E_x + k^2 E_x = 0$$
$$\nabla^2 E_y + k^2 E_y = 0$$
$$\nabla^2 E_z + k^2 E_z = 0$$

These equations are solved independently.

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# Expand Maxwell's Equations

Maxwell's equations are used to analyze waveguides.

$$\nabla \times \vec{E} = -j\omega\mu\vec{H} \qquad \nabla \times \vec{H} = j\omega\varepsilon\vec{E}$$

The two curl equations expand into a set of six coupled partial differential equations.

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -j\omega\mu H_x \qquad \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = j\omega\varepsilon E_x$$
$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -j\omega\mu H_y \qquad \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = j\omega\varepsilon E_y$$
$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -j\omega\mu H_z \qquad \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = j\omega\varepsilon E_z$$

There are six field components to solve for: $E_x$, $E_y$, $E_z$, $H_x$, $H_y$, and $H_z$.

Yikes!! ☹

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# General Form of Solution for Waveguides

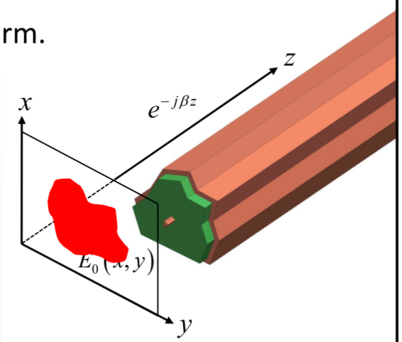
A mode in a waveguide has the following general mathematical form.

$$\vec{E}(x,y,z) = \vec{E}_0(x,y)e^{-j\beta z}$$

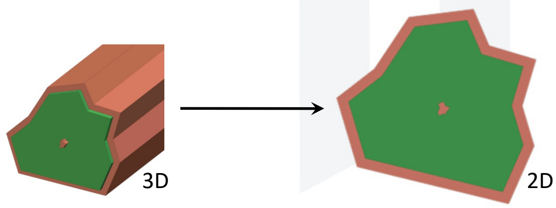
complex amplitude, mode shape

accumulation of phase in $z$ direction

$$\beta \equiv \text{phase constant}$$




This means the problem can be solved by just analyzing the cross section in the $x$-$y$ plane. The problem reduces mathematically to two dimensions.

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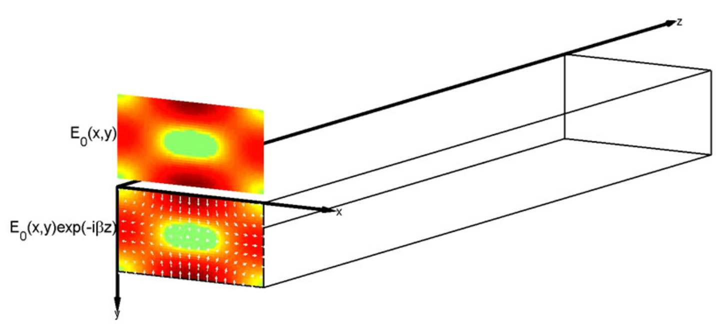
# Animation of a Waveguide Mode

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# Assume the Form of the Solution

For a waveguide uniform in the $z$ direction, the solution will have the form

$$\vec{E}(x,y,z)=\vec{E}_0(x,y)e^{-j\beta z} \qquad \vec{H}(x,y,z)=\vec{H}_0(x,y)e^{-j\beta z}$$

Substituting this solution into the set of six equations gives

$$\frac{\partial E_{0,z}}{\partial y}+j\beta E_{0,y}=-j\omega\mu H_{0,x} \qquad \frac{\partial H_{0,z}}{\partial y}+j\beta H_{0,y}=j\omega\varepsilon E_{0,x}$$

$$-j\beta E_{0,x}-\frac{\partial E_{0,z}}{\partial x}=-j\omega\mu H_{0,y} \qquad -j\beta H_{0,x}-\frac{\partial H_{0,z}}{\partial x}=j\omega\varepsilon E_{0,y}$$

$$\frac{\partial E_{0,y}}{\partial x}-\frac{\partial E_{0,x}}{\partial y}=-j\omega\mu H_{0,z} \qquad \frac{\partial H_{0,y}}{\partial x}-\frac{\partial H_{0,x}}{\partial y}=j\omega\varepsilon E_{0,z}$$

Things are a little simpler, but there are still six field components to solve for. ☹

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# Reduced Set of Equations for Waveguide Analysis

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# Reducing Number of Terms

It is possible to put $E_{0,x}$, $E_{0,y}$, $H_{0,x}$, and $H_{0,y}$ in terms of just $E_{0,z}$ and $H_{0,z}$.

$$\frac{\partial E_{0,z}}{\partial y}+j\beta E_{0,y}=-j\omega\mu H_{0,x}$$
$$-j\beta E_{0,x}-\frac{\partial E_{0,z}}{\partial x}=-j\omega\mu H_{0,y}$$
$$\frac{\partial E_{0,y}}{\partial x}-\frac{\partial E_{0,x}}{\partial y}=-j\omega\mu H_{0,z}$$

$$\frac{\partial H_{0,z}}{\partial y}+j\beta H_{0,y}=j\omega\varepsilon E_{0,x}$$
$$-j\beta H_{0,x}-\frac{\partial H_{0,z}}{\partial x}=j\omega\varepsilon E_{0,y}$$
$$\frac{\partial H_{0,y}}{\partial x}-\frac{\partial H_{0,x}}{\partial y}=j\omega\varepsilon E_{0,z}$$

$\Rightarrow$

$$E_{0,x}=-\frac{j}{k^2-\beta^2}\left(\beta\frac{\partial E_{0,z}}{\partial x}+\omega\mu\frac{\partial H_{0,z}}{\partial y}\right)$$
$$E_{0,y}=\frac{j}{k^2-\beta^2}\left(-\beta\frac{\partial E_{0,z}}{\partial y}+\omega\mu\frac{\partial H_{0,z}}{\partial x}\right)$$
$$H_{0,x}=\frac{j}{k^2-\beta^2}\left(\omega\varepsilon\frac{\partial E_{0,z}}{\partial y}-\beta\frac{\partial H_{0,z}}{\partial x}\right)$$
$$H_{0,y}=-\frac{j}{k^2-\beta^2}\left(\omega\varepsilon\frac{\partial E_{0,z}}{\partial x}+\beta\frac{\partial H_{0,z}}{\partial y}\right)$$

Now to analyze a waveguide, it is only necessary to solve for $E_{0,z}$ and $H_{0,z}$.

