Advanced Electromagnetics:
21st Century Electromagnetics

The Grating Equation

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Lecture Outline

- The grating equation
- Diffraction configurations
- Dependence of grating on diffraction orders
- Analysis of gratings via the grating equation

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The Grating Equation

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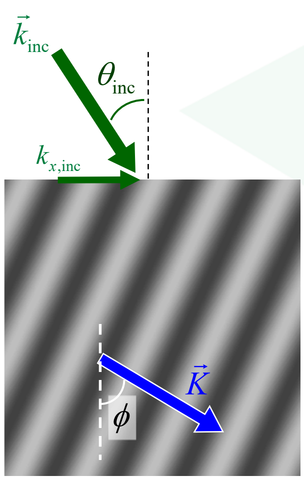
Phase Matching Into a Diffraction Grating



Boundary conditions require the tangential component of the wave vector be continuous across an interface.

$$k_{x,\text{trn}} \stackrel{?}{=} k_{x,\text{inc}}$$

The wave is entering a grating, so the phase matching condition must apply to all of the diffraction orders.

$$k_x(m) = k_{x,\text{inc}} - mK_x$$

The longitudinal wave vector component is calculated from the dispersion relation.

$$k_z^2(m) = \left(k_0 n_{\text{avg}}\right)^2 - k_x^2(m)$$

For large values of m, $k_z(m)$ can become imaginary. This indicates that the highest diffraction-orders are evanescent, or cut off.

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Derivation of the Grating Equation

Start with the phase matching condition.

$$k_x(m) = k_{x,\text{inc}} - mK_x$$

Recognize that $k_x(m) = k_0 n_{\text{avg}} \sin[\theta(m)]$ and $k_{x,\text{inc}} = k_0 n_{\text{inc}} \sin\theta_{\text{inc}}$.

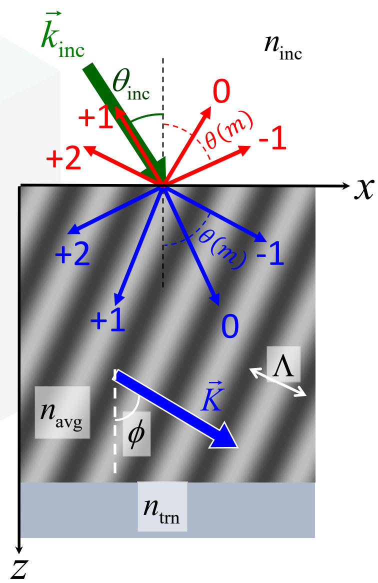
$$k_0 n_{\text{avg}} \sin\left[\theta(m)\right] = k_0 n_{\text{inc}} \sin\theta_{\text{inc}} - mK_x$$

Divide equation by k_0.

$$n_{\text{avg}} \sin\left[\theta(m)\right] = n_{\text{inc}} \sin\theta_{\text{inc}} - m\frac{K_x}{k_0}$$

Recognize that $k_0 = 2\pi/\lambda_0$ and $K_x = 2\pi/\Lambda_x$.

$$n_{\text{avg}} \sin\left[\theta(m)\right] = n_{\text{inc}} \sin\theta_{\text{inc}} - m\frac{\lambda_0}{\Lambda_x}$$



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Accounting for Grating Slant ϕ

When phase matching into a grating, it is only the tangential components of $\vec{k}$ and $\vec{K}$ that need to be considered.

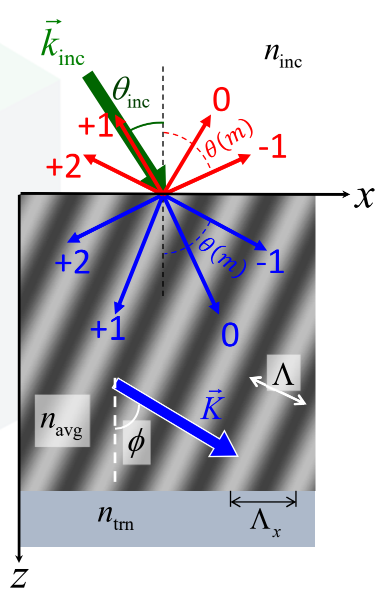
$$K_x = \left|\vec{K}\right| \sin\phi \qquad\qquad k_{x,\text{inc}} = k_0 n_{\text{inc}} \sin\theta_{\text{inc}}$$

