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Advanced Computation:
Computational Electromagnetics

4×4 Matrix Formulation

1

Outline

- Formulation of 4×4 matrix equation
- Solution in a LHI medium
- Interpreting the solution

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Slide 2

2

Formulation of 4×4 Matrix Equation

Slide 3

3

Waves in Homogeneous Media

The positive sign convention will be used. So a wave propagation in the $+z$ direction is written as

$$\vec{E}\left(\vec{r}\right)=\vec{E}_0 e^{jkz}$$

$$\vec{\tilde{H}}\left(\vec{r}\right)=\vec{\tilde{H}}_0 e^{jkz}$$

If follows that the electric and magnetic field components of a wave propagating in the $\vec{k}$ direction is written as

$$\vec{E}\left(\vec{r}\right)=\vec{E}_0 e^{j\vec{k}\bullet\vec{r}}=\vec{E}_0 e^{jk_x x}e^{jk_y y}e^{jk_z z}$$

$$\vec{\tilde{H}}\left(\vec{r}\right)=\vec{\tilde{H}}_0 e^{j\vec{k}\bullet\vec{r}}=\vec{\tilde{H}}_0 e^{jk_x x}e^{jk_y y}e^{jk_z z}$$

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4

Spatial Derivatives of Plane Waves

$$\vec{E}(\vec{r}) = \vec{E}_0 e^{j\vec{k}\bullet\vec{r}} = \vec{E}_0 e^{jk_x x} e^{jk_y y} e^{jk_z z}$$

What if a spatial derivative is calculated is the x direction?

$$\frac{\partial}{\partial x}\vec{E}(\vec{r}) = \frac{\partial}{\partial x}\left(\vec{E}_0 e^{jk_x x} e^{jk_y y} e^{jk_z z}\right) = jk_x \vec{E}_0 e^{jk_y y} e^{jk_z z} e^{jk_x x} = jk_x \vec{E}(\vec{r}) \quad \rightarrow \quad \boxed{\frac{\partial}{\partial x} = jk_x}$$

Similarly, a spatial derivative in the y direction is

$$\frac{\partial}{\partial y}\vec{E}(\vec{r}) = \frac{\partial}{\partial y}\left(\vec{E}_0 e^{jk_x x} e^{jk_y y} e^{jk_z z}\right) = jk_y \vec{E}_0 e^{jk_y y} e^{jk_z z} e^{jk_x x} = jk_y \vec{E}(\vec{r}) \quad \rightarrow \quad \boxed{\frac{\partial}{\partial y} = jk_y}$$

In the framework of TMM, it cannot be said that $\partial/\partial z = jk_z$ because the structure is not homogeneous in the z direction.

$$\boxed{\frac{\partial}{\partial z} \neq jk_z}$$

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Slide 5

5

Starting Point

Start with Maxwell's equations in the following form. Linear, homogeneous and isotropic (LHI) materials are assumed and the positive sign convention for waves is adopted.

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = k_0 \mu_r \tilde{H}_x \qquad \frac{\partial \tilde{H}_z}{\partial y} - \frac{\partial \tilde{H}_y}{\partial z} = k_0 \varepsilon_r E_x$$

$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = k_0 \mu_r \tilde{H}_y \qquad \frac{\partial \tilde{H}_x}{\partial z} - \frac{\partial \tilde{H}_z}{\partial x} = k_0 \varepsilon_r E_y$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = k_0 \mu_r \tilde{H}_z \qquad \frac{\partial \tilde{H}_y}{\partial x} - \frac{\partial \tilde{H}_x}{\partial y} = k_0 \varepsilon_r E_z$$

