



Advanced Electromagnetics:  
21<sup>st</sup> Century Electromagnetics

# The Plane Wave Spectrum



## Lecture Outline

- Complex Fourier series
- The plane wave spectrum
- Plane wave spectrum for crossed gratings

# Complex Fourier Series



Jean Baptiste Joseph Fourier

**Born:** March 21, 1768 in Yonne, France.

**Died:** May 16, 1830 in Paris, France.

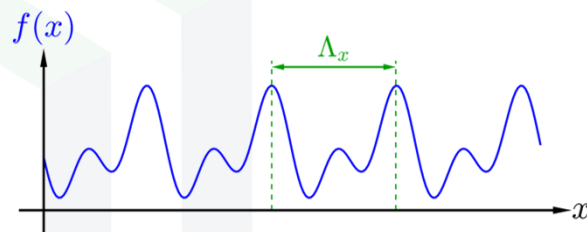
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## 1D Complex Fourier Series

If a function  $f(x)$  is periodic with period  $\Lambda_x$ , it can be expanded into a complex Fourier series.

$$f(x) = \sum_{m=-\infty}^{\infty} a(m) e^{j \frac{2\pi m x}{\Lambda_x}}$$

$$a(m) = \frac{1}{\Lambda_x} \int_{-\Lambda_x/2}^{\Lambda_x/2} f(x) e^{-j \frac{2\pi m x}{\Lambda_x}} dx$$



In computer analysis, only a finite number of terms can be retained in the expansion.

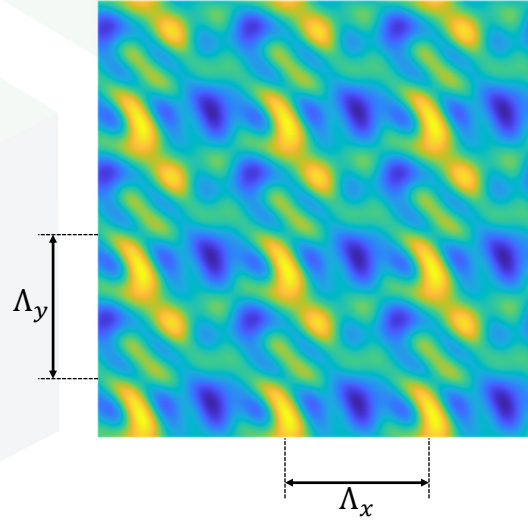
$$f(x) \cong \sum_{m=-M}^M a(m) e^{j \frac{2\pi m x}{\Lambda_x}}$$

## 2D Complex Fourier Series

For 2D periodic functions, the complex Fourier series generalizes to

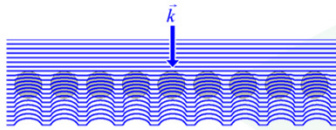
$$f(x, y) = \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} a(p, q) e^{j\left(\frac{2\pi px}{\Lambda_x} + \frac{2\pi qy}{\Lambda_y}\right)}$$

$$a(p, q) = \frac{1}{A} \iint_A f(x, y) e^{-j\left(\frac{2\pi px}{\Lambda_x} + \frac{2\pi qy}{\Lambda_y}\right)} dx dy$$

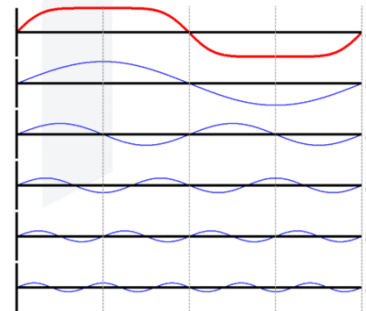
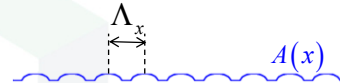


## The Plane Wave Spectrum

## Periodic Functions Can Be Expanded into a Fourier Series



Examine a cross section



Waves in periodic structures obey Bloch's equation.