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# Reduce the Number of Terms to Solve (1 of 2)

$$\frac{\partial E_{0,z}}{\partial y}+j\beta E_{0,y}=-j\omega\mu H_{0,x} \quad \text{Eq. (1a)}$$
$$-j\beta E_{0,x}-\frac{\partial E_{0,z}}{\partial x}=-j\omega\mu H_{0,y} \quad \text{Eq. (1b)}$$
$$\frac{\partial E_{0,y}}{\partial x}-\frac{\partial E_{0,x}}{\partial y}=-j\omega\mu H_{0,z} \quad \text{Eq. (1c)}$$
$$\frac{\partial H_{0,z}}{\partial y}+j\beta H_{0,y}=j\omega\varepsilon E_{0,x} \quad \text{Eq. (1d)}$$
$$-j\beta H_{0,x}-\frac{\partial H_{0,z}}{\partial x}=j\omega\varepsilon E_{0,y} \quad \text{Eq. (1e)}$$
$$\frac{\partial H_{0,y}}{\partial x}-\frac{\partial H_{0,x}}{\partial y}=j\omega\varepsilon E_{0,z} \quad \text{Eq. (1f)}$$

Step 1 – Solve Eq. (1e) for $E_{0,y}$.

$$E_{0,y}=\frac{1}{j\omega\varepsilon}\left(-j\beta H_{0,x}-\frac{\partial H_{0,z}}{\partial x}\right)$$

Step 2 – Substitute this expression into Eq. (1a) to eliminate $E_{0,y}$.

$$\frac{\partial E_{0,z}}{\partial y}+j\beta\left[\frac{1}{j\omega\varepsilon}\left(-j\beta H_{0,x}-\frac{\partial H_{0,z}}{\partial x}\right)\right]=-j\omega\mu H_{0,x}$$

Step 3 – Recall that $k^2=\omega^2\mu\varepsilon$ and solve this new expression for $H_{0,x}$.

$$H_{0,x}=\frac{j}{k^2-\beta^2}\left(\omega\varepsilon\frac{\partial E_{0,z}}{\partial y}-\beta\frac{\partial H_{0,z}}{\partial x}\right)$$

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## Reduce the Number of Terms to Solve (2 of 2)

Step 4 – Derive three more similar equations.

Solve Eq. (1d) for $E_{0,x}$, substitute that expression into Eq. (1b) and solve for $H_{0,y}$.

$$H_{0,y} = -\frac{j}{k^2-\beta^2}\left(\omega\varepsilon\frac{\partial E_{0,z}}{\partial x}+\beta\frac{\partial H_{0,z}}{\partial y}\right)$$

Solve Eq. (1b) for $H_{0,y}$, substitute that expression into Eq. (1d) and solve for $E_{0,x}$.

$$E_{0,x} = -\frac{j}{k^2-\beta^2}\left(\beta\frac{\partial E_{0,z}}{\partial x}+\omega\mu\frac{\partial H_{0,z}}{\partial y}\right)$$

Solve Eq. (1a) for $H_{0,x}$, substitute that expression into Eq. (1e) and solve for $E_{0,y}$.

$$E_{0,y} = \frac{j}{k^2-\beta^2}\left(-\beta\frac{\partial E_{0,z}}{\partial y}+\omega\mu\frac{\partial H_{0,z}}{\partial x}\right)$$

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## Reduced Set of Equations

Step 5 – Define the *cutoff wave number* $k_c$ as

$$k_c^2 = k^2 - \beta^2$$

This term will have more meaning later on.

Now all of the transverse field components $E_{0,x}$, $E_{0,y}$, $H_{0,x}$ and $H_{0,y}$ are expressed in terms of just the two longitudinal components $E_{0,z}$ and $H_{0,z}$.

$$H_{0,x} = \frac{j}{k_c^2}\left(\omega\varepsilon\frac{\partial E_{0,z}}{\partial y}-\beta\frac{\partial H_{0,z}}{\partial x}\right) \qquad E_{0,x} = -\frac{j}{k_c^2}\left(\beta\frac{\partial E_{0,z}}{\partial x}+\omega\mu\frac{\partial H_{0,z}}{\partial y}\right)$$

$$H_{0,y} = -\frac{j}{k_c^2}\left(\omega\varepsilon\frac{\partial E_{0,z}}{\partial x}+\beta\frac{\partial H_{0,z}}{\partial y}\right) \qquad E_{0,y} = \frac{j}{k_c^2}\left(-\beta\frac{\partial E_{0,z}}{\partial y}+\omega\mu\frac{\partial H_{0,z}}{\partial x}\right)$$

Analyzing a waveguide reduces to just solving for $E_{0,z}$ and $H_{0,z}$. The remaining field components can be calculated directly from these two terms.

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# How To Find $E_{0,z}$ and $H_{0,z}$?

Recall that in LHI media, the wave equation simplified to

$$\nabla \times \left( \mu^{-1} \nabla \times \vec{E} \right) = \omega^2 \varepsilon \vec{E} \qquad \nabla \times \left( \varepsilon^{-1} \nabla \times \vec{H} \right) = \omega^2 \mu \vec{H}$$

$$\downarrow \qquad\qquad \downarrow$$

$$\nabla^2 \vec{E} + k^2 \vec{E} = 0 \qquad \nabla^2 \vec{H} + k^2 \vec{H} = 0$$

$$\downarrow \qquad\qquad \downarrow$$

$$\nabla^2 E_x + k^2 E_x = 0 \qquad \nabla^2 \vec{H}_x + k^2 \vec{H}_x = 0$$

$$\nabla^2 E_y + k^2 E_y = 0 \qquad \nabla^2 \vec{H}_y + k^2 \vec{H}_y = 0$$

$$\nabla^2 E_z + k^2 E_z = 0 \qquad \nabla^2 \vec{H}_z + k^2 \vec{H}_z = 0$$

Substituting the solution $E_z = E_{0,z} e^{-j\beta z}$ into the bottom equations above gives

$$\boxed{\nabla^2 E_{0,z} + k_c^2 E_{0,z} = 0} \qquad \boxed{\nabla^2 H_{0,z} + k_c^2 H_{0,z} = 0}$$

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# Solution Categories

**Hybrid Mode Analysis**

$E_{0,z} \neq 0$ and $H_{0,z} \neq 0$

Can analyze any waveguide.
Usually very complicated.