$\vec{\tilde{H}} = +j\eta_0 \vec{H}$ Normalized magnetic field

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6

Reduction of Maxwell's Equations to 1D

Let $\partial/\partial x = jk_x$ and $\partial/\partial y = jk_y$, and Maxwell's equations become

$$jk_y E_z - \frac{\partial E_y}{\partial z} = k_0 \mu_r \tilde{H}_x \qquad jk_y \tilde{H}_z - \frac{\partial \tilde{H}_y}{\partial z} = k_0 \varepsilon_r E_x$$

$$\frac{\partial E_x}{\partial z} - jk_x E_z = k_0 \mu_r \tilde{H}_y \qquad \frac{\partial \tilde{H}_x}{\partial z} - jk_x \tilde{H}_z = k_0 \varepsilon_r E_y$$

$$jk_x E_y - jk_y E_x = k_0 \mu_r \tilde{H}_z \qquad jk_x \tilde{H}_y - jk_y \tilde{H}_x = k_0 \varepsilon_r E_z$$

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7

Reduction of Maxwell's Equations to 1D

The only independent variable left is z so the derivate becomes ordinary, $\partial/\partial z = d/dz$.

$$jk_y E_z - \frac{dE_y}{dz} = k_0 \mu_r \tilde{H}_x \qquad jk_y \tilde{H}_z - \frac{d\tilde{H}_y}{dz} = k_0 \varepsilon_r E_x$$

$$\frac{dE_x}{dz} - jk_x E_z = k_0 \mu_r \tilde{H}_y \qquad \frac{d\tilde{H}_x}{dz} - jk_x \tilde{H}_z = k_0 \varepsilon_r E_y$$

$$jk_x E_y - jk_y E_x = k_0 \mu_r \tilde{H}_z \qquad jk_x \tilde{H}_y - jk_y \tilde{H}_x = k_0 \varepsilon_r E_z$$

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8

Normalize Spatial Coordinate z

The spatial coordinate z is normalized according to $z' = k_0 z$.

$$jk_y E_z - k_0 \frac{dE_y}{dz'} = k_0 \mu_r \tilde{H}_x \qquad jk_y \tilde{H}_z - k_0 \frac{d\tilde{H}_y}{dz'} = k_0 \varepsilon_r E_x$$

$$k_0 \frac{dE_x}{dz'} - jk_x E_z = k_0 \mu_r \tilde{H}_y \qquad k_0 \frac{d\tilde{H}_x}{dz'} - jk_x \tilde{H}_z = k_0 \varepsilon_r E_y$$

$$jk_x E_y - jk_y E_x = k_0 \mu_r \tilde{H}_z \qquad jk_x \tilde{H}_y - jk_y \tilde{H}_x = k_0 \varepsilon_r E_z$$

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9

Normalize Wave Vector Components k_x, k_y and k_z

The wave vector components are normalized according to $\tilde{k}_x = k_x / k_0$, $\tilde{k}_y = k_y / k_0$ and $\tilde{k}_z = k_z / k_0$.

$$j\tilde{k}_y E_z - \frac{dE_y}{dz'} = \mu_r \tilde{H}_x \qquad j\tilde{k}_y \tilde{H}_z - \frac{d\tilde{H}_y}{dz'} = \varepsilon_r E_x$$

$$\frac{dE_x}{dz'} - j\tilde{k}_x E_z = \mu_r \tilde{H}_y \qquad \frac{d\tilde{H}_x}{dz'} - j\tilde{k}_x \tilde{H}_z = \varepsilon_r E_y$$

$$j\tilde{k}_x E_y - j\tilde{k}_y E_x = \mu_r \tilde{H}_z \qquad j\tilde{k}_x \tilde{H}_y - j\tilde{k}_y \tilde{H}_x = \varepsilon_r E_z$$

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10

Solve for the Longitudinal Components $\tilde{H}_z$ and E_z

Solve the third and sixth equations for the longitudinal field components $\tilde{H}_z$ and E_z.

