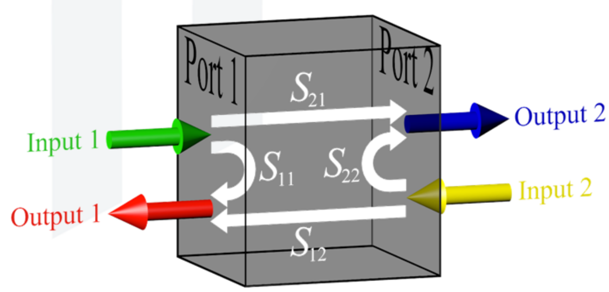
Advanced Computation:
Computational Electromagnetics

# Advanced Networking Concepts

1

# Outline

- Longitudinally periodic structures
- Dispersion analysis
- Alternatives to scattering matrices




Slide 2

2

# Longitudinally Periodic Devices

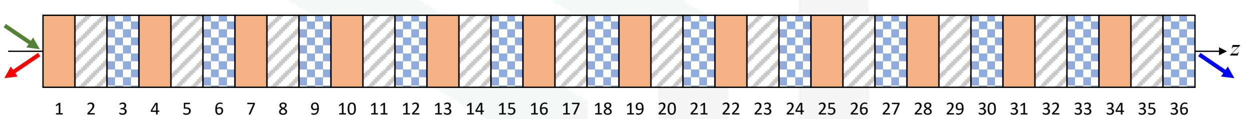
EMPossible

Slide 3

3

## Longitudinally Periodic Devices (1 of 2)

Suppose it is desired to simulate the following device.




The device has 36 layers. Does this require 36 $S$-matrix calculations and 35 star products?

$$\mathbf{S}^{(\times 36)} = \mathbf{S}^{(1)} \otimes \mathbf{S}^{(2)} \otimes \mathbf{S}^{(3)} \otimes \mathbf{S}^{(4)} \otimes \mathbf{S}^{(5)} \otimes \mathbf{S}^{(6)} \otimes \mathbf{S}^{(7)} \otimes \mathbf{S}^{(8)} \otimes \mathbf{S}^{(9)} \otimes \mathbf{S}^{(10)} \otimes \mathbf{S}^{(11)} \otimes \mathbf{S}^{(12)} \otimes \mathbf{S}^{(13)} \otimes \mathbf{S}^{(14)} \otimes \mathbf{S}^{(15)} \otimes \mathbf{S}^{(16)} \otimes \mathbf{S}^{(17)} \otimes \mathbf{S}^{(18)}$$
$$\otimes \mathbf{S}^{(19)} \otimes \mathbf{S}^{(20)} \otimes \mathbf{S}^{(21)} \otimes \mathbf{S}^{(22)} \otimes \mathbf{S}^{(23)} \otimes \mathbf{S}^{(24)} \otimes \mathbf{S}^{(25)} \otimes \mathbf{S}^{(26)} \otimes \mathbf{S}^{(27)} \otimes \mathbf{S}^{(28)} \otimes \mathbf{S}^{(29)} \otimes \mathbf{S}^{(30)} \otimes \mathbf{S}^{(31)} \otimes \mathbf{S}^{(32)} \otimes \mathbf{S}^{(33)} \otimes \mathbf{S}^{(34)} \otimes \mathbf{S}^{(35)} \otimes \mathbf{S}^{(36)}$$

This is very slow and inefficient!

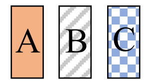
EMPossible

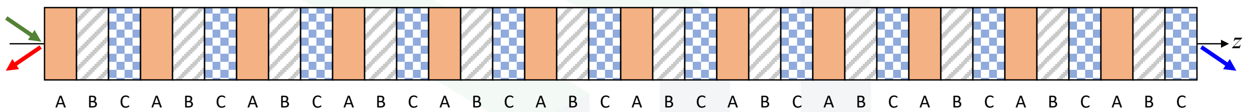
Slide 4

4

# Longitudinally Periodic Devices (2 of 2)

What if it is recognized that there exist only three unique layers?