- No simplifications to the analysis are possible.
- Solutions contain all six field components

**TEM Analysis**

$E_{0,z} = H_{0,z} = 0$

- Must be a transmission line,
- and must have a homogeneous dielectric

- **Transverse electromagnetic (TEM) mode**
- Both electric and magnetic fields are transverse to the direction of propagation.

**TE & TM Analysis**

$E_{0,z} = 0$ or $H_{0,z} = 0$, but not both

- Homogeneous dielectric,
- or has a dielectric uniform in one direction

- **Transverse electric (TE) mode**
- $E_{0,z} = 0$ so the electric field is completely transverse to the direction of propagation.

- **Transverse magnetic (TM) mode**
- $H_{0,z} = 0$ so the magnetic field is completely transverse to the direction of propagation.

**Slab Waveguide Analysis**

- Material properties uniform and infinite along one direction

- Maxwell's equations simplify and decouple into two independent sets of modes.

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# Setup for Analyzing Hybrid Modes

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# Eliminate $E_{0,z}$ and $H_{0,z}$

To setup for hybrid modes, back up to Maxwell's equations in linear and isotropic media (i.e. can be inhomogeneous).

$$\frac{\partial E_{0,z}}{\partial y} + j\beta E_{0,y} = -j\omega\mu H_{0,x} \quad \text{Eq. (1a)}$$

$$-j\beta E_{0,x} - \frac{\partial E_{0,z}}{\partial x} = -j\omega\mu H_{0,y} \quad \text{Eq. (1b)}$$

$$\frac{\partial E_{0,y}}{\partial x} - \frac{\partial E_{0,x}}{\partial y} = -j\omega\mu H_{0,z} \quad \text{Eq. (1c)}$$

$$\frac{\partial H_{0,z}}{\partial y} + j\beta H_{0,y} = j\omega\varepsilon E_{0,x} \quad \text{Eq. (2a)}$$

$$-j\beta H_{0,x} - \frac{\partial H_{0,z}}{\partial x} = j\omega\varepsilon E_{0,y} \quad \text{Eq. (2b)}$$

$$\frac{\partial H_{0,y}}{\partial x} - \frac{\partial H_{0,x}}{\partial y} = j\omega\varepsilon E_{0,z} \quad \text{Eq. (2c)}$$

Solve Eq. (1c) for $H_{0,z}$ and solve Eq. (2c) for $E_{0,z}$.

$$H_{0,z} = -\frac{1}{j\omega\mu}\left(\frac{\partial E_{0,y}}{\partial x} - \frac{\partial E_{0,x}}{\partial y}\right) \quad \text{Eq. (3a)}$$

$$E_{0,z} = \frac{1}{j\omega\varepsilon}\left(\frac{\partial H_{0,y}}{\partial x} - \frac{\partial H_{0,x}}{\partial y}\right) \quad \text{Eq. (3b)}$$

Substitute Eq. (3a) into Eqs. (2a) and (2b), & substitute Eq. (3b) into Eqs. (1a) and (1b).

$$\frac{1}{\omega}\frac{\partial}{\partial x}\left[\frac{1}{\varepsilon}\left(\frac{\partial H_{0,y}}{\partial x} - \frac{\partial H_{0,x}}{\partial y}\right)\right] + \omega\mu H_{0,y} = \beta E_{0,x} \quad \text{Eq. (4a)}$$

$$\frac{1}{\omega}\frac{\partial}{\partial y}\left[\frac{1}{\varepsilon}\left(\frac{\partial H_{0,y}}{\partial x} - \frac{\partial H_{0,x}}{\partial y}\right)\right] + \omega\mu H_{0,x} = \beta E_{0,y} \quad \text{Eq. (4b)}$$

$$-\frac{1}{\omega}\frac{\partial}{\partial x}\left[\frac{1}{\mu}\left(\frac{\partial E_{0,y}}{\partial x} - \frac{\partial E_{0,x}}{\partial y}\right)\right] - \omega\varepsilon E_{0,y} = \beta H_{0,x} \quad \text{Eq. (5a)}$$

$$-\frac{1}{\omega}\frac{\partial}{\partial y}\left[\frac{1}{\mu}\left(\frac{\partial E_{0,y}}{\partial x} - \frac{\partial E_{0,x}}{\partial y}\right)\right] - \omega\varepsilon E_{0,x} = \beta H_{0,y} \quad \text{Eq. (5b)}$$

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# Form a Matrix Equation

The four remaining equations can be written more compactly as

$$\begin{bmatrix} -\frac{1}{\omega}\frac{\partial}{\partial x}\frac{1}{\varepsilon}\frac{\partial}{\partial y} & \frac{1}{\omega}\frac{\partial}{\partial x}\frac{1}{\varepsilon}\frac{\partial}{\partial x}+\omega\mu \\ -\left(\frac{1}{\omega}\frac{\partial}{\partial y}\frac{1}{\varepsilon}\frac{\partial}{\partial y}+\omega\mu\right) & \frac{1}{\omega}\frac{\partial}{\partial y}\frac{1}{\varepsilon}\frac{\partial}{\partial x}\end{bmatrix}\begin{bmatrix}H_{0,x}\\ H_{0,y}\end{bmatrix}=\beta\begin{bmatrix}E_{0,x}\\ E_{0,y}\end{bmatrix} \quad \text{Eq. (6)}$$

$$\begin{bmatrix} \frac{1}{\omega}\frac{\partial}{\partial x}\frac{1}{\mu}\frac{\partial}{\partial y} & -\left(\frac{1}{\omega}\frac{\partial}{\partial x}\frac{1}{\mu}\frac{\partial}{\partial x}+\omega\varepsilon\right) \\ \frac{1}{\omega}\frac{\partial}{\partial y}\frac{1}{\mu}\frac{\partial}{\partial y}+\omega\varepsilon & -\frac{1}{\omega}\frac{\partial}{\partial y}\frac{1}{\mu}\frac{\partial}{\partial x}\end{bmatrix}\begin{bmatrix}E_{0,x}\\ E_{0,y}\end{bmatrix}=\beta\begin{bmatrix}H_{0,x}\\ H_{0,y}\end{bmatrix} \quad \text{Eq. (7)}$$

