Advanced Electromagnetics:
21st Century Electromagnetics

The Grating Equation

Lecture Outline

• The grating equation
• Diffraction configurations
• Dependence of grating on diffraction orders
• Analysis of gratings via the grating equation
Phase Matching Into a Diffraction Grating

Boundary conditions require the tangential component of the wave vector be continuous across an interface.

\[ k_{x,\text{tm}} = k_{x,\text{inc}} \]

The wave is entering a grating, so the phase matching condition must apply to all of the diffraction orders.

\[ k_x(m) = k_{x,\text{inc}} - mK_x \]

The longitudinal wave vector component is calculated from the dispersion relation.

\[ k_z^2(m) = \left(k_0n_{\text{avg}}\right)^2 - k_x^2(m) \]

For large values of \( m \), \( k_z(m) \) can become imaginary. This indicates that the highest diffraction-orders are evanescent, or cut off.
Derivation of the Grating Equation

Start with the phase matching condition.
\[ k_x (m) = k_{x, \text{inc}} - mK_x \]
Recognize that \( k_x (m) = k_0 n_{\text{avg}} \sin[\theta (m)] \) and \( k_{x, \text{inc}} = k_0 n_{\text{inc}} \sin \theta_{\text{inc}} \).
\[ k_0 n_{\text{avg}} \sin[\theta (m)] = k_0 n_{\text{inc}} \sin \theta_{\text{inc}} - mK_x \]
Divide equation by \( k_0 \).
\[ n_{\text{avg}} \sin[\theta (m)] = n_{\text{inc}} \sin \theta_{\text{inc}} - m\frac{K_x}{k_0} \]
Recognize that \( k_0 = 2\pi/\lambda_0 \) and \( K_x = 2\pi/\Lambda_x \).
\[ n_{\text{avg}} \sin[\theta (m)] = n_{\text{inc}} \sin \theta_{\text{inc}} - m\frac{\lambda_0}{\Lambda_x} \]

Accounting for Grating Slant \( \phi \)

When phase matching into a grating, it is only the tangential components of \( \vec{k} \) and \( \vec{\Lambda} \) that need to be considered.
\[ K_x = |\vec{\Lambda}| \sin \phi \quad k_{x, \text{inc}} = k_0 n_{\text{inc}} \sin \theta_{\text{inc}} \]
From \( K_x = |\vec{\Lambda}| \sin \phi \), it follows that the tangential component of the grating period \( \Lambda_x \) is
\[ \frac{2\pi}{\Lambda_x} = \frac{2\pi}{\Lambda} \sin \phi \quad \Lambda_x = \frac{\Lambda}{\sin \phi} \]
Perhaps the most general form of the grating equation is
\[ n_{\text{avg}} \sin[\theta (m)] = n_{\text{inc}} \sin \theta_{\text{inc}} - m\frac{\lambda_0}{\Lambda_x} \sin \phi \]
Grating Equation for Planar Diffraction

The angles of the diffraction orders $\theta(m)$ are related to the free space wavelength $\lambda_0$, grating period $\Lambda_x$, angle of incidence $\theta_{\text{inc}}$, and refractive indices ($n_{\text{inc}}$, $n_{\text{ref}}$ and $n_{\text{trm}}$) through the famous grating equation.

The grating equation only predicts the directions of the diffraction orders, not how much power is in them.

**Reflection Region**

$$n_{\text{ref}} \sin[\theta(m)] = n_{\text{inc}} \sin \theta_{\text{inc}} - m \frac{\lambda_0}{\Lambda_x}$$

**Transmission Region**

$$n_{\text{inc}} \sin[\theta(m)] = n_{\text{inc}} \sin \theta_{\text{inc}} - m \frac{\lambda_0}{\Lambda_x}$$

RefRACTive index where diffraction is being observed.

Diffraction Configurations
Diffraction in Two Dimensions

• Everything is known about the direction of diffracted waves just from the angle of incidence $\theta_{inc}$, grating period $\Lambda$, wavelength $\lambda_0$, and refractive indices. It is not necessary to solve Maxwell’s equation to determine direction of the diffraction orders.

- Square
- Oblique
- Hexagonal

Diffraction tends to occur primarily along the axes of the lattice.