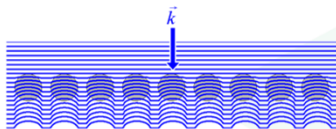
$$E(x, y) = A(x) e^{j\vec{\beta} \cdot \vec{r}}$$

The envelope  $A(x)$  is periodic along  $x$  with period  $\Lambda_x$  so it can be expanded into a Fourier series.

$$A(x) = \sum_{m=-\infty}^{\infty} S(m) e^{-j \frac{2\pi m x}{\Lambda_x}}$$

$$S(m) = \int_{\Lambda} A(x) e^{j \frac{2\pi m x}{\Lambda_x}} dx$$

## Rearrange the Fourier Series (1 of 2)



Examine a cross section



Waves in periodic structures obey Bloch's equation.

$$E(x, y) = A(x) e^{j\vec{\beta} \cdot \vec{r}}$$

$$= \left[ \sum_{m=-\infty}^{\infty} S(m) e^{-j \frac{2\pi m x}{\Lambda_x}} \right] e^{j\vec{\beta} \cdot \vec{r}}$$

Envelope term  $A(x)$  is expanded into a complex Fourier series.

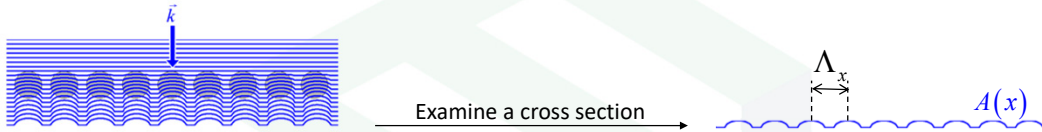
$$= \sum_{m=-\infty}^{\infty} S(m) e^{j\vec{\beta} \cdot \vec{r}} e^{-j \frac{2\pi m x}{\Lambda_x}}$$

Plane wave term  $e^{j\vec{\beta} \cdot \vec{r}}$  is brought inside of the summation.

$$= \sum_{m=-\infty}^{\infty} S(m) e^{j\beta_x x} e^{j\beta_y y} e^{j\beta_z z} e^{-j \frac{2\pi m x}{\Lambda_x}}$$

The dot product  $\vec{\beta} \cdot \vec{r}$  is expanded.

## Rearrange the Fourier Series (2 of 2)



$\beta_x$  can be combined with the last complex exponential.

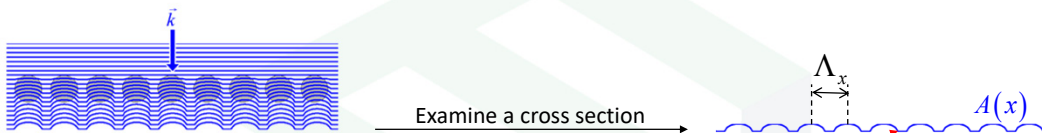
$$E(x, y) = \sum_{m=-\infty}^{\infty} S(m) e^{j\beta_x x} e^{j\beta_y y} e^{j\beta_z z} e^{-j\frac{2\pi m x}{\Lambda_x}}$$

$$= \sum_{m=-\infty}^{\infty} S(m) e^{j\left(\beta_x - \frac{2\pi m}{\Lambda_x}\right)x} e^{j\beta_y y} e^{j\beta_z z}$$

Now let  $k_x(m) = \beta_x - \frac{2\pi m}{\Lambda_x}$ ,  $k_y(m) = \beta_y$  and  $k_z(m) = \beta_z$ .

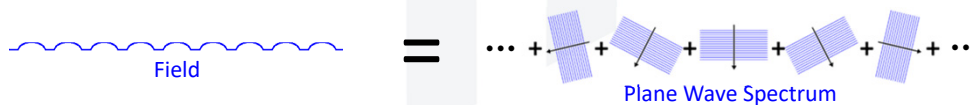
$$E(x, y) = \sum_{m=-\infty}^{\infty} S(m) e^{j\vec{k}(m) \cdot \vec{r}} \quad \vec{k}(m) = \left( \beta_x - \frac{2\pi m}{\Lambda_x} \right) \hat{a}_x + \beta_y \hat{a}_y + \beta_z \hat{a}_z$$

## The Plane Wave Spectrum



$$E(x, y) = \sum_{m=-\infty}^{\infty} S(m) e^{j\vec{k}(m) \cdot \vec{r}}$$

Terms were rearranged to show that a periodic field can also be expressed as an infinite sum of plane waves. This is the "plane wave spectrum" of a periodic field.