$$j\tilde{k}_y E_z - \frac{dE_y}{dz'} = \mu_r \tilde{H}_x$$

$$\frac{dE_x}{dz'} - j\tilde{k}_x E_z = \mu_r \tilde{H}_y$$

$$j\tilde{k}_x E_y - j\tilde{k}_y E_x = \mu_r \tilde{H}_z \quad \rightarrow \quad \tilde{H}_z = \frac{j}{\mu_r}\left(\tilde{k}_x E_y - \tilde{k}_y E_x\right)$$

$$j\tilde{k}_y \tilde{H}_z - \frac{d\tilde{H}_y}{dz'} = \varepsilon_r E_x$$

$$\frac{d\tilde{H}_x}{dz'} - j\tilde{k}_x \tilde{H}_z = \varepsilon_r E_y$$

$$j\tilde{k}_x \tilde{H}_y - j\tilde{k}_y \tilde{H}_x = \varepsilon_r E_z \quad \rightarrow \quad E_z = \frac{j}{\varepsilon_r}\left(\tilde{k}_x \tilde{H}_y - \tilde{k}_y \tilde{H}_x\right)$$

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Eliminate the Longitudinal Components $\tilde{H}_z$ and E_z

Eliminate the longitudinal field components $\tilde{H}_z$ and E_z by substituting them back into the remaining equations.

$$j\tilde{k}_y E_z - \frac{dE_y}{dz'} = \mu_r \tilde{H}_x$$

$$\frac{dE_x}{dz'} - j\tilde{k}_x E_z = \mu_r \tilde{H}_y$$

$$\tilde{H}_z = \frac{j}{\mu_r}\left(\tilde{k}_x E_y - \tilde{k}_y E_x\right)$$

$$j\tilde{k}_y \tilde{H}_z - \frac{d\tilde{H}_y}{dz'} = \varepsilon_r E_x$$

$$\frac{d\tilde{H}_x}{dz'} - j\tilde{k}_x \tilde{H}_z = \varepsilon_r E_y$$

$$E_z = \frac{j}{\varepsilon_r}\left(\tilde{k}_x \tilde{H}_y - \tilde{k}_y \tilde{H}_x\right)$$

$$\tilde{k}_y^2 \tilde{H}_x - \tilde{k}_x \tilde{k}_y \tilde{H}_y - \varepsilon_r \frac{dE_y}{dz'} = \mu_r \varepsilon_r \tilde{H}_x$$

$$\varepsilon_r \frac{dE_x}{dz'} + \tilde{k}_x^2 \tilde{H}_y - \tilde{k}_x \tilde{k}_y \tilde{H}_x = \mu_r \varepsilon_r \tilde{H}_y$$

$$\tilde{k}_y^2 E_x - \tilde{k}_x \tilde{k}_y E_y - \mu_r \frac{d\tilde{H}_y}{dz'} = \mu_r \varepsilon_r E_x$$

$$\mu_r \frac{d\tilde{H}_x}{dz'} + \tilde{k}_x^2 E_y - \tilde{k}_x \tilde{k}_y E_x = \mu_r \varepsilon_r E_y$$

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Rearrange the Equations

Rearrange the terms and the order of the equations.

$$\tilde{k}_y^2 \tilde{H}_x - \tilde{k}_x \tilde{k}_y \tilde{H}_y - \varepsilon_r \frac{dE_y}{dz'} = \mu_r \varepsilon_r \tilde{H}_x$$

$$\varepsilon_r \frac{dE_x}{dz'} + \tilde{k}_x^2 \tilde{H}_y - \tilde{k}_x \tilde{k}_y \tilde{H}_x = \mu_r \varepsilon_r \tilde{H}_y$$

$$\frac{dE_x}{dz'} = \frac{\tilde{k}_x \tilde{k}_y}{\varepsilon_r} \tilde{H}_x + \left(\mu_r - \frac{\tilde{k}_x^2}{\varepsilon_r} \right) \tilde{H}_y$$

$$\frac{dE_y}{dz'} = \left(\frac{\tilde{k}_y^2}{\varepsilon_r} - \mu_r \right) \tilde{H}_x - \frac{\tilde{k}_x \tilde{k}_y}{\varepsilon_r} \tilde{H}_y$$

$$\tilde{k}_y^2 E_x - \tilde{k}_x \tilde{k}_y E_y - \mu_r \frac{d\tilde{H}_y}{dz'} = \mu_r \varepsilon_r E_x$$