Now the overall scattering matrix calculation becomes

$$\mathbf{S}^{(\times 36)} = \mathbf{S}^{(A)} \otimes \mathbf{S}^{(B)} \otimes \mathbf{S}^{(C)} \otimes \mathbf{S}^{(A)} \otimes \mathbf{S}^{(B)} \otimes \mathbf{S}^{(C)} \otimes \mathbf{S}^{(A)} \otimes \mathbf{S}^{(B)} \otimes \mathbf{S}^{(C)} \otimes \mathbf{S}^{(A)} \otimes \mathbf{S}^{(B)} \otimes \mathbf{S}^{(C)} \otimes \mathbf{S}^{(A)} \otimes \mathbf{S}^{(B)} \otimes \mathbf{S}^{(C)} \otimes \mathbf{S}^{(A)} \otimes \mathbf{S}^{(B)} \otimes \mathbf{S}^{(C)}$$
$$\otimes \mathbf{S}^{(A)} \otimes \mathbf{S}^{(B)} \otimes \mathbf{S}^{(C)} \otimes \mathbf{S}^{(A)} \otimes \mathbf{S}^{(B)} \otimes \mathbf{S}^{(C)} \otimes \mathbf{S}^{(A)} \otimes \mathbf{S}^{(B)} \otimes \mathbf{S}^{(C)} \otimes \mathbf{S}^{(A)} \otimes \mathbf{S}^{(B)} \otimes \mathbf{S}^{(C)} \otimes \mathbf{S}^{(A)} \otimes \mathbf{S}^{(B)} \otimes \mathbf{S}^{(C)} \otimes \mathbf{S}^{(A)} \otimes \mathbf{S}^{(B)} \otimes \mathbf{S}^{(C)}$$

This is a little faster, but is still very slow and inefficient!

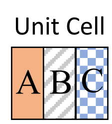
EMPossible

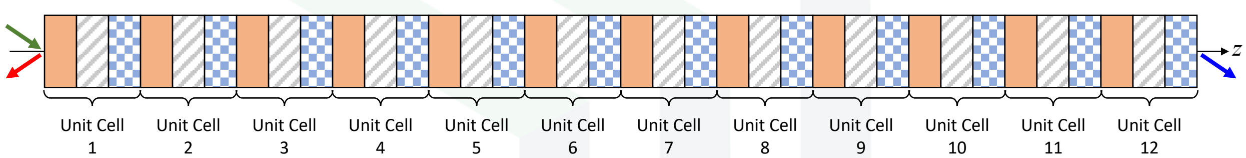
Slide 5

5

# Concept of the Unit cell

Observe that is device is composed a group of three repeating layers.




Maybe it would be best to first calculate the scattering matrix of the unit cell and then combine these to calculate the overall scattering matrix.

$$\mathbf{S}^{(ABC)} = \mathbf{S}^{(A)} \otimes \mathbf{S}^{(B)} \otimes \mathbf{S}^{(C)}$$

$$\mathbf{S}^{(\times 36)} = \mathbf{S}^{(ABC)} \otimes \mathbf{S}^{(ABC)} \otimes \mathbf{S}^{(ABC)} \otimes \mathbf{S}^{(ABC)} \otimes \mathbf{S}^{(ABC)} \otimes \mathbf{S}^{(ABC)} \otimes \mathbf{S}^{(ABC)} \otimes \mathbf{S}^{(ABC)} \otimes \mathbf{S}^{(ABC)} \otimes \mathbf{S}^{(ABC)} \otimes \mathbf{S}^{(ABC)} \otimes \mathbf{S}^{(ABC)}$$

This is better, but is still slow and inefficient when there are hundreds or more unit cells.

EMPossible

Slide 6

6

# Cascading and Doubling Algorithm

Cascading and double provides a very fast and efficient way to handle devices with many thousands of repeating unit cells.

$$\mathbf{S}^{(\times 1)} = \mathbf{S}^{(A)} \otimes \mathbf{S}^{(B)} \otimes \mathbf{S}^{(C)}$$

Start by calculating the scattering matrix for a single unit cell.

$$\mathbf{S}^{(\times 2)} = \mathbf{S}^{(\times 1)} \otimes \mathbf{S}^{(\times 1)}$$

Combine this with itself to describe two unit cells.