## Longitudinal Wave Vector Components of the Plane Wave Spectrum

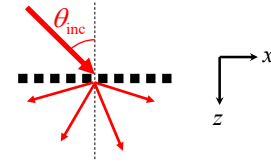
The wave incident on a 1D grating can be written as

$$E_{\text{inc}}(x, y) = E_0 e^{j(k_{x,\text{inc}}x + k_{z,\text{inc}}z)}$$

$$k_{x,\text{inc}} = k_0 n_{\text{inc}} \sin \theta_{\text{inc}}$$

$$k_{y,\text{inc}} = 0$$

$$k_{z,\text{inc}} = k_0 n_{\text{inc}} \cos \theta_{\text{inc}}$$



Phase matching into the grating leads to

$$k_x(m) = k_{x,\text{inc}} - m \frac{2\pi}{\Lambda_x} \quad m = \dots, -2, -1, 0, 1, 2, \dots$$

Note:  $k_x(m)$  is always real.

Each plane wave must satisfy the dispersion relation.

$$k_x^2(m) + k_z^2(m) = (k_0 n_{\text{grat}})^2$$



$$k_z(m) = \sqrt{(k_0 n_{\text{grat}})^2 - k_x^2(m)}$$

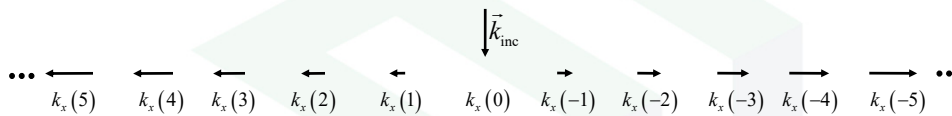
There are two possible solutions here.

1. Purely real  $k_z(m)$
2. Purely imaginary  $k_z(m)$

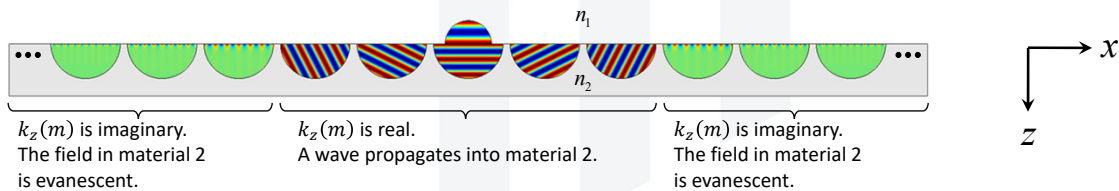


## Visualizing Phase Matching into the Grating

The wave vector expansion for the first 11 diffraction orders can be visualized as...



Each of these is phase matched into material 2. The longitudinal component of the wave vector is calculated using the dispersion relation in material 2.