• The grating equation says nothing about how much power is in the diffracted modes. This information must come from solving Maxwell’s equations.

Diffraction Configurations

Planar Diffraction from a Ruled Grating
- Diffraction is confined within a plane
- Numerically much simpler than other cases
- E and H modes are independent

Conical Diffraction from a Ruled Grating
- Diffraction is no longer confined to a plane
- Almost same analytical complexity as crossed grating case, but simpler numerically
- E and H modes are coupled

Conical Diffraction from a Crossed Grating with Planar Incidence
- Diffraction occurs in all directions
- Almost same numerical complexity as next case
- E and H modes are coupled

Conical Diffraction from a Crossed Grating
- Diffraction occurs in all directions
- Most complicated case numerically
- E and H modes are coupled
- Essentially the same as previous case
Dependence of Grating on Diffraction Orders

Effect of Grating Period $\Lambda_x$

- **Subwavelength Grating**
  \[ \Lambda_x < \frac{\lambda_0}{n_{avg}} \]

- **“Subwavelength” Grating**
  \[ \frac{\lambda_0}{n_{avg}} < \Lambda_x < \frac{\lambda_0}{n_{inc}} \]

- **Low Order Grating**
  \[ \Lambda_x > \frac{\lambda_0}{n_{inc}} \]

- **High Order Grating**
  \[ \Lambda_x >> \frac{\lambda_0}{n_{inc}} \]
**Animation of Ruled Grating Diffraction (1 of 3)**

**What is Being Visualized**
Diffraction orders resulting from normal incidence as the period of the grating is adjusted relative to the wavelength.

**Conclusions**
- All waves occur within the same plane.
- Longer periods produce higher number of diffraction orders.
- More diffraction orders exist in regions where refractive index is higher.
- Zero order modes are the typical reflected and refracted waves at an interface.
- Diffraction is symmetric about the normal.

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**Animation of Ruled Grating Diffraction (2 of 3)**

**What is Being Visualized**
Diffraction orders resulting from oblique incidence as the period of the grating is adjusted relative to the wavelength.

**Conclusions**
- All waves occur within the same plane.
- Longer periods produce higher number of diffraction orders.
- More diffraction orders exist in regions where refractive index is higher.
- Zero order modes are the typical reflected and refracted waves at an interface.
- Diffraction is asymmetric about the normal due to oblique incidence.
### Animation of Ruled Grating Diffraction (3 of 3)

**What is Being Visualized**
Diffraction orders resulting from a ruled grating as the angle of incidence is changed.

**Conclusions**
- All waves occur within the same plane.
- More diffraction orders exist in regions where refractive index is higher.
- Zero order modes are the typical reflected and refracted waves at an interface.
- Diffraction is asymmetric about the normal due to oblique incidence.

### Animation of Crossed Grating Diffraction (1 of 3)

**What is Being Visualized**
Diffraction orders resulting from normal incidence as the period of the grating is adjusted relative to the wavelength.

**Conclusions**
- Diffraction occurs in all directions.
- Longer periods produce higher number of diffraction orders.
- More diffraction orders exist in regions where refractive index is higher.
- Zero order modes are the typical reflected and refracted waves at an interface.
- Diffraction is symmetric about the normal.
Animation of Crossed Grating Diffraction (2 of 3)

**What is Being Visualized**
Diffraction orders resulting from oblique incidence as the period of the grating is adjusted relative to the wavelength.

**Conclusions**
- Diffraction occurs in all directions.
- Longer periods produce higher number of diffraction orders.
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- Zero order modes are the typical reflected and refracted waves at an interface.
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Animation of Crossed Grating Diffraction (3 of 3)

**What is Being Visualized**
Diffraction orders resulting from a crossed grating as the angle of incidence is meandered.

**Conclusions**
- Diffraction occurs in all directions.
- Longer periods produce higher number of diffraction orders.
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- Diffraction is symmetric about the normal.
Analysis of Gratings via the Grating Equation

Grating Cutoff Wavelength

When $\theta(m)$ becomes imaginary, the $m$th diffraction order is evanescent and cut off.

