Advanced Computation:
Computational Electromagnetics

Formulation of the Plane Wave Expansion Method

Outline

- Formulation of the basic 3D eigen-value problem
- Formulation of efficient 2D PWEM

Formulation of the Basic 3D Eigen-Value Problem

Block Matrix Form

Start with the Fourier-space Maxwell's equations in matrix form.

$$\mathbf{K}_y\mathbf{u}_z - \mathbf{K}_z\mathbf{u}_y = jk_0[\![\varepsilon_r]\!]\mathbf{s}_x$$
$$\mathbf{K}_z\mathbf{u}_x - \mathbf{K}_x\mathbf{u}_z = jk_0[\![\varepsilon_r]\!]\mathbf{s}_y$$
$$\mathbf{K}_x\mathbf{u}_y - \mathbf{K}_y\mathbf{u}_x = jk_0[\![\varepsilon_r]\!]\mathbf{s}_z$$

$$\mathbf{K}_y\mathbf{s}_z - \mathbf{K}_z\mathbf{s}_y = jk_0[\![\mu_r]\!]\mathbf{u}_x$$
$$\mathbf{K}_z\mathbf{s}_x - \mathbf{K}_x\mathbf{s}_z = jk_0[\![\mu_r]\!]\mathbf{u}_y$$
$$\mathbf{K}_x\mathbf{s}_y - \mathbf{K}_y\mathbf{s}_x = jk_0[\![\mu_r]\!]\mathbf{u}_z$$

These can be written in block matrix form as

$$\begin{bmatrix} \mathbf{0} & -\mathbf{K}_z & \mathbf{K}_y \\ \mathbf{K}_z & \mathbf{0} & -\mathbf{K}_x \\ -\mathbf{K}_y & \mathbf{K}_x & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{u}_x \\ \mathbf{u}_y \\ \mathbf{u}_z \end{bmatrix} = jk_0 \begin{bmatrix} [\![\varepsilon_r]\!] & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & [\![\varepsilon_r]\!] & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & [\![\varepsilon_r]\!] \end{bmatrix} \begin{bmatrix} \mathbf{s}_x \\ \mathbf{s}_y \\ \mathbf{s}_z \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{0} & -\mathbf{K}_z & \mathbf{K}_y \\ \mathbf{K}_z & \mathbf{0} & -\mathbf{K}_x \\ -\mathbf{K}_y & \mathbf{K}_x & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{s}_x \\ \mathbf{s}_y \\ \mathbf{s}_z \end{bmatrix} = jk_0 \begin{bmatrix} [\![\mu_r]\!] & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & [\![\mu_r]\!] & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & [\![\mu_r]\!] \end{bmatrix} \begin{bmatrix} \mathbf{u}_x \\ \mathbf{u}_y \\ \mathbf{u}_z \end{bmatrix}$$

Compact Block Matrix Notation

These equations can be written even more compactly as...

$$\begin{bmatrix} \mathbf{0} & -\mathbf{K}_z & \mathbf{K}_y \\ \mathbf{K}_z & \mathbf{0} & -\mathbf{K}_x \\ -\mathbf{K}_y & \mathbf{K}_x & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{u}_x \\ \mathbf{u}_y \\ \mathbf{u}_z \end{bmatrix} = jk_0 \begin{bmatrix} [\![\varepsilon_r]\!] & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & [\![\varepsilon_r]\!] & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & [\![\varepsilon_r]\!] \end{bmatrix} \begin{bmatrix} \mathbf{s}_x \\ \mathbf{s}_y \\ \mathbf{s}_z \end{bmatrix} \Rightarrow \left[\vec{\mathbf{K}}\times\right]\vec{\mathbf{u}} = jk_0 \left[[\![\varepsilon_r]\!]\right]\vec{\mathbf{s}}$$

$$\begin{bmatrix} \mathbf{0} & -\mathbf{K}_z & \mathbf{K}_y \\ \mathbf{K}_z & \mathbf{0} & -\mathbf{K}_x \\ -\mathbf{K}_y & \mathbf{K}_x & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{s}_x \\ \mathbf{s}_y \\ \mathbf{s}_z \end{bmatrix} = jk_0 \begin{bmatrix} [\![\mu_r]\!] & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & [\![\mu_r]\!] & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & [\![\mu_r]\!] \end{bmatrix} \begin{bmatrix} \mathbf{u}_x \\ \mathbf{u}_y \\ \mathbf{u}_z \end{bmatrix} \Rightarrow \left[\vec{\mathbf{K}}\times\right]\vec{\mathbf{s}} = jk_0 \left[[\![\mu_r]\!]\right]\vec{\mathbf{u}}$$

$$\vec{\mathbf{s}} = \begin{bmatrix} \mathbf{s}_x \\ \mathbf{s}_y \\ \mathbf{s}_z \end{bmatrix} \qquad \vec{\mathbf{u}} = \begin{bmatrix} \mathbf{u}_x \\ \mathbf{u}_y \\ \mathbf{u}_z \end{bmatrix}$$

$$\left[\vec{\mathbf{K}}\times\right] = \begin{bmatrix} \mathbf{0} & -\mathbf{K}_z & \mathbf{K}_y \\ \mathbf{K}_z & \mathbf{0} & -\mathbf{K}_x \\ -\mathbf{K}_y & \mathbf{K}_x & \mathbf{0} \end{bmatrix} \qquad \left[[\![\mu_r]\!]\right] = \begin{bmatrix} [\![\mu_r]\!] & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & [\![\mu_r]\!] & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & [\![\mu_r]\!] \end{bmatrix} \qquad \left[[\![\varepsilon_r]\!]\right] = \begin{bmatrix} [\![\varepsilon_r]\!] & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & [\![\varepsilon_r]\!] & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & [\![\varepsilon_r]\!] \end{bmatrix}$$