Full Wave Analysis

Solve Eq. (7) for the magnetic field components.

$$\begin{bmatrix}H_{0,x}\\ H_{0,y}\end{bmatrix}=\frac{1}{\beta}\begin{bmatrix} \frac{1}{\omega}\frac{\partial}{\partial x}\frac{1}{\mu}\frac{\partial}{\partial y} & -\left(\frac{1}{\omega}\frac{\partial}{\partial x}\frac{1}{\mu}\frac{\partial}{\partial x}+\omega\varepsilon\right) \\ \frac{1}{\omega}\frac{\partial}{\partial y}\frac{1}{\mu}\frac{\partial}{\partial y}+\omega\varepsilon & -\frac{1}{\omega}\frac{\partial}{\partial y}\frac{1}{\mu}\frac{\partial}{\partial x}\end{bmatrix}\begin{bmatrix}E_{0,x}\\ E_{0,y}\end{bmatrix} \quad \text{Eq. (8)}$$

Solve Eq. (8) into Eq. (6) to arrive at the final wave equation to be solved.

$$\begin{bmatrix} -\frac{1}{\omega}\frac{\partial}{\partial x}\frac{1}{\varepsilon}\frac{\partial}{\partial y} & \frac{1}{\omega}\frac{\partial}{\partial x}\frac{1}{\varepsilon}\frac{\partial}{\partial x}+\omega\mu \\ -\left(\frac{1}{\omega}\frac{\partial}{\partial y}\frac{1}{\varepsilon}\frac{\partial}{\partial y}+\omega\mu\right) & \frac{1}{\omega}\frac{\partial}{\partial y}\frac{1}{\varepsilon}\frac{\partial}{\partial x}\end{bmatrix}\begin{bmatrix} \frac{1}{\omega}\frac{\partial}{\partial x}\frac{1}{\mu}\frac{\partial}{\partial y} & -\left(\frac{1}{\omega}\frac{\partial}{\partial x}\frac{1}{\mu}\frac{\partial}{\partial x}+\omega\varepsilon\right) \\ \frac{1}{\omega}\frac{\partial}{\partial y}\frac{1}{\mu}\frac{\partial}{\partial y}+\omega\varepsilon & -\frac{1}{\omega}\frac{\partial}{\partial y}\frac{1}{\mu}\frac{\partial}{\partial x}\end{bmatrix}\begin{bmatrix}E_{0,x}\\ E_{0,y}\end{bmatrix}-\beta^2\begin{bmatrix}E_{0,x}\\ E_{0,y}\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

Yikes!! ☹ This is typically solved numerically on a computer.

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# Quasi-LP Analysis

Recognizing that the hybrid modes tend to be strongly linearly polarized, a simplifying approximation can be made. The approximation is that the cross coupling between $E_{0,x}$ and $E_{0,y}$ is weak and can be neglected. Under this condition, the governing equation separates into two independent equations, one for each LP mode.

$$\begin{bmatrix}\Omega_{xx} & \cancel{\Omega_{xy}}\\ \cancel{\Omega_{yx}} & \Omega_{yy}\end{bmatrix}\begin{bmatrix}E_{0,x}\\ E_{0,y}\end{bmatrix}-\beta^2\begin{bmatrix}E_{0,x}\\ E_{0,y}\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix} \quad \rightarrow \quad \begin{aligned}\Omega_{xx}E_{0,x}-\beta^2E_{0,x}&=0\\ \Omega_{yy}E_{0,y}-\beta^2E_{0,y}&=0\end{aligned}$$

$$\Omega_{xx}=\frac{1}{\omega}\left(\frac{\partial}{\partial x}\frac{1}{\varepsilon}\frac{\partial}{\partial x}+\omega^2\mu\right)\left(\frac{\partial}{\partial y}\frac{1}{\mu}\frac{\partial}{\partial y}+\omega^2\varepsilon\right)-\frac{1}{\omega^2}\left(\frac{\partial}{\partial x}\frac{1}{\varepsilon}\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x}\frac{1}{\mu}\frac{\partial}{\partial y}\right)$$

$$\Omega_{yy}=\frac{1}{\omega}\left(\frac{\partial}{\partial y}\frac{1}{\varepsilon}\frac{\partial}{\partial y}+\omega^2\mu\right)\left(\frac{\partial}{\partial x}\frac{1}{\mu}\frac{\partial}{\partial x}+\omega^2\varepsilon\right)-\frac{1}{\omega^2}\left(\frac{\partial}{\partial y}\frac{1}{\varepsilon}\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial y}\frac{1}{\mu}\frac{\partial}{\partial x}\right)$$

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# Setup for TEM Analysis

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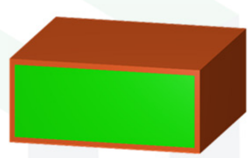
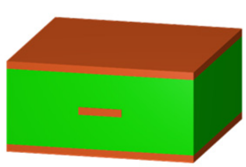
## Existence Conditions for TEM

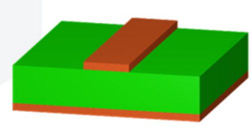
TEM modes only exist in transmission lines embedded in a homogeneous fill.