From $K_x = \left|\vec{K}\right| \sin\phi$, it follows that the tangential component of the grating period Λ_x is

$$\frac{2\pi}{\Lambda_x} = \frac{2\pi}{\Lambda}\sin\phi \longrightarrow \Lambda_x = \frac{\Lambda}{\sin\phi}$$

Perhaps the most general form of the grating equation is

$$\boxed{n_{\text{avg}} \sin\left[\theta(m)\right] = n_{\text{inc}} \sin\theta_{\text{inc}} - m\frac{\lambda_0}{\Lambda}\sin\phi}$$



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Grating Equation for Planar Diffraction

The angles of the diffraction orders $\theta(m)$ are related to the free space wavelength λ_0, grating period Λ_x, angle of incidence θ_{inc}, and refractive indices (n_{inc}, n_{ref} and n_{trn}) through the famous grating equation.

The grating equation only predicts the directions of the diffraction orders, not how much power is in them.

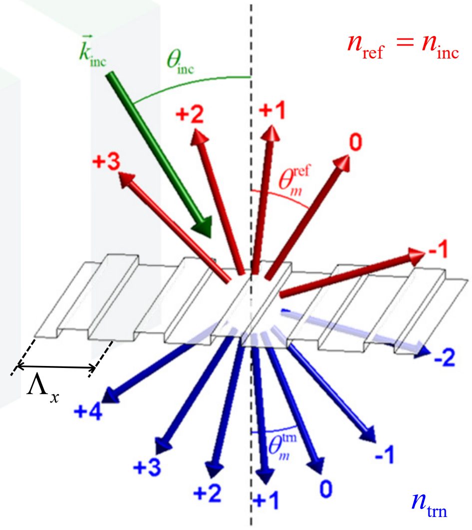
Reflection Region

$$n_{\text{ref}} \sin\left[\theta(m)\right] = n_{\text{inc}} \sin\theta_{\text{inc}} - m\frac{\lambda_0}{\Lambda_x}$$

Transmission Region

$$n_{\text{trn}} \sin\left[\theta(m)\right] = n_{\text{inc}} \sin\theta_{\text{inc}} - m\frac{\lambda_0}{\Lambda_x}$$

Refractive index where diffraction is being observed.



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Diffraction Configurations

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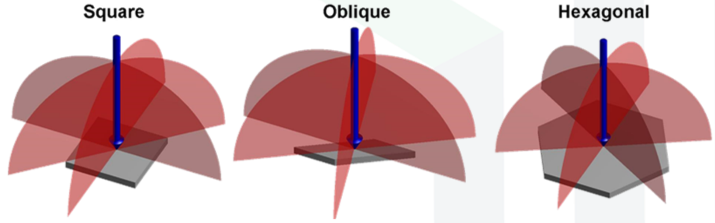
Diffraction in Two Dimensions

- Everything is known about the direction of diffracted waves just from the angle of incidence θ_{inc}, grating period Λ, wavelength λ_0, and refractive indices. It is not necessary to solve Maxwell's equation to determine direction of the diffraction orders.



Diffraction tends to occur primarily along the axes of the lattice.

- The grating equation says nothing about how much power is in the diffracted modes. This information must come from solving Maxwell's equations.

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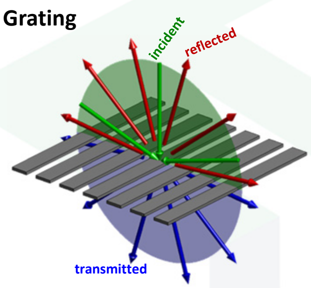
Diffraction Configurations

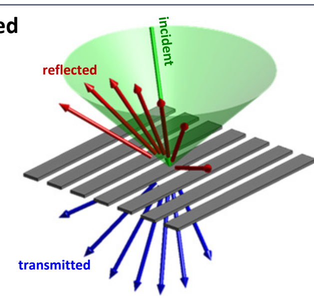
Planar Diffraction from a Ruled Grating

- Diffraction is confined within a plane
- Numerically much simpler than other cases
- E and H modes are independent