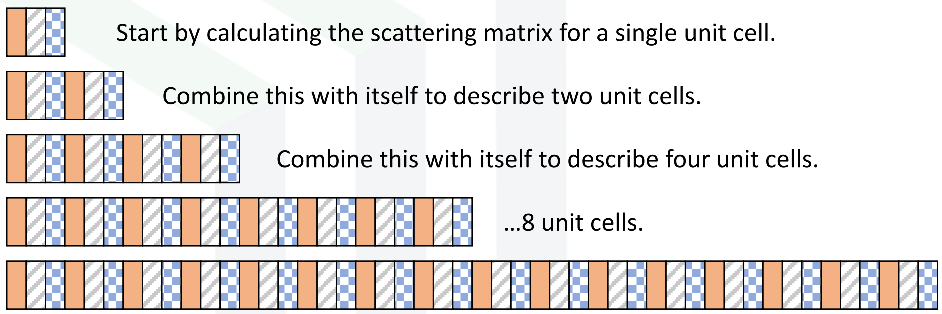
$$\mathbf{S}^{(\times 4)} = \mathbf{S}^{(\times 2)} \otimes \mathbf{S}^{(\times 2)}$$

Combine this with itself to describe four unit cells.

$$\mathbf{S}^{(\times 8)} = \mathbf{S}^{(\times 4)} \otimes \mathbf{S}^{(\times 4)}$$

…8 unit cells.

$$\mathbf{S}^{(\times 16)} = \mathbf{S}^{(\times 8)} \otimes \mathbf{S}^{(\times 8)}$$



…16 unit cells.

⋮

…and so on.

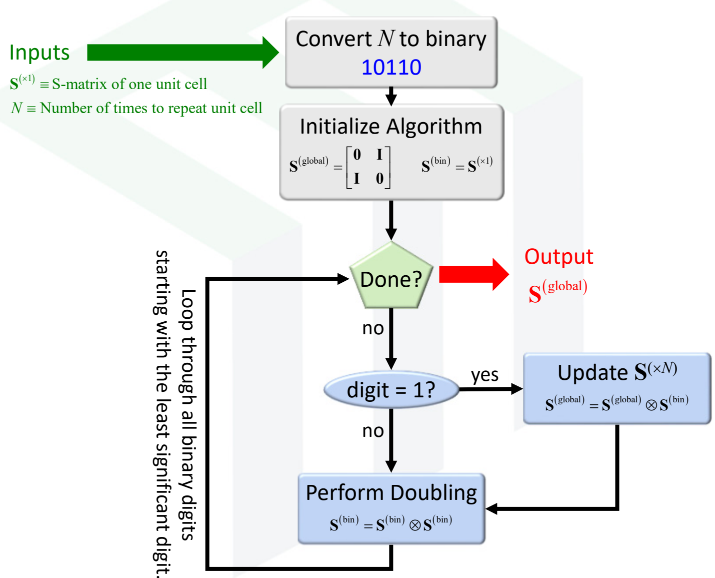
What if there is a non-integer power of 2 number of layers?

EMPossible

Slide 7



# Block Diagram for Modified Cascading and Doubling Algorithm



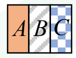

EMPossible

Slide 8

# Example of Cascading and Doubling Algorithm

**Step 0 – Calculate scattering matrix for one unit cell**

$$\mathbf{S}^{(\times 1)} = \mathbf{S}^{(A)} \otimes \mathbf{S}^{(B)} \otimes \mathbf{S}^{(C)}$$




**Inputs to algorithm:**

$\mathbf{S}^{(\times 1)} \equiv$ scattering matrix for a single unit cell

$N = 22 \equiv$ number of unit cells to combine

---

**Step 1 – Convert $N$ to binary**

| | 16's | 8's | 4's | 2's | 1's |
|---|---|---|---|---|---|
| $22 \rightarrow$ | 1 | 0 | 1 | 1 | 0 |

**Step 2 – Initialize binary and global scattering matrices**

$$\mathbf{S}^{(\text{bin})} = \mathbf{S}^{(\times 1)}$$

$$\mathbf{S}_{11}^{(\text{global})} \equiv \mathbf{0}$$
$$\mathbf{S}_{12}^{(\text{global})} \equiv \mathbf{I}$$
$$\mathbf{S}_{21}^{(\text{global})} \equiv \mathbf{I}$$
$$\mathbf{S}_{22}^{(\text{global})} \equiv \mathbf{0}$$