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Eliminate the Magnetic Field

Start with the following equations.

$$\left[\vec{\mathbf{K}}\times\right]\vec{\mathbf{s}} = jk_0 \left[[\![\mu_r]\!]\right]\vec{\mathbf{u}} \qquad \left[\vec{\mathbf{K}}\times\right]\vec{\mathbf{u}} = jk_0 \left[[\![\varepsilon_r]\!]\right]\vec{\mathbf{s}}$$

Solve for the magnetic field $\vec{\mathbf{u}}$

$$\vec{\mathbf{u}} = \frac{1}{jk_0}\left[[\![\mu_r]\!]\right]^{-1}\left[\vec{\mathbf{K}}\times\right]\vec{\mathbf{s}}$$

Substitute expression for $\vec{\mathbf{u}}$ into second equation.

$$\left[\vec{\mathbf{K}}\times\right]\left(\frac{1}{jk_0}\left[[\![\mu_r]\!]\right]^{-1}\left[\vec{\mathbf{K}}\times\right]\vec{\mathbf{s}}\right) = jk_0 \left[[\![\varepsilon_r]\!]\right]\vec{\mathbf{s}}$$

Simplify

$$\left[\vec{\mathbf{K}}\times\right]\left[[\![\mu_r]\!]\right]^{-1}\left[\vec{\mathbf{K}}\times\right]\vec{\mathbf{s}} = -k_0^2 \left[[\![\varepsilon_r]\!]\right]\vec{\mathbf{s}}$$

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The 3D Eigen-Value Problem

The 3D matrix equation in terms of the electric field is

$$\left[\vec{\mathbf{K}}\times\right]\left[\left[\mu_{\mathrm{r}}\right]\right]^{-1}\left[\vec{\mathbf{K}}\times\right]\vec{\mathbf{s}}=-k_0^2\left[\left[\varepsilon_{\mathrm{r}}\right]\right]\vec{\mathbf{s}}$$

$$\left[\vec{\mathbf{K}}\times\right]=\begin{bmatrix}\mathbf{0} & -\mathbf{K}_z & \mathbf{K}_y\\ \mathbf{K}_z & \mathbf{0} & -\mathbf{K}_x\\ -\mathbf{K}_y & \mathbf{K}_x & \mathbf{0}\end{bmatrix}\qquad \left[\left[\mu_{\mathrm{r}}\right]\right]^{-1}=\begin{bmatrix}\left[\mu_{\mathrm{r}}\right]^{-1} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \left[\mu_{\mathrm{r}}\right]^{-1} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \left[\mu_{\mathrm{r}}\right]^{-1}\end{bmatrix}\qquad \left[\left[\varepsilon_{\mathrm{r}}\right]\right]=\begin{bmatrix}\left[\varepsilon_{\mathrm{r}}\right] & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \left[\varepsilon_{\mathrm{r}}\right] & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \left[\varepsilon_{\mathrm{r}}\right]\end{bmatrix}\qquad \vec{\mathbf{s}}=\begin{bmatrix}\mathbf{s}_x\\ \mathbf{s}_y\\ \mathbf{s}_z\end{bmatrix}$$

This has the form of a generalized eigen-value problem $\mathbf{Ax}=\lambda\mathbf{Bx}$

$$\mathbf{A}=\left[\vec{\mathbf{K}}\times\right]\left[\left[\mu_{\mathrm{r}}\right]\right]^{-1}\left[\vec{\mathbf{K}}\times\right]$$
$$\mathbf{B}=\left[\left[\varepsilon_{\mathrm{r}}\right]\right]$$
$$\mathbf{x}=\vec{\mathbf{s}}$$
$$\lambda=-k_0^2$$

```
[V,D] = eig(A,B);
```

Notes:

1. It is more common to see this expressed in terms of the magnetic field because it is an ordinary eigen-value problem.
2. This formulation does not enforce transversality (i.e. zero divergence) of the fields $\nabla\cdot\vec{E}=0$.
3. It is possible to reduce this to a 2×2 block matrix equation that does enforce transversality. See PWEM Extras lecture.

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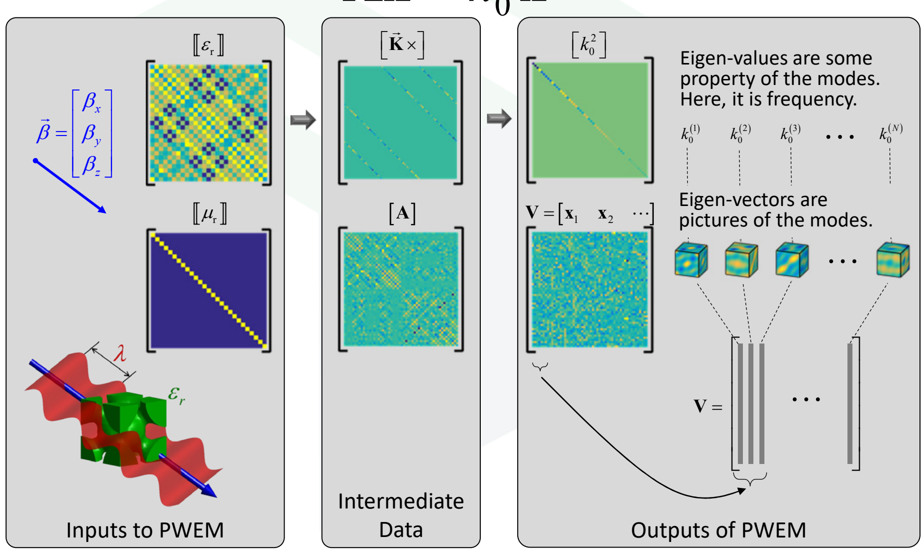