$$\mu_r \frac{d\tilde{H}_x}{dz'} + \tilde{k}_x^2 E_y - \tilde{k}_x \tilde{k}_y E_x = \mu_r \varepsilon_r E_y$$

$$\frac{d\tilde{H}_x}{dz'} = \frac{\tilde{k}_x \tilde{k}_y}{\mu_r} E_x + \left(\varepsilon_r - \frac{\tilde{k}_x^2}{\mu_r} \right) E_y$$

$$\frac{d\tilde{H}_y}{dz'} = \left(\frac{\tilde{k}_y^2}{\mu_r} - \varepsilon_r \right) E_x - \frac{\tilde{k}_x \tilde{k}_y}{\mu_r} E_y$$

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Slide 13

13

Matrix Form of Maxwell's Equations

The remaining four equations can be written in matrix form as

$$\frac{dE_x}{dz'} = \frac{\tilde{k}_x \tilde{k}_y}{\varepsilon_r} \tilde{H}_x + \left(\mu_r - \frac{\tilde{k}_x^2}{\varepsilon_r} \right) \tilde{H}_y$$

$$\frac{dE_y}{dz'} = \left(\frac{\tilde{k}_y^2}{\varepsilon_r} - \mu_r \right) \tilde{H}_x - \frac{\tilde{k}_x \tilde{k}_y}{\varepsilon_r} \tilde{H}_y$$

$$\frac{d\tilde{H}_x}{dz'} = \frac{\tilde{k}_x \tilde{k}_y}{\mu_r} E_x + \left(\varepsilon_r - \frac{\tilde{k}_x^2}{\mu_r} \right) E_y$$

$$\frac{d\tilde{H}_y}{dz'} = \left(\frac{\tilde{k}_y^2}{\mu_r} - \varepsilon_r \right) E_x - \frac{\tilde{k}_x \tilde{k}_y}{\mu_r} E_y$$

4×4 matrix equation

$$\frac{d}{dz'} \begin{bmatrix} E_x \\ E_y \\ \tilde{H}_x \\ \tilde{H}_y \end{bmatrix} = \begin{bmatrix} 0 & 0 & \frac{\tilde{k}_x \tilde{k}_y}{\varepsilon_r} & \mu_r - \frac{\tilde{k}_x^2}{\varepsilon_r} \\ 0 & 0 & \frac{\tilde{k}_y^2}{\varepsilon_r} - \mu_r & -\frac{\tilde{k}_x \tilde{k}_y}{\varepsilon_r} \\ \frac{\tilde{k}_x \tilde{k}_y}{\mu_r} & \varepsilon_r - \frac{\tilde{k}_x^2}{\mu_r} & 0 & 0 \\ \frac{\tilde{k}_y^2}{\mu_r} - \varepsilon_r & -\frac{\tilde{k}_x \tilde{k}_y}{\mu_r} & 0 & 0 \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ \tilde{H}_x \\ \tilde{H}_y \end{bmatrix}$$