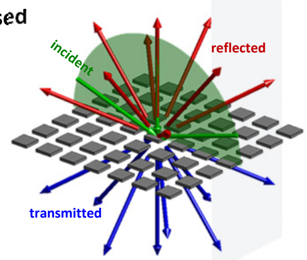
Conical Diffraction from a Ruled Grating

- Diffraction is no longer confined to a plane
- Almost same analytical complexity as crossed grating case, but simpler numerically
- E and H modes are coupled



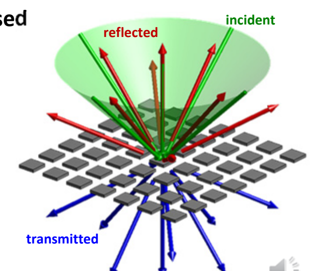
Conical Diffraction from a Crossed Grating with Planar Incidence

- Diffraction occurs in all directions
- Almost same numerical complexity as next case
- E and H modes are coupled



Conical Diffraction from a Crossed Grating

- Diffraction occurs in all directions
- Most complicated case numerically
- E and H modes are coupled
- Essentially the same as previous case



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Dependence of Grating on Diffraction Orders

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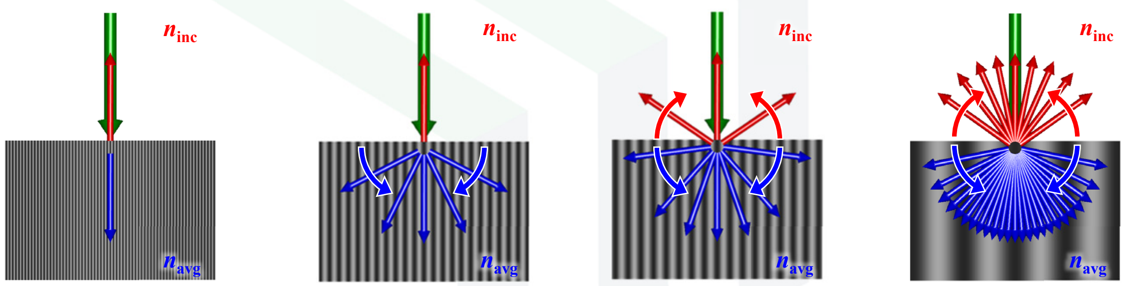
Effect of Grating Period Λ_x

Subwavelength Grating | "Subwavelength" Grating | Low Order Grating | High Order Grating



$\Lambda_x < \dfrac{\lambda_0}{n_{\text{avg}}}$ | $\dfrac{\lambda_0}{n_{\text{avg}}} < \Lambda_x < \dfrac{\lambda_0}{n_{\text{inc}}}$ | $\Lambda_x > \dfrac{\lambda_0}{n_{\text{inc}}}$ | $\Lambda_x >> \dfrac{\lambda_0}{n_{\text{inc}}}$

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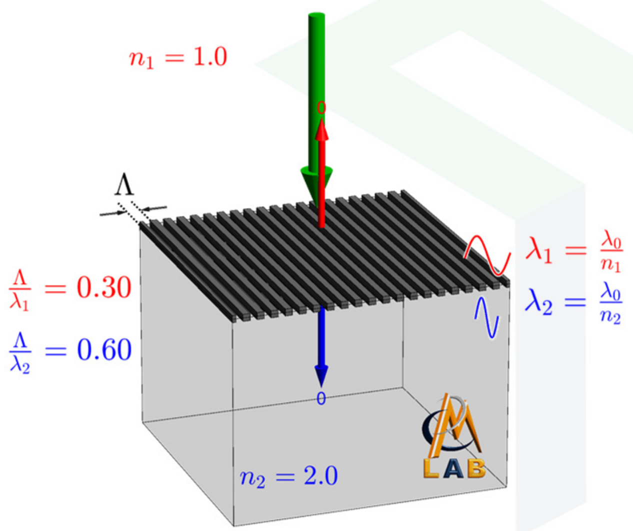
Animation of Ruled Grating Diffraction (1 of 3)



What is Being Visualized

Diffraction orders resulting from normal incidence as the period of the grating is adjusted relative to the wavelength.