Does Not Support TEM



Supports TEM



Does Not Support TEM

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# TEM Analysis (1 of 3)

For TEM waves, $E_{0,z} = H_{0,z} = 0$. Under this condition, Maxwell's equations reduce to

$$\cancel{\frac{\partial E_{0,z}}{\partial y}} + j\beta E_{0,y} = -j\omega\mu H_{0,x} \qquad \text{Eq. (1a)}$$

$$-j\beta E_{0,x} - \cancel{\frac{\partial E_{0,z}}{\partial x}} = -j\omega\mu H_{0,y} \qquad \text{Eq. (1b)}$$

$$\frac{\partial E_{0,y}}{\partial x} - \frac{\partial E_{0,x}}{\partial y} = -\cancel{j\omega\mu H_{0,z}} \qquad \text{Eq. (1c)}$$

$$\cancel{\frac{\partial H_{0,z}}{\partial y}} + j\beta H_{0,y} = j\omega\varepsilon E_{0,x} \qquad \text{Eq. (1d)}$$

$$-j\beta H_{0,x} - \cancel{\frac{\partial H_{0,z}}{\partial x}} = j\omega\varepsilon E_{0,y} \qquad \text{Eq. (1e)}$$

$$\frac{\partial H_{0,y}}{\partial x} - \frac{\partial H_{0,x}}{\partial y} = \cancel{j\omega\varepsilon E_{0,z}} \qquad \text{Eq. (1f)}$$

⇩

$$j\beta E_{0,y} = -j\omega\mu H_{0,x} \qquad \text{Eq. (2a)}$$

$$-j\beta E_{0,x} = -j\omega\mu H_{0,y} \qquad \text{Eq. (2b)}$$

$$\frac{\partial E_{0,y}}{\partial x} - \frac{\partial E_{0,x}}{\partial y} = 0 \qquad \text{Eq. (2c)}$$

$$j\beta H_{0,y} = j\omega\varepsilon E_{0,x} \qquad \text{Eq. (2d)}$$

$$-j\beta H_{0,x} = j\omega\varepsilon E_{0,y} \qquad \text{Eq. (2e)}$$

$$\frac{\partial H_{0,y}}{\partial x} - \frac{\partial H_{0,x}}{\partial y} = 0 \qquad \text{Eq. (2f)}$$

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# TEM Analysis (2 of 3)

From the previous slide…

$$j\beta E_{0,y} = -j\omega\mu H_{0,x} \qquad \text{Eq. (2a)}$$

$$-j\beta E_{0,x} = -j\omega\mu H_{0,y} \qquad \text{Eq. (2b)}$$

$$\frac{\partial E_{0,y}}{\partial x} - \frac{\partial E_{0,x}}{\partial y} = 0 \qquad \text{Eq. (2c)}$$

$$j\beta H_{0,y} = j\omega\varepsilon E_{0,x} \qquad \text{Eq. (2d)}$$

$$-j\beta H_{0,x} = j\omega\varepsilon E_{0,y} \qquad \text{Eq. (2e)}$$

$$\frac{\partial H_{0,y}}{\partial x} - \frac{\partial H_{0,x}}{\partial y} = 0 \qquad \text{Eq. (2f)}$$

Solve Eq. (2d) for $H_{0,y}$.

$$H_{0,y} = \frac{\omega\varepsilon}{\beta} E_{0,x}$$

Substitute $H_{0,y}$ into Eq. (2b).

$$-j\beta E_{0,x} = -j\omega\mu\left(\frac{\omega\varepsilon}{\beta} E_{0,x}\right)$$

$$\beta^2 E_{0,x} = \omega^2 \mu\varepsilon E_{0,x}$$

$$\beta^2 E_{0,x} = k^2 E_{0,x}$$

This shows that for TEM analysis

$$\beta = k$$

Previously, the cutoff wave number was defined as $k_c^2 = k^2 - \beta^2$.

If $\beta = k$, then $k_c = 0$ indicating that there is no cutoff frequency for the TEM mode.

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# TEM Analysis (3 of 3)

In LHI media, recall that the wave equation was

$$\nabla^2 \vec{E}_{0,xy} + k_c^2 \vec{E}_{0,xy} = 0$$

But for the TEM mode, $k_c = 0$.

$$\nabla^2 \vec{E}_{0,xy} + \cancel{k_c^2 \vec{E}_{0,xy}} = 0$$

$$\nabla^2 \vec{E}_{0,xy} = 0$$

→ The wave equation reduces to Laplace's equation from electrostatics.

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# Alternate Derivation of TEM Analysis

The TEM mode in a transmission line has no cutoff frequency ($k_c = 0$).
This means that it can be analyzed as $\omega \rightarrow 0$ and the problem reduces to an electrostatics problem.

Derivation

Maxwell's equations for electrostatics

$\nabla \times \vec{E} = 0$ Eq. (3a)

$\nabla \bullet \vec{D} = 0$ Eq. (3b)

$\vec{D} = [\varepsilon]\vec{E}$ Eq. (3c)

$\vec{E} = -\nabla V$ Eq. (3d)

Substitute Eq. (3c) into Eq. (3b).

$\nabla \bullet ([\varepsilon]\vec{E}) = 0$ Eq. (4)

Substitute Eq. (3d) into Eq. (4).

$\nabla \bullet \{[\varepsilon](\nabla V)\} = 0$

For isotropic dielectrics

$\nabla \bullet [\varepsilon(\nabla V)] = 0$

For homogeneous dielectrics

$$\boxed{\nabla^2 V = 0}$$

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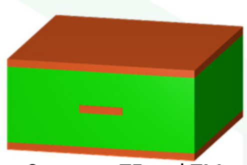
# Setup for TE & TM Analysis



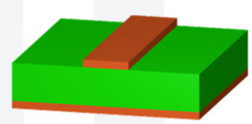


## Existence Conditions for TE and TM Modes

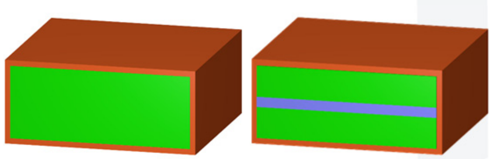
TE and TM modes only exist in waveguides with a homogeneous fill or in waveguides with a uniform axis like slabs and circularly symmetric guides.



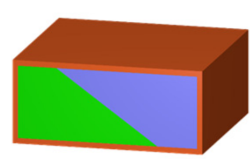
Supports TE and TM



Does Not Support TE or TM



Supports TE and TM



Does Not Support TE or TM

# TE Analysis in LHI Media

Choose to set $E_{0,z} = 0$ and $H_{0,z} \neq 0$. This means it is only necessary to solve for $H_{0,z}$.

$$\cancel{\nabla^2 E_{0,z} + k_c^2 E_{0,z} = 0} \qquad \boxed{\nabla^2 H_{0,z} + k_c^2 H_{0,z} = 0}$$

An added benefit of this solution approach is that $H_{0,z}$ is tangential to all boundaries in a waveguide.