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Slide 14

14

BTW…for Fully Anisotropic Materials

$$\frac{\partial}{\partial z'}\begin{bmatrix} E_x \\ E_y \\ \tilde{H}_x \\ \tilde{H}_y \end{bmatrix} = \begin{bmatrix} -j\left(\tilde{k}_y \frac{\mu_{yz}}{\mu_{zz}} + \tilde{k}_x \frac{\varepsilon_{zx}}{\varepsilon_{zz}}\right) & j\tilde{k}_x\left(\frac{\mu_{yz}}{\mu_{zz}} - \frac{\varepsilon_{zy}}{\varepsilon_{zz}}\right) & \left(\frac{\tilde{k}_x \tilde{k}_y}{\varepsilon_{zz}} + \mu_{yx} - \frac{\mu_{yz}\mu_{zx}}{\mu_{zz}}\right) & \left(-\frac{\hat{k}_x^2}{\varepsilon_{zz}} + \mu_{yy} - \frac{\mu_{yz}\mu_{zy}}{\mu_{zz}}\right) \\ j\tilde{k}_y\left(\frac{\mu_{xz}}{\mu_{zz}} - \frac{\varepsilon_{zx}}{\varepsilon_{zz}}\right) & -j\left(\tilde{k}_x \frac{\mu_{xz}}{\mu_{zz}} + \tilde{k}_y \frac{\varepsilon_{zy}}{\varepsilon_{zz}}\right) & \left(\frac{\tilde{k}_y^2}{\varepsilon_{zz}} - \mu_{xx} + \frac{\mu_{xz}\mu_{zx}}{\mu_{zz}}\right) & \left(-\frac{\tilde{k}_x \tilde{k}_y}{\varepsilon_{zz}} - \mu_{xy} + \frac{\mu_{xz}\mu_{zy}}{\mu_{zz}}\right) \\ \left(\frac{\tilde{k}_x \tilde{k}_y}{\mu_{zz}} + \varepsilon_{yx} - \frac{\varepsilon_{yz}\varepsilon_{zx}}{\varepsilon_{zz}}\right) & \left(-\frac{\tilde{k}_x^2}{\mu_{zz}} + \varepsilon_{yy} - \frac{\varepsilon_{yz}\varepsilon_{zy}}{\varepsilon_{zz}}\right) & -j\left(\tilde{k}_y \frac{\varepsilon_{yz}}{\varepsilon_{zz}} + \tilde{k}_x \frac{\mu_{zx}}{\mu_{zz}}\right) & j\tilde{k}_x\left(\frac{\varepsilon_{yz}}{\varepsilon_{zz}} - \frac{\mu_{zy}}{\mu_{zz}}\right) \\ \left(\frac{\tilde{k}_y^2}{\mu_{zz}} - \varepsilon_{xx} + \frac{\varepsilon_{xz}\varepsilon_{zx}}{\varepsilon_{zz}}\right) & \left(-\frac{\tilde{k}_x \tilde{k}_y}{\mu_{zz}} - \varepsilon_{xy} + \frac{\varepsilon_{xz}\varepsilon_{zy}}{\varepsilon_{zz}}\right) & j\tilde{k}_y\left(\frac{\varepsilon_{xz}}{\varepsilon_{zz}} - \frac{\mu_{zx}}{\mu_{zz}}\right) & -j\left(\tilde{k}_x \frac{\varepsilon_{xz}}{\varepsilon_{zz}} + \tilde{k}_y \frac{\mu_{zy}}{\mu_{zz}}\right) \end{bmatrix}\begin{bmatrix} E_x \\ E_y \\ \tilde{H}_x \\ \tilde{H}_y \end{bmatrix}$$

Note: This is for the $e^{+j\beta z}$ sign convention.

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Slide 15

15

Solution in an LHI Medium

Slide 16

16

Compact 4×4 Matrix Equation

The 4×4 matrix equation can be written more compactly as

$$\frac{d\boldsymbol{\psi}}{dz'} - \boldsymbol{\Omega}\boldsymbol{\psi} = \mathbf{0}$$

$$\boldsymbol{\psi}(z') = \begin{bmatrix} E_x(z') \\ E_y(z') \\ \tilde{H}_x(z') \\ \tilde{H}_y(z') \end{bmatrix} \qquad \boldsymbol{\Omega} = \begin{bmatrix} 0 & 0 & \dfrac{\tilde{k}_x \tilde{k}_y}{\varepsilon_r} & \mu_r - \dfrac{\tilde{k}_x^2}{\varepsilon_r} \\ 0 & 0 & \dfrac{\tilde{k}_y^2}{\varepsilon_r} - \mu_r & -\dfrac{\tilde{k}_x \tilde{k}_y}{\varepsilon_r} \\ \dfrac{\tilde{k}_x \tilde{k}_y}{\mu_r} & \varepsilon_r - \dfrac{\tilde{k}_x^2}{\mu_r} & 0 & 0 \\ \dfrac{\tilde{k}_y^2}{\mu_r} - \varepsilon_r & -\dfrac{\tilde{k}_x \tilde{k}_y}{\mu_r} & 0 & 0 \end{bmatrix}$$