Assuming normal incidence (i.e. $\theta_{inc} = 0^\circ$), the grating equation reduces to

$$ n \sin \left[ \theta(m) \right] = -m \frac{\lambda_0}{\Lambda_x} $$

The first diffraction orders to appear are $m = \pm 1$.

The cutoff for the first-order modes happens when $\theta(\pm1) = 90^\circ$.

$$ \theta(\pm1) = 90^\circ $$

$$ \sin[90^\circ] = 1 = \frac{\lambda_{0,c}}{n\Lambda_x} $$

$$ \Lambda_x = \frac{\lambda_{0,c}}{n} = \lambda_c $$

To prevent the first-order modes, the grating must be subwavelength.

$$ \Lambda_x < \lambda_c = \frac{\lambda_{0,c}}{n} $$

To ensure first-order modes exist, the grating period has to be larger than the wavelength.

$$ \Lambda_x > \lambda_c = \frac{\lambda_{0,c}}{n} $$
Total Number of Diffraction Orders

Given the grating period $\Lambda_x$ and the wavelength $\lambda_0$, the number of diffraction orders $M$ can be determined.

Assuming normal incidence (i.e. $\theta_{inc} = 0^\circ$), the grating equation reduces to

$$\sin[\theta(m)] = -\frac{m\lambda_0}{n_{avg}\Lambda_x} \quad \rightarrow \quad \left|\sin[\theta(m)]\right| = \left|\frac{m\lambda_0}{n_{avg}\Lambda_x}\right| < 1$$

Condition to keep $\theta(m)$ purely real.

Therefore, a maximum value for $m$ that keeps $\theta(m)$ purely real is

$$m_{max} = \frac{n_{avg}\Lambda_x}{\lambda_0}$$

The total number of possible diffracted modes $M$ is then $2m_{max} + 1$.

$$M = \frac{2n_{avg}\Lambda_x}{\lambda_0} + 1$$

$2m_{max} \rightarrow$ diffraction is symmetric (i.e. both positive and negative values of $m$).

$+1 \rightarrow$ the zero-order must count as a diffraction order.

Determining Grating Cutoff Conditions

<table>
<thead>
<tr>
<th>Condition</th>
<th>Requirements</th>
</tr>
</thead>
<tbody>
<tr>
<td>0-order mode</td>
<td>Always exists unless there is total internal reflection</td>
</tr>
<tr>
<td>No 1&lt;sup&gt;st&lt;/sup&gt;-order modes</td>
<td>Grating period must be shorter than what causes $\theta(\pm 1) = 90^\circ$</td>
</tr>
<tr>
<td>Ensure 1&lt;sup&gt;st&lt;/sup&gt;-order modes</td>
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<tr>
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</tr>
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<td>Grating period must be larger than what causes $\theta(\pm 2) = 90^\circ$</td>
</tr>
<tr>
<td>No $m^{th}$-order modes</td>
<td>Grating period must be shorter than what causes $\theta(\pm m) = 90^\circ$</td>
</tr>
<tr>
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</tbody>
</table>
Analysis of Diffraction Gratings

方向的衍射级数

\[ n \sin \left( \theta(m) \right) = n_{inc} \sin \theta_{inc} - m \frac{\lambda}{\Lambda} \sin \phi \]

衍射效率和 polarization

Maxwell’s equation must be solved to determine amplitude and polarization of the diffraction orders.

\[ \nabla \times \vec{E} = -j \omega \mu \vec{H} \]
\[ \nabla \times \vec{H} = j \omega \varepsilon \vec{E} \]
\[ \nabla \cdot (\varepsilon \vec{E}) = 0 \]
\[ \nabla \cdot (\mu \vec{H}) = 0 \]

Three Modes of Operation for 1D Gratings

**布拉格格栅**
- 耦合电流间反向传播的波。

**衍射格栅**
- 耦合波在不同的角度。

**长周期格栅**
- 耦合同向传播波。

**应用**
- 薄膜光学滤波器
- 光纤 gratings
- 波长分割复用
- 电介质镜子
- 光子晶体波导

**应用**
- 感应器
- 模式化的 fanout gratings
- 激光锁定
- 光谱分析
- 反反射
- 频率选择表面
- 光栅器