For TE analysis, the other field components are calculated just from $H_{0,z}$.

$$H_{0,x} = -\frac{j\beta}{k_c^2}\frac{\partial H_{0,z}}{\partial x} \qquad E_{0,x} = -\frac{j\omega\mu}{k_c^2}\frac{\partial H_{0,z}}{\partial y}$$

$$H_{0,y} = -\frac{j\beta}{k_c^2}\frac{\partial H_{0,z}}{\partial y} \qquad E_{0,y} = \frac{j\omega\mu}{k_c^2}\frac{\partial H_{0,z}}{\partial x}$$

From this, the characteristic impedance $Z_{\text{TE}}$ is

$$Z_{\text{TE}} = \frac{E_{0,x}}{H_{0,y}} = \frac{-\frac{j\omega\mu}{k_c^2}\frac{\partial H_{0,z}}{\partial y}}{-\frac{j\beta}{k_c^2}\frac{\partial H_{0,z}}{\partial y}} = \frac{\omega\mu}{\beta} = \eta\frac{k}{\beta}$$

$\beta$ is found by solving the wave equation.

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# TM Analysis in LHI Media

Choose to set $E_{0,z} \neq 0$ and $H_{0,z} = 0$. This means it is only necessary to solve for $E_{0,z}$.

$$\boxed{\nabla^2 E_{0,z} + k_c^2 E_{0,z} = 0} \qquad \cancel{\nabla^2 H_{0,z} + k_c^2 H_{0,z} = 0}$$

An added benefit of this solution approach is that $E_{0,z}$ is tangential to all boundaries in a waveguide.

For TM analysis, the other field components are calculated just from $E_{0,z}$.

$$H_{0,x} = \frac{j\omega\varepsilon}{k_c^2}\frac{\partial E_{0,z}}{\partial y} \qquad E_{0,x} = -\frac{j\beta}{k_c^2}\frac{\partial E_{0,z}}{\partial x}$$

$$H_{0,y} = -\frac{j\omega\varepsilon}{k_c^2}\frac{\partial E_{0,z}}{\partial x} \qquad E_{0,y} = -\frac{j\beta}{k_c^2}\frac{\partial E_{0,z}}{\partial y}$$

From this, the characteristic impedance $Z_{\text{TM}}$ is

$$Z_{\text{TM}} = \frac{E_{0,x}}{H_{0,y}} = \frac{-\frac{j\beta}{k_c^2}\frac{\partial E_{0,z}}{\partial x}}{-\frac{j\omega\varepsilon}{k_c^2}\frac{\partial E_{0,z}}{\partial x}} = \frac{\beta}{\omega\varepsilon} = \eta\frac{\beta}{k}$$

$\beta$ is found by solving the wave equation.

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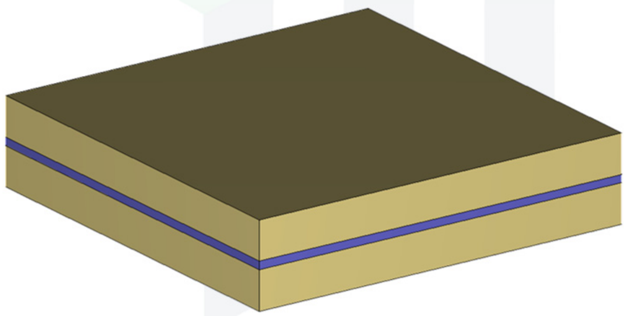
# Setup for Analyzing Slab Waveguides



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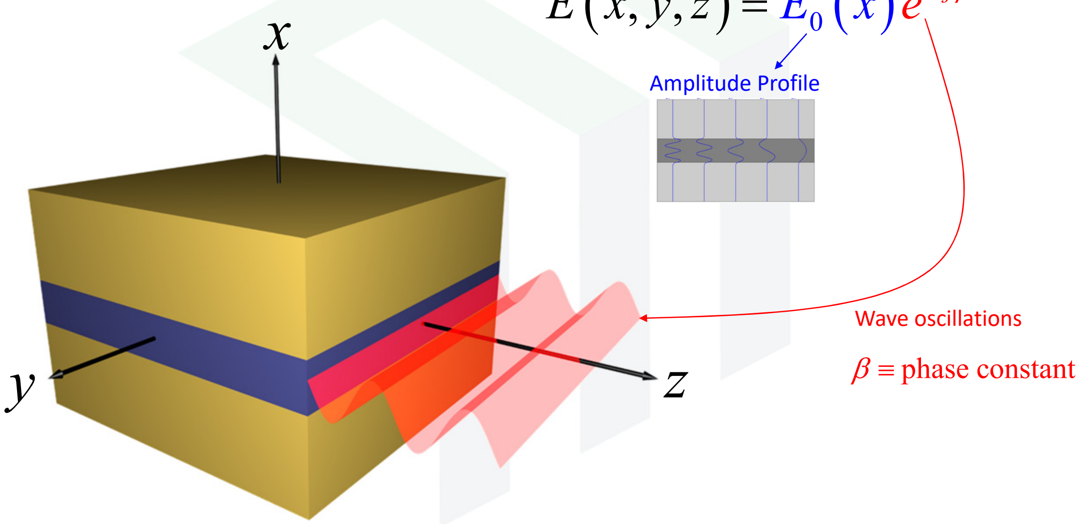
## Geometry and Solution

$$\vec{E}(x,y,z) = \vec{E}_0(x)e^{-j\beta z}$$

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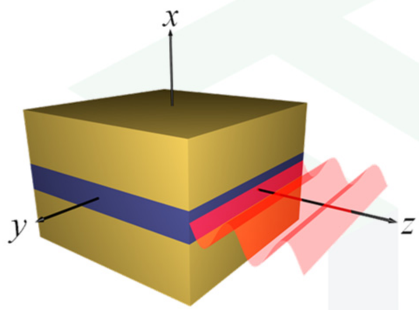
# Origin of TE and TM Modes (1 of 2)