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Slide 17

17

General Solution

The 4×4 matrix differential equation is actually a matrix differential equation.

$$\frac{d\boldsymbol{\psi}}{dz'} - \boldsymbol{\Omega}\boldsymbol{\psi} = \mathbf{0}$$

It has the following general solution.

$$\boldsymbol{\psi}(z') = e^{\boldsymbol{\Omega} z'}\boldsymbol{\psi}(0)$$

This is easy to write, but how is the exponential of a matrix $e^{\boldsymbol{\Omega} z'}$ calculated?

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18

Functions of Matrices (1 of 2)

It is sometimes necessary to evaluate the function of a matrix $f(\mathbf{A})$.

$$f(\mathbf{A}) = ?$$

It is NOT correct to calculate the function applied to every element in the matrix $\mathbf{A}$ individually. A different technique must be used.

$$f(\mathbf{A}) \neq \begin{bmatrix} f(A_{11}) & f(A_{12}) & \cdots & f(A_{1N}) \\ f(A_{21}) & f(A_{22}) & \cdots & f(A_{2N}) \\ \vdots & \vdots & \ddots & \\ f(A_{M1}) & f(A_{M2}) & & f(A_{MN}) \end{bmatrix}$$

WRONG!!

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19

Functions of Matrices (2 of 2)

To calculate $f(\mathbf{A})$ correctly, first calculate the eigen-vectors and eigen-values of the matrix $\mathbf{A}$.

$$\mathbf{A} \rightarrow \begin{matrix} \mathbf{V} \text{ eigen-vector matrix of } \mathbf{A} \\ \mathbf{D} \text{ eigen-value matrix of } \mathbf{A} \end{matrix}$$

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[V,D] = eig(A);
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$$\mathbf{V} = \begin{bmatrix} v_{11} & v_{12} & \cdots & v_{1M} \\ v_{21} & v_{22} & \cdots & v_{2M} \\ \vdots & \vdots & \ddots & \\ v_{M1} & v_{M2} & & v_{MM} \end{bmatrix} \qquad \mathbf{D} = \begin{bmatrix} D_1 & 0 & \cdots & 0 \\ 0 & D_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & & D_M \end{bmatrix}$$

Given the eigen-vector matrix $\mathbf{V}$ and the eigen-value matrix $\mathbf{D}$, the function of the matrix is evaluated as

$$f(\mathbf{A}) = \mathbf{V} \cdot f(\mathbf{D}) \cdot \mathbf{V}^{-1} \qquad f(\mathbf{D}) = \begin{bmatrix} f(D_1) & 0 & \cdots & 0 \\ 0 & f(D_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & f(D_M) \end{bmatrix}$$

$f(\mathbf{D})$ is very easy to evaluate because $\mathbf{D}$ is a diagonal matrix so the function only has to be performed individually on the diagonal elements.

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20

Revised Solution (1 of 2)

So far, the following matrix differential equation had a general solution of

$$\frac{d\boldsymbol{\psi}}{dz'} - \boldsymbol{\Omega}\boldsymbol{\psi} = \mathbf{0} \quad \rightarrow \quad \boldsymbol{\psi}(z') = e^{\boldsymbol{\Omega}z'}\boldsymbol{\psi}(0)$$

It is now possible to evaluate the matrix exponential using the eigen-values and eigen-vectors of the matrix $\boldsymbol{\Omega}$.

$$\boldsymbol{\Omega} \quad \rightarrow \quad \begin{matrix} \mathbf{W} \text{ eigen-vector matrix} \\ \boldsymbol{\lambda} \text{ eigen-value matrix} \end{matrix}$$

$$e^{\boldsymbol{\Omega}z'} = \mathbf{W}e^{\boldsymbol{\lambda}z'}\mathbf{W}^{-1} \qquad e^{\boldsymbol{\lambda}z'} = \begin{bmatrix} e^{\lambda_1 z'} & 0 & 0 & 0 \\ 0 & e^{\lambda_2 z'} & 0 & 0 \\ 0 & 0 & e^{\lambda_3 z'} & 0 \\ 0 & 0 & 0 & e^{\lambda_4 z'} \end{bmatrix}$$