Conclusions

- All waves occur within the same plane.
- Longer periods produce higher number of diffraction orders.
- More diffraction orders exist in regions where refractive index is higher.
- Zero order modes are the typical reflected and refracted waves at an interface.
- Diffraction is symmetric about the normal.

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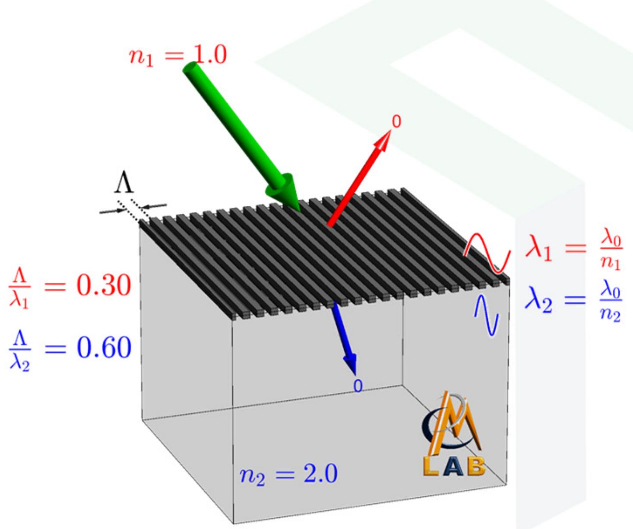
Animation of Ruled Grating Diffraction (2 of 3)



What is Being Visualized

Diffraction orders resulting from oblique incidence as the period of the grating is adjusted relative to the wavelength.

Conclusions

- All waves occur within the same plane.
- Longer periods produce higher number of diffraction orders.
- More diffraction orders exist in regions where refractive index is higher.
- Zero order modes are the typical reflected and refracted waves at an interface.
- Diffraction is asymmetric about the normal due to oblique incidence.

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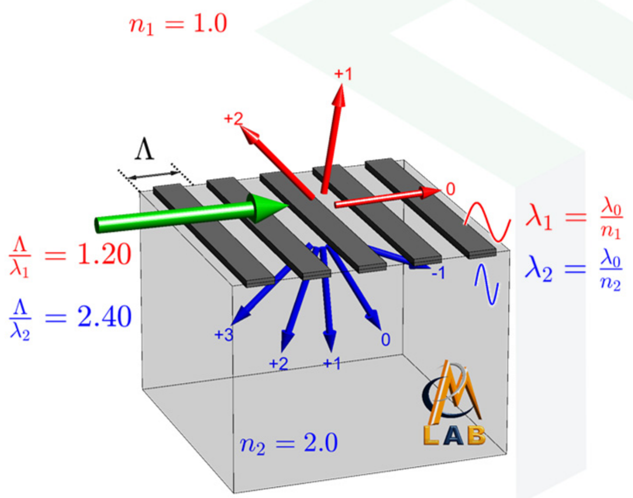
Animation of Ruled Grating Diffraction (3 of 3)



What is Being Visualized

Diffraction orders resulting from a ruled grating as the angle of incidence is changed.

Conclusions

- All waves occur within the same plane.
- More diffraction orders exist in regions where refractive index is higher.
- Zero order modes are the typical reflected and refracted waves at an interface.
- Diffraction is asymmetric about the normal due to oblique incidence.

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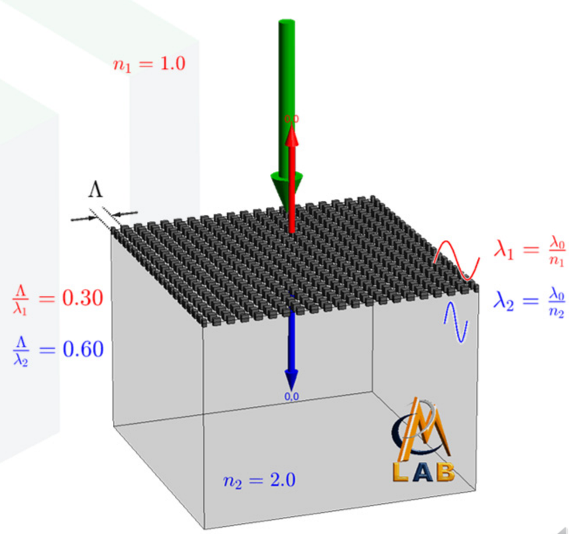
Animation of Crossed Grating Diffraction (1 of 3)

What is Being Visualized

Diffraction orders resulting from normal incidence as the period of the grating is adjusted relative to the wavelength.