Given this geometry

$$\frac{\partial}{\partial y} = 0$$

$$\cancel{\frac{\partial E_{0,z}}{\partial y}} + j\beta E_{0,y} = -j\omega\mu H_{0,x} \quad \text{Eq. (1a)}$$

$$-j\beta E_{0,x} - \frac{\partial E_{0,z}}{\partial x} = -j\omega\mu H_{0,y} \quad \text{Eq. (1b)}$$

$$\frac{\partial E_{0,y}}{\partial x} - \cancel{\frac{\partial E_{0,x}}{\partial y}} = -j\omega\mu H_{0,z} \quad \text{Eq. (1c)}$$

$$\cancel{\frac{\partial H_{0,z}}{\partial y}} + j\beta H_{0,y} = j\omega\varepsilon E_{0,x} \quad \text{Eq. (1d)}$$

$$-j\beta H_{0,x} - \frac{\partial H_{0,z}}{\partial x} = j\omega\varepsilon E_{0,y} \quad \text{Eq. (1e)}$$

$$\frac{\partial H_{0,y}}{\partial x} - \cancel{\frac{\partial H_{0,x}}{\partial y}} = j\omega\varepsilon E_{0,z} \quad \text{Eq. (1f)}$$

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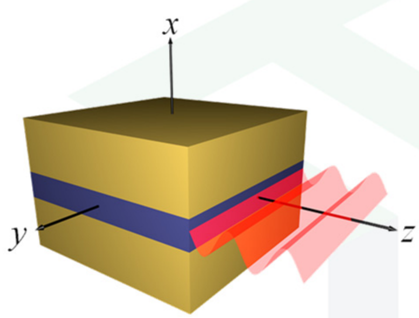
# Origin of TE and TM Modes (1 of 2)




Given this geometry

$$\frac{\partial}{\partial y} = 0$$

$$j\beta E_{0,y} = -j\omega\mu H_{0,x} \quad \text{Eq. (1a)}$$

$$-j\beta E_{0,x} - \frac{\partial E_{0,z}}{\partial x} = -j\omega\mu H_{0,y} \quad \text{Eq. (1b)}$$

$$\frac{\partial E_{0,y}}{\partial x} = -j\omega\mu H_{0,z} \quad \text{Eq. (1c)}$$

$$j\beta H_{0,y} = j\omega\varepsilon E_{0,x} \quad \text{Eq. (1d)}$$

$$-j\beta H_{0,x} - \frac{\partial H_{0,z}}{\partial x} = j\omega\varepsilon E_{0,y} \quad \text{Eq. (1e)}$$

$$\frac{\partial H_{0,y}}{\partial x} = j\omega\varepsilon E_{0,z} \quad \text{Eq. (1f)}$$

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# Origin of TE and TM Modes (2 of 2)

Maxwell's equations have decoupled into two independent sets of equations.

TE Mode (i.e. $E_{0,z}=0$)

$$j\beta E_{0,y}=-j\omega\mu H_{0,x} \quad \text{Eq. (2a)}$$

$$\frac{\partial E_{0,y}}{\partial x}=-j\omega\mu H_{0,z} \quad \text{Eq. (2c)}$$

$$-j\beta H_{0,x}-\frac{\partial H_{0,z}}{\partial x}=j\omega\varepsilon E_{0,y} \quad \text{Eq. (2e)}$$

TM Mode (i.e. $H_{0,z}=0$)

$$j\beta H_{0,y}=j\omega\varepsilon E_{0,x} \quad \text{Eq. (2d)}$$

$$\frac{\partial H_{0,y}}{\partial x}=j\omega\varepsilon E_{0,z} \quad \text{Eq. (2f)}$$

$$-j\beta E_{0,x}-\frac{\partial E_{0,z}}{\partial x}=-j\omega\mu H_{0,y} \quad \text{Eq. (2b)}$$

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# Origin of TE and TM Modes (2 of 2)

Maxwell's equations have decoupled into two independent sets of equations.

TE Mode (i.e. $E_{0,z}=0$)

$$-j\beta H_{0,x}-\frac{\partial H_{0,z}}{\partial x}=j\omega\varepsilon E_{0,y} \quad \text{Eq. (3a)}$$

$$j\beta E_{0,y}=-j\omega\mu H_{0,x} \quad \text{Eq. (3b)}$$

$$\frac{\partial E_{0,y}}{\partial x}=-j\omega\mu H_{0,z} \quad \text{Eq. (3c)}$$

TM Mode (i.e. $H_{0,z}=0$)

$$-j\beta E_{0,x}-\frac{\partial E_{0,z}}{\partial x}=-j\omega\mu H_{0,y} \quad \text{Eq. (3d)}$$

$$j\beta H_{0,y}=j\omega\varepsilon E_{0,x} \quad \text{Eq. (3e)}$$

$$\frac{\partial H_{0,y}}{\partial x}=j\omega\varepsilon E_{0,z} \quad \text{Eq. (3f)}$$

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# TE Wave Equation

Solve Eq. (3b) for $H_{0,x}$ and solve Eq. (3c) for $H_{0,z}$.

TE Mode (i.e. $E_{0,z} = 0$)

$$-j\beta H_{0,x} - \frac{\partial H_{0,z}}{\partial x} = j\omega\varepsilon E_{0,y} \qquad \text{Eq. (3a)}$$

$$j\beta E_{0,y} = -j\omega\mu H_{0,x} \qquad \text{Eq. (3b)} \longrightarrow H_{0,x} = -\frac{\beta}{\omega\mu} E_{0,y} \qquad \text{Eq. (4a)}$$

$$\frac{\partial E_{0,y}}{\partial x} = -j\omega\mu H_{0,z} \qquad \text{Eq. (3c)} \longrightarrow H_{0,z} = -\frac{1}{j\omega\mu}\frac{\partial E_{0,y}}{\partial x} \qquad \text{Eq. (4b)}$$

Substitute Eq. (4a) and (4b) into Eq. (3a) to obtain an equation that only contains $E_{0,y}$.

$$-j\beta H_{0,x} - \frac{\partial H_{0,z}}{\partial x} = j\omega\varepsilon E_{0,y} \quad \Rightarrow \quad -j\beta\left(-\frac{\beta}{\omega\mu} E_{0,y}\right) - \frac{\partial}{\partial x}\left(-\frac{1}{j\omega\mu}\frac{\partial E_{0,y}}{\partial x}\right) = j\omega\varepsilon E_{0,y} \quad \Rightarrow \quad \boxed{\mu\frac{\partial}{\partial x}\left(\frac{1}{\mu}\frac{\partial E_{0,y}}{\partial x}\right) + k_c^2 E_{0,y} = 0}$$

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# TM Wave Equation

Solve Eq. (3e) for $E_{0,x}$ and solve Eq. (3f) for $E_{0,z}$.