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21

Revised Solution (2 of 2)

The solution to the matrix differential equation is therefore

$$\frac{d\boldsymbol{\psi}}{dz'} - \boldsymbol{\Omega}\boldsymbol{\psi} = \mathbf{0} \quad \rightarrow \quad \boldsymbol{\psi}(z') = e^{\boldsymbol{\Omega}z'}\boldsymbol{\psi}(0) = \mathbf{W}e^{\boldsymbol{\lambda}z'}\underbrace{\mathbf{W}^{-1}\boldsymbol{\psi}(0)}_{\mathbf{c}}$$

The unknown initial values $\boldsymbol{\psi}(0)$ can be combined with $\mathbf{W}^{-1}$ because that product just leads to another column vector of unknown constants.

The final solution is then

$$\frac{d\boldsymbol{\psi}}{dz'} - \boldsymbol{\Omega}\boldsymbol{\psi} = \mathbf{0} \quad \rightarrow \quad \boldsymbol{\psi}(z') = \mathbf{W}e^{\boldsymbol{\lambda}z'}\mathbf{c}$$

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Slide 22

22

Interpreting the Solution

23

Anatomy of the Solution

$$\boldsymbol{\psi}(z') = \mathbf{W}e^{\boldsymbol{\lambda} z'}\mathbf{c}$$

$\boldsymbol{\psi}(z')$ – Overall solution which is the sum of all the modes at plane z'.

c – Column vector containing the amplitude coefficient of each of the modes. This quantifies how much power is in each mode.

W – Square matrix who's column vectors describe the "modes" that can exist in the material. These are essentially pictures of the modes which quantify the relative amplitudes of E_x, E_y, H_x, and H_y.

$e^{\lambda z'}$ – Diagonal matrix describing how the modes propagate. This includes accumulation of phase as well as decaying (loss) or growing (gain) amplitude.

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24

An Example…

For a layer with $\varepsilon_r = 9.0$ and $\mu_r = 1.0$ (i.e. $n = 3.0$) and a wave at normal incidence, the $\mathbf{\Omega}$ matrix will be

$$\mathbf{\Omega} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 9 & 0 & 0 \\ -9 & 0 & 0 & 0 \end{bmatrix}$$

The matrix $\mathbf{\Omega}$ has the following eigen-vectors $\mathbf{W}$ and eigen-values $\boldsymbol{\lambda}$.

$$\mathbf{W} = \begin{bmatrix} -j0.32 & j0.32 & 0 & 0 \\ 0 & 0 & j0.32 & -j0.32 \\ 0 & 0 & 0.95 & 0.95 \\ 0.95 & 0.95 & 0 & 0 \end{bmatrix} \qquad \boldsymbol{\lambda} = \begin{bmatrix} j3.0 & 0 & 0 & 0 \\ 0 & -j3.0 & 0 & 0 \\ 0 & 0 & j3.0 & 0 \\ 0 & 0 & 0 & -j3.0 \end{bmatrix}$$