Conclusions

- Diffraction occurs in all directions.
- Longer periods produce higher number of diffraction orders.
- More diffraction orders exist in regions where refractive index is higher.
- Zero order modes are the typical reflected and refracted waves at an interface.
- Diffraction is symmetric about the normal.



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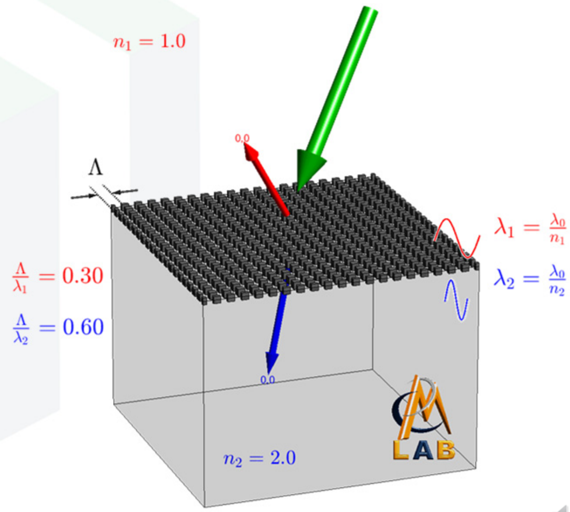
Animation of Crossed Grating Diffraction (2 of 3)

What is Being Visualized

Diffraction orders resulting from oblique incidence as the period of the grating is adjusted relative to the wavelength.

Conclusions

- Diffraction occurs in all directions.
- Longer periods produce higher number of diffraction orders.
- More diffraction orders exist in regions where refractive index is higher.
- Zero order modes are the typical reflected and refracted waves at an interface.
- Diffraction is asymmetric about the normal.



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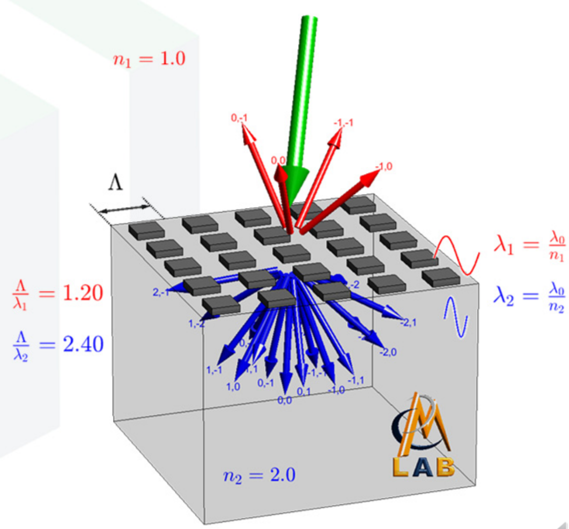
Animation of Crossed Grating Diffraction (3 of 3)

What is Being Visualized

Diffraction orders resulting from a crossed grating as the angle of incidence is meandered.

Conclusions

- Diffraction occurs in all directions.
- Longer periods produce higher number of diffraction orders.
- More diffraction orders exist in regions where refractive index is higher.
- Zero order modes are the typical reflected and refracted waves at an interface.
- Diffraction is symmetric about the normal.



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Analysis of Gratings via the Grating Equation

Grating Cutoff Wavelength

When $\theta(m)$ becomes imaginary, the mth diffraction order is evanescent and cut off.

Assuming normal incidence (i.e. $\theta_{\text{inc}} = 0°$), the grating equation reduces to

$$n \sin\left[\theta(m)\right] = -m \frac{\lambda_0}{\Lambda_x}$$

The first diffraction orders to appear are $m = \pm 1$.