TM Mode (i.e. $H_{0,z} = 0$)

$$-j\beta E_{0,x} - \frac{\partial E_{0,z}}{\partial x} = -j\omega\mu H_{0,y} \qquad \text{Eq. (3d)}$$

$$j\beta H_{0,y} = j\omega\varepsilon E_{0,x} \qquad \text{Eq. (3e)} \longrightarrow E_{0,x} = \frac{\beta}{\omega\varepsilon} H_{0,y} \qquad \text{Eq. (5a)}$$

$$\frac{\partial H_{0,y}}{\partial x} = j\omega\varepsilon E_{0,z} \qquad \text{Eq. (3f)} \longrightarrow E_{0,z} = \frac{1}{j\omega\varepsilon}\frac{\partial H_{0,y}}{\partial x} \qquad \text{Eq. (5b)}$$

Substitute Eq. (5a) and (5b) into Eq. (3d) to obtain an equation that only contains $H_{0,y}$.

$$-j\beta E_{0,x} - \frac{\partial E_{0,z}}{\partial x} = -j\omega\mu H_{0,y} \quad \Rightarrow \quad -j\beta\left(\frac{\beta}{\omega\varepsilon} H_{0,y}\right) - \frac{\partial}{\partial x}\left(\frac{1}{j\omega\varepsilon}\frac{\partial H_{0,y}}{\partial x}\right) = -j\omega\mu H_{0,y} \quad \Rightarrow \quad \boxed{\varepsilon\frac{\partial}{\partial x}\left(\frac{1}{\varepsilon}\frac{\partial H_{0,y}}{\partial x}\right) + k_c^2 H_{0,y} = 0}$$

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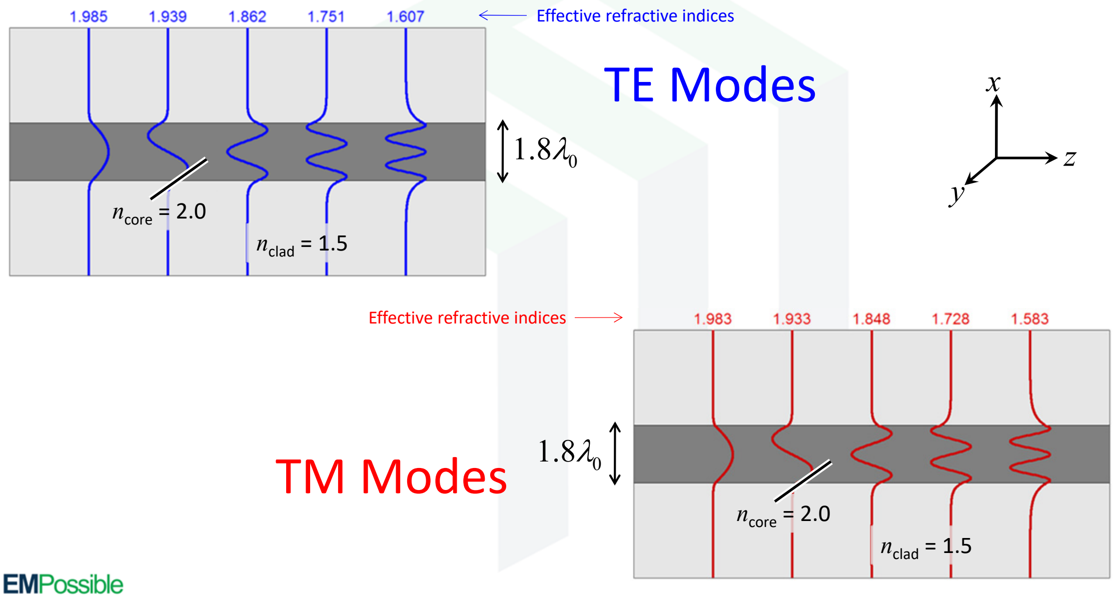
# Typical Modes in a Slab Waveguide

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# Remarks About Slab Waveguide Analysis

- Waves are confined in only one transverse direction.
- Waves are free to spread out in the uniform transverse direction
- Propagation within the slab can be restricted to the $z$ direction without loss of generality.

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# Summary of This Lecture

- Identify what types of modes are supported
  - TEM, TE, TM, or hybrid
- Analysis setup

TEM

$$\nabla^2 V = 0$$

TE

$$\nabla^2 H_{0,z} + k_c^2 H_{0,z} = 0$$

TM

$$\nabla^2 E_{0,z} + k_c^2 E_{0,z} = 0$$

Hybrid

$$\begin{bmatrix} -\frac{1}{\omega}\frac{\partial}{\partial x}\frac{1}{\varepsilon}\frac{\partial}{\partial y} & \frac{1}{\omega}\frac{\partial}{\partial x}\frac{1}{\varepsilon}\frac{\partial}{\partial x}+\omega\mu \\ -\left(\frac{1}{\omega}\frac{\partial}{\partial y}\frac{1}{\varepsilon}\frac{\partial}{\partial y}+\omega\mu\right) & \frac{1}{\omega}\frac{\partial}{\partial y}\frac{1}{\varepsilon}\frac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} \frac{1}{\omega}\frac{\partial}{\partial x}\frac{1}{\mu}\frac{\partial}{\partial y} & -\left(\frac{1}{\omega}\frac{\partial}{\partial x}\frac{1}{\mu}\frac{\partial}{\partial x}+\omega\varepsilon\right) \\ \frac{1}{\omega}\frac{\partial}{\partial y}\frac{1}{\mu}\frac{\partial}{\partial y}+\omega\varepsilon & -\frac{1}{\omega}\frac{\partial}{\partial y}\frac{1}{\mu}\frac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} E_{0,x} \\ E_{0,y} \end{bmatrix} - \beta^2 \begin{bmatrix} E_{0,x} \\ E_{0,y} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Slab

$$\text{TE:}\ \mu\frac{\partial}{\partial x}\left(\frac{1}{\mu}\frac{\partial E_{0,y}}{\partial x}\right)+k_c^2 E_{0,y}=0$$

$$\text{TM:}\ \varepsilon\frac{\partial}{\partial x}\left(\frac{1}{\varepsilon}\frac{\partial H_{0,y}}{\partial x}\right)+k_c^2 H_{0,y}=0$$