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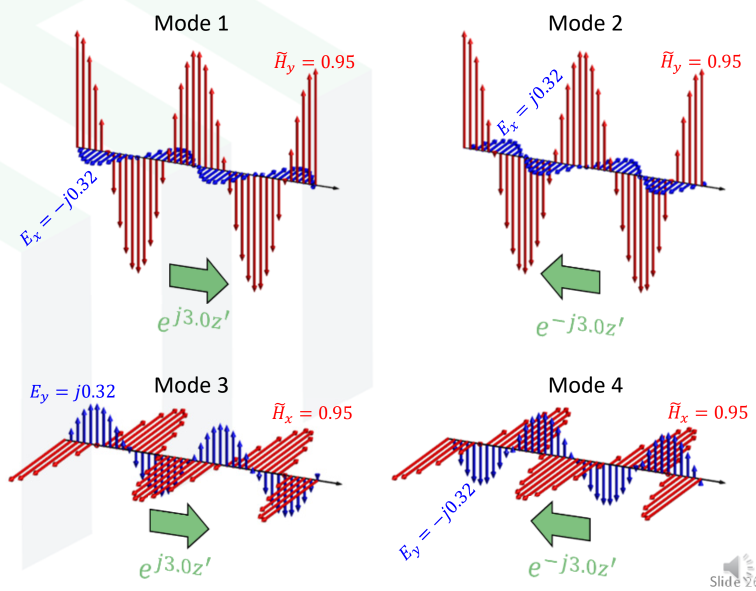
25

Visualizing the Modes

$$\mathbf{W} = \begin{bmatrix} -j0.32 & j0.32 & 0 & 0 \\ 0 & 0 & j0.32 & -j0.32 \\ 0 & 0 & 0.95 & 0.95 \\ 0.95 & 0.95 & 0 & 0 \end{bmatrix} \begin{matrix} \leftarrow E_x \\ \leftarrow E_y \\ \leftarrow \tilde{H}_x \\ \leftarrow \tilde{H}_y \end{matrix}$$

$$\boldsymbol{\lambda} = \begin{bmatrix} j3.0 & 0 & 0 & 0 \\ 0 & -j3.0 & 0 & 0 \\ 0 & 0 & j3.0 & 0 \\ 0 & 0 & 0 & -j3.0 \end{bmatrix}$$

Mode 1 | Mode 2 | Mode 3 | Mode 4



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26

Physical Meaning of the Eigen-Vector Numbers

It can be observed that the modes occur as either an E_x-$\tilde{H}_y$ or an E_y-$\tilde{H}_x$ pair. This is consistent with plane waves. Due to the normalization, they are 90° out of phase. The sign differences indicate forward and backward waves. Only the relative amplitudes of $\vec{E}$ and $\vec{\tilde{H}}$ is important here.

$$\mathbf{W} = \begin{bmatrix} -j0.32 & j0.32 & 0 & 0 \\ 0 & 0 & j0.32 & -j0.32 \\ 0 & 0 & 0.95 & 0.95 \\ 0.95 & 0.95 & 0 & 0 \end{bmatrix} \begin{matrix} \leftarrow E_x \\ \leftarrow E_y \\ \leftarrow \tilde{H}_x \\ \leftarrow \tilde{H}_y \end{matrix}$$

$$\frac{E}{\tilde{H}} = \sqrt{\frac{\mu_r}{\varepsilon_r}} = \frac{1}{3} \qquad \frac{0.32}{0.95} \cong \frac{1}{3}$$

The modes in **W** only contain information about the relative amplitudes of the field components.

$$\mathbf{W} = \begin{bmatrix} -j0.64 & j0.32 & 0 & 0 \\ 0 & 0 & j0.32 & -j0.32 \\ 0 & 0 & 0.95 & 0.95 \\ 1.9 & 0.95 & 0 & 0 \end{bmatrix}$$

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Slide 27

27

Physical Meaning of the Eigen-Value Numbers

The eigen-values are the complex propagation constants. They are essentially j times the refractive index of the medium.

The refractive index is known ($n = 3.0$), so the eigen-values are $\pm jn$. The sign indicates forward and backward waves.

$$\boldsymbol{\lambda} = \begin{bmatrix} j3.0 & 0 & 0 & 0 \\ 0 & -j3.0 & 0 & 0 \\ 0 & 0 & j3.0 & 0 \\ 0 & 0 & 0 & -j3.0 \end{bmatrix}$$

$$e^{\lambda z'} \leftrightarrow e^{jn\cos\theta_{\text{inc}} z'}$$

$$\lambda = jn\cos\theta_{\text{inc}}$$

$$n = \sqrt{\mu_r \varepsilon_r} = 3$$

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Slide 28

28