EMPossible

Slide 9

9

# Example of Cascading and Doubling Algorithm

**Step 3 – Loop through binary digits**

$22 \rightarrow 10110$

1's digit = 0
→ Do not update $\mathbf{S}^{(\text{global})}$ — $\mathbf{S}^{(\text{global})}$ still encompasses 0 unit cells
→ Double $\mathbf{S}^{(\text{bin})}$ — $\mathbf{S}^{(\text{bin})} \equiv \mathbf{S}^{(\text{bin})} \otimes \mathbf{S}^{(\text{bin})}$ — $\mathbf{S}^{(\text{bin})}$ now represents 2 unit cells

2's digit = 1
→ Update $\mathbf{S}^{(\text{global})}$ — $\mathbf{S}^{(\text{global})} \equiv \mathbf{S}^{(\text{global})} \otimes \mathbf{S}^{(\text{bin})}$ — $\mathbf{S}^{(\text{global})}$ now encompasses 2 unit cells
→ Double $\mathbf{S}^{(\text{bin})}$ — $\mathbf{S}^{(\text{bin})} \equiv \mathbf{S}^{(\text{bin})} \otimes \mathbf{S}^{(\text{bin})}$ — $\mathbf{S}^{(\text{bin})}$ now represents 4 unit cells

4's digit = 1
→ Update $\mathbf{S}^{(\text{global})}$ — $\mathbf{S}^{(\text{global})} \equiv \mathbf{S}^{(\text{global})} \otimes \mathbf{S}^{(\text{bin})}$ — $\mathbf{S}^{(\text{global})}$ now encompasses 6 unit cells
→ Double $\mathbf{S}^{(\text{bin})}$ — $\mathbf{S}^{(\text{bin})} \equiv \mathbf{S}^{(\text{bin})} \otimes \mathbf{S}^{(\text{bin})}$ — $\mathbf{S}^{(\text{bin})}$ now represents 8 unit cells

8's digit = 0
→ Do not update $\mathbf{S}^{(\text{global})}$ — $\mathbf{S}^{(\text{global})}$ still encompasses 6 unit cells
→ Double $\mathbf{S}^{(\text{bin})}$ — $\mathbf{S}^{(\text{bin})} \equiv \mathbf{S}^{(\text{bin})} \otimes \mathbf{S}^{(\text{bin})}$ — $\mathbf{S}^{(\text{bin})}$ now represents 16 unit cells

16's digit = 1
→ Update $\mathbf{S}^{(\text{global})}$ — $\mathbf{S}^{(\text{global})} \equiv \mathbf{S}^{(\text{global})} \otimes \mathbf{S}^{(\text{bin})}$ — $\mathbf{S}^{(\text{global})}$ now encompasses 22 unit cells
→ Double $\mathbf{S}^{(\text{bin})}$ — $\mathbf{S}^{(\text{bin})} \equiv \mathbf{S}^{(\text{bin})} \otimes \mathbf{S}^{(\text{bin})}$ — $\mathbf{S}^{(\text{bin})}$ now represents 32 unit cells

Oops! This algorithm performs one unnecessary doubling operation. How can this be fixed?

EMPossible

Slide 10

10

# Dispersion Analysis





## Dispersion Analysis (1 of 2)

An overall scattering matrix is calculated that describes the unit cell.

$$\begin{bmatrix} \mathbf{c}_0^- \\ \mathbf{c}_{N+1}^+ \end{bmatrix} = \begin{bmatrix} \mathbf{S}_{11}^{(\mathrm{uc})} & \mathbf{S}_{12}^{(\mathrm{uc})} \\ \mathbf{S}_{21}^{(\mathrm{uc})} & \mathbf{S}_{22}^{(\mathrm{uc})} \end{bmatrix} \begin{bmatrix} \mathbf{c}_0^+ \\ \mathbf{c}_{N+1}^- \end{bmatrix}$$

$\mathbf{S}^{(\mathrm{uc})}$ same as $\mathbf{S}^{(\times 1)}$ on previous slide

The terms are rearranged into "almost" the form of a transfer matrix.