The cutoff for the first-order modes happens when $\theta(\pm 1) = 90°$.

$$\theta(\pm 1) = 90°$$

$$\sin[90°] = 1 = \frac{\lambda_{0,\text{c}}}{n\Lambda_x}$$

$$\boxed{\Lambda_x = \frac{\lambda_{0,\text{c}}}{n} = \lambda_\text{c}}$$

To prevent the first-order modes, the grating must be subwavelength.

$$\Lambda_x < \lambda_\text{c} = \frac{\lambda_{0,\text{c}}}{n}$$

To ensure first-order modes exist, the grating period has to be larger than the wavelength.

$$\Lambda_x > \lambda_\text{c} = \frac{\lambda_{0,\text{c}}}{n}$$

Total Number of Diffraction Orders

Given the grating period Λ_x and the wavelength λ_0, the number of diffraction orders M can be determined.

Assuming normal incidence (i.e. $\theta_{\text{inc}} = 0°$), the grating equation reduces to

$$\sin\left[\theta(m)\right] = -\frac{m\lambda_0}{n_{\text{avg}}\Lambda_x} \quad \rightarrow \quad \left|\sin\left[\theta(m)\right]\right| = \left|\frac{m\lambda_0}{n_{\text{avg}}\Lambda_x}\right| < 1$$

Condition to keep $\theta(m)$ purely real.

Therefore, a maximum value for m that keeps $\theta(m)$ purely real is

$$m_{\max} = \frac{n_{\text{avg}}\Lambda_x}{\lambda_0}$$

The total number of possible diffracted modes M is then $2m_{\max}+1$.

$$\boxed{M = \frac{2n_{\text{avg}}\Lambda_x}{\lambda_0} + 1}$$

$2m_{\max}$ → diffraction is symmetric (i.e. both positive and negative values of m).

+1 → the zero-order must count as a diffraction order.

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Determining Grating Cutoff Conditions

Condition	Requirements
0-order mode	Always exists unless there is total-internal reflection
No 1st-order modes	Grating period must be shorter than what causes $\theta(\pm 1)$ = 90°
Ensure 1st-order modes	Grating period must be larger than what causes $\theta(\pm 1)$ = 90°
No 2nd-order modes	Grating period must be shorter than what causes $\theta(\pm 2)$ = 90°
Ensure 2nd-order modes	Grating period must be larger than what causes $\theta(\pm 2)$ = 90°
No mth-order modes	Grating period must be shorter than what causes $\theta(\pm m)$ = 90°
Ensure mth-order modes	Grating period must be larger than what causes $\theta(\pm m)$ = 90°

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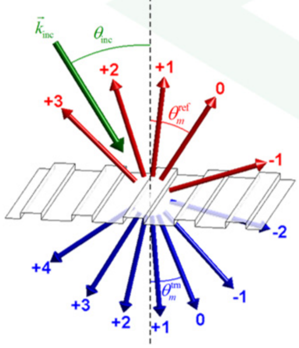
Analysis of Diffraction Gratings

Direction of the Diffraction Orders



$$n\sin\left[\theta(m)\right] = n_{\text{inc}}\sin\theta_{\text{inc}} - m\frac{\lambda_0}{\Lambda}\sin\phi$$

Diffraction Efficiency and Polarization of the Diffraction Orders

Maxwell's equation must be solved to determine amplitude and polarization of the diffraction orders.

$$\nabla\times\vec{E} = -j\omega\mu\vec{H}$$
$$\nabla\times\vec{H} = j\omega\varepsilon\vec{E}$$
$$\nabla\bullet\left(\varepsilon\vec{E}\right) = 0$$
$$\nabla\bullet\left(\mu\vec{H}\right) = 0$$

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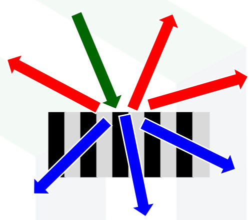
Three Modes of Operation for 1D Gratings

Bragg Grating
Couples power between counter-propagating waves.

Applications
- Thin film optical filters
- Fiber optic gratings
- Wavelength division multiplexing
- Dielectric mirrors
- Photonic crystal waveguides

Diffraction Grating
Couples power between waves at different angles.



Applications
- Beam splitters
- Patterned fanout gratings
- Laser locking
- Spectrometry
- Sensing
- Anti-reflection
- Frequency selective surfaces
- Grating couplers

Long Period Grating
Couples power between co-propagating waves.

Applications
- Sensing
- Directional coupling

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