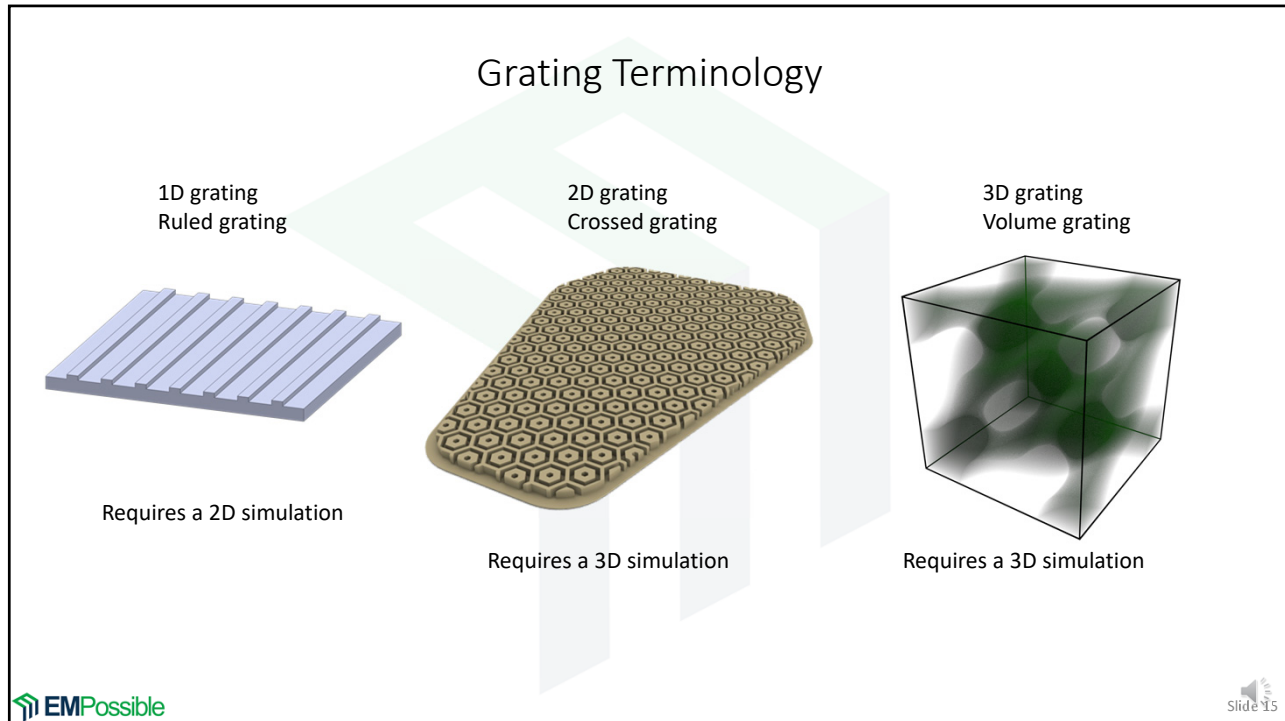


Note: The "evanescent" fields in material 2 do not have a completely imaginary wave vector. They have a purely real  $k_x(m)$  so power does flow in the transverse direction.

## Conclusions About the Plane Wave Spectrum

- Fields in periodic media take on the same periodicity as the media they are in.
- Periodic fields can be expanded into a Fourier series.
- Each term of the Fourier series represents a diffraction order.
- Since there are an infinite number of terms in the Fourier series, there are an infinite number of diffraction orders.
- Only a few of the diffraction orders are propagating waves and only these transport power to and from a device.

## Plane Wave Spectrum from Crossed Gratings



## Diffraction from Crossed Gratings

Doubly-periodic gratings, also called crossed gratings, can diffract waves into many directions.

Square gratings are described by two grating vectors,  $\vec{K}_x$  and  $\vec{K}_y$ .

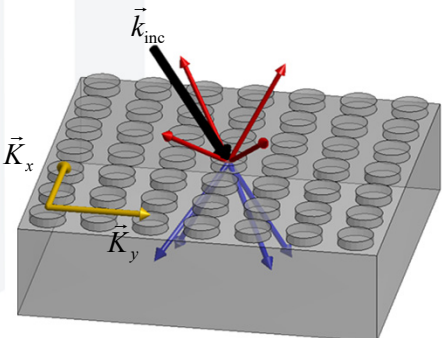
Two boundary conditions are necessary here.

$$k_x(m) = k_{x,\text{inc}} - mK_x \quad m = \dots, -2, -1, 0, 1, 2, \dots$$

$$k_y(n) = k_{y,\text{inc}} - nK_y \quad n = \dots, -2, -1, 0, 1, 2, \dots$$

$$\vec{K}_x = \frac{2\pi}{\Lambda_x} \hat{x}$$

$$\vec{K}_y = \frac{2\pi}{\Lambda_y} \hat{y}$$



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## Transverse Wave Vector Expansion (1 of 2)

Crossed gratings can diffract in all directions.

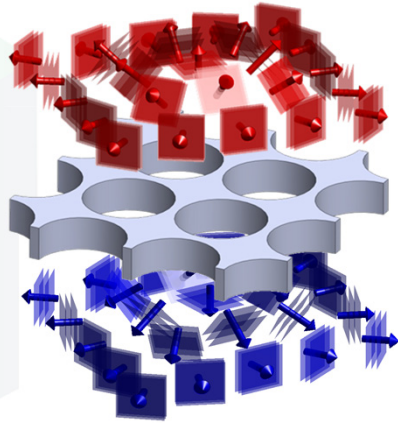
To quantify diffraction for crossed gratings, an expansion must be calculated for both  $k_x$  and  $k_y$ .

$$k_x(m) = k_{x,\text{inc}} - \frac{2\pi m}{\Lambda_x} \quad m = -\infty, \dots, -2, -1, 0, 1, 2, \dots \infty$$

$$k_y(n) = k_{y,\text{inc}} - \frac{2\pi n}{\Lambda_y} \quad n = -\infty, \dots, -2, -1, 0, 1, 2, \dots \infty$$

$$\vec{k}_t(m, n) = k_x(m)\hat{x} + k_y(n)\hat{y}$$

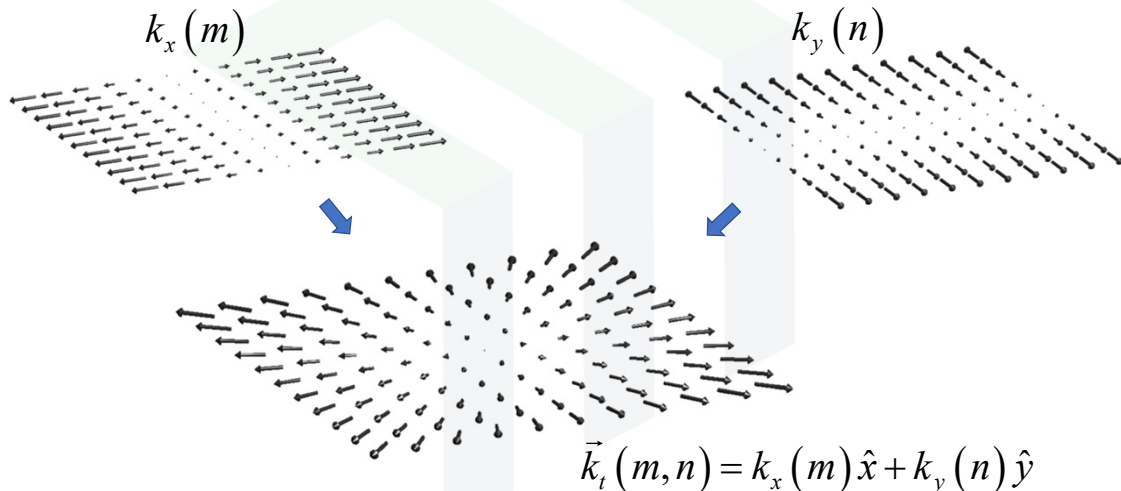
```
% TRANSVERSE WAVE VECTOR EXPANSION
M = [-floor(Nx/2):floor(Nx/2)]';
N = [-floor(Ny/2):floor(Ny/2)]';
kx = kxinc - 2*pi*M/Lx;
ky = kyinc - 2*pi*N/Ly;
[ky, kx] = meshgrid(ky, kx);
```



This code is used in many numerical methods including 2D PWEM, 3D RCWA, 3D FDTD, 3D FDFD, 3D MoL, and more.

## Transverse Wave Vector Expansion (2 of 2)

The vector expansions can be visualized this way...



## Longitudinal Wave Vector Expansion (1 of 2)

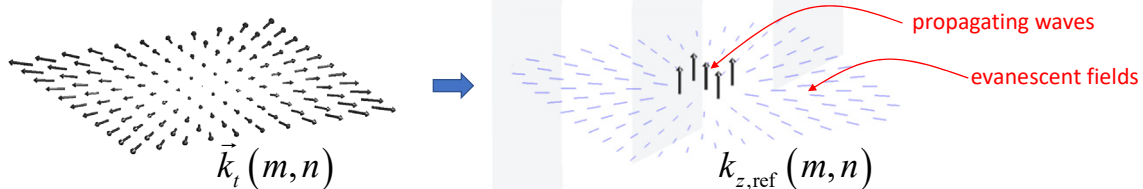
The longitudinal components of the wave vectors are computed as

$$k_{z,\text{ref}}(m,n) = -\sqrt{(k_0 n_{\text{ref}})^2 - k_x^2(m) - k_y^2(m)}$$

$$k_{z,\text{tm}}(m,n) = \sqrt{(k_0 n_{\text{tm}})^2 - k_x^2(m) - k_y^2(m)}$$

```
% Compute Longitudinal Wave Vector Components
k0    = 2*pi/lam;
kzinc = k0*nref;
kzref = -sqrt((k0*nref)^2 - kx.^2 - ky.^2);
kztrn = +sqrt((k0*ntrn)^2 - kx.^2 - ky.^2);
```

The lowest-order modes will have real  $k_z$ 's. These correspond to propagating waves. The others will have imaginary  $k_z$ 's and correspond to evanescent waves that do not transport power away from the diffraction grating.



## Longitudinal Wave Vector Expansion (2 of 2)

The overall wave vector expansion can be visualized this way...