$$\begin{bmatrix} \mathbf{0} & -\mathbf{S}_{12}^{(\mathrm{uc})} \\ \mathbf{I} & -\mathbf{S}_{22}^{(\mathrm{uc})} \end{bmatrix} \begin{bmatrix} \mathbf{c}_{N+1}^+ \\ \mathbf{c}_{N+1}^- \end{bmatrix} = \begin{bmatrix} \mathbf{S}_{11}^{(\mathrm{uc})} & -\mathbf{I} \\ \mathbf{S}_{21}^{(\mathrm{uc})} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{c}_0^+ \\ \mathbf{c}_0^- \end{bmatrix}$$

If the device is infinitely periodic in the $z$ direction, then the following periodic boundary condition must hold.

$$\begin{bmatrix} \mathbf{c}_{N+1}^+ \\ \mathbf{c}_{N+1}^- \end{bmatrix} = e^{j\beta\Lambda_z} \begin{bmatrix} \mathbf{c}_0^+ \\ \mathbf{c}_0^- \end{bmatrix}$$

Here $\beta$ is the *effective propagation constant* of the mode.

# Dispersion Analysis (2 of 2)

Substitute the periodic boundary condition into the rearranged equation to get

$$\begin{bmatrix} \mathbf{S}_{11}^{(\mathrm{uc})} & -\mathbf{I} \\ \mathbf{S}_{21}^{(\mathrm{uc})} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{c}_0^+ \\ \mathbf{c}_0^- \end{bmatrix} = e^{j\beta\Lambda_z} \begin{bmatrix} \mathbf{0} & -\mathbf{S}_{12}^{(\mathrm{uc})} \\ \mathbf{I} & -\mathbf{S}_{22}^{(\mathrm{uc})} \end{bmatrix} \begin{bmatrix} \mathbf{c}_0^+ \\ \mathbf{c}_0^- \end{bmatrix}$$

This is a generalized eigen-value problem.

$$\mathbf{A}\mathbf{x} = \lambda \mathbf{B}\mathbf{x}$$

$$\mathbf{A} = \begin{bmatrix} \mathbf{S}_{11}^{(\mathrm{uc})} & -\mathbf{I} \\ \mathbf{S}_{21}^{(\mathrm{uc})} & \mathbf{0} \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} \mathbf{c}_0^+ \\ \mathbf{c}_0^- \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} \mathbf{0} & -\mathbf{S}_{12}^{(\mathrm{uc})} \\ \mathbf{I} & -\mathbf{S}_{22}^{(\mathrm{uc})} \end{bmatrix} \qquad \lambda = e^{j\beta\Lambda_z}$$

```
[V,D] = eig(A,B);
```

Eigen vectors
Bloch modes

Eigen values
$\beta$'s

EMPossible

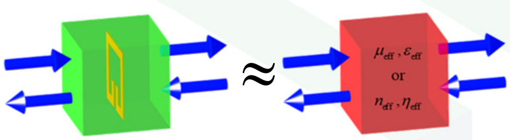
Slide 13

13

# Who Cares?

Given $\beta$, it is possible to…

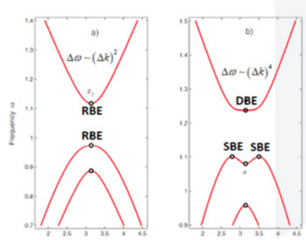
1. Calculate the effective properties of the unit cell.




$$\beta = \omega\sqrt{\mu_{\mathrm{r,eff}}\varepsilon_{\mathrm{r,eff}}}$$

This is an over simplification of parameter retrieval and beyond the scope of this course.

2. Construct band diagrams.




EMPossible

Slide 14

14

# Alternatives to Scattering Matrices

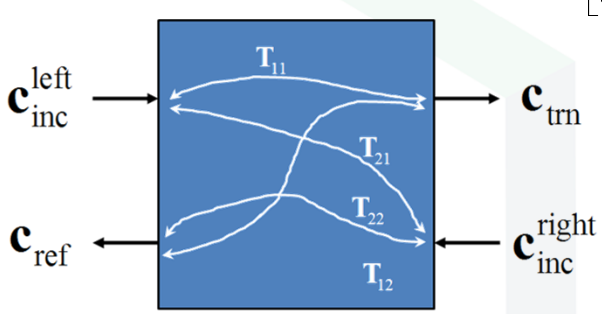


15

# Transmittance Matrices (T-Matrices)

The *T*-matrix method is the transfer matrix method where forward and backward waves are distinguished.

$$\begin{bmatrix} \mathbf{c}_{\text{trn}} \\ \mathbf{c}_{\text{inc}}^{\text{right}} \end{bmatrix} = \begin{bmatrix} \mathbf{T}_{11} & \mathbf{T}_{12} \\ \mathbf{T}_{21} & \mathbf{T}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{c}_{\text{inc}}^{\text{left}} \\ \mathbf{c}_{\text{ref}} \end{bmatrix}$$




Benefits
- Much faster (5 to 10 times)
- Unconditionally stable

Drawbacks
- Less memory efficient
- Cannot exploit longitudinal periodicity
- Less popular in the literature

M. G. Moharam, Drew A. Pommet, Eric B. Grann, "Stable implementation of the rigorous coupled-wave analysis for surface-relief gratings: enhanced transmittance matrix approach," J. Opt. Soc. Am. A, Vol. 12, No. 5, pp. 1077-1086, 1995.



16

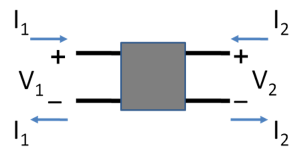
# Hybrid Matrices (H-Matrices)

The *H*-matrix method is borrowed from electrical two-port networks.

$$\begin{bmatrix} V_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ V_2 \end{bmatrix}$$

$$h_{11} \equiv \left. \frac{V_1}{I_2} \right|_{V_2=0} \qquad h_{12} \equiv \left. \frac{V_1}{V_2} \right|_{I_1=0}$$

$$h_{21} \equiv \left. \frac{I_2}{I_1} \right|_{V_2=0} \qquad h_{22} \equiv \left. \frac{I_2}{V_2} \right|_{I_1=0}$$




In the framework of fields, the *H*-matrix is defined as

$$\begin{bmatrix} E_{x,i-1} \\ E_{y,i-1} \\ \tilde{H}_{x,i} \\ \tilde{H}_{y,i} \end{bmatrix} = \begin{bmatrix} \mathbf{H}_{11}^{(i)} & \mathbf{H}_{12}^{(i)} \\ \mathbf{H}_{21}^{(i)} & \mathbf{H}_{22}^{(i)} \end{bmatrix} \begin{bmatrix} \tilde{H}_{x,i-1} \\ \tilde{H}_{y,i-1} \\ E_{x,i} \\ E_{y,i} \end{bmatrix}$$

Claimed Benefits
- Improved numerical stability
- More concise formulation
- Simpler to implement
- Improved numerical efficiency (~30% better than ETM)
- Unconditionally stable

Eng L. Tan, "Hybrid-matrix algorithm for rigorous coupled-wave analysis of multilayered diffraction gratings," J. Mod. Opt., Vol. 53, No. 4, pp. 417-428, 2006.

EMPossible Slide 17

17

# Reflection Matrices (R-Matrices)

The *R*-matrix method build on reflection $r$ and transmission $t$ matrices.

$$\begin{bmatrix} \mathbf{u}(z_0) \\ \mathbf{d}(z_{j+1}) \end{bmatrix} = \begin{bmatrix} \mathbf{T}_{uu}^{(j)} & \mathbf{R}_{ud}^{(j)} \\ \mathbf{R}_{du}^{(j)} & \mathbf{T}_{dd}^{(j)} \end{bmatrix} \begin{bmatrix} \mathbf{u}(z_{j+1}) \\ \mathbf{d}(z_0) \end{bmatrix}$$

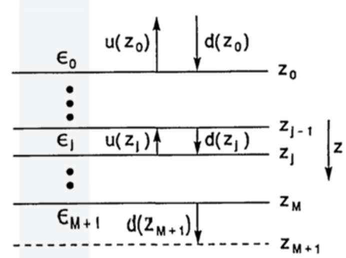



Fig. 1. *M*-layer stratified grating. The periodic variations of the media in the horizontal direction are not shown.

Claimed Benefits
- Unconditionally stable
- Improved numerical efficiency

Lifeng Li, "Bremmer series, *R*-matrix propagation algorithm, and numerical modeling of diffraction gratings," J. Opt. Soc. Am. A, Vol. 11, No. 11, pp. 2829-2836, 1994.

EMPossible Slide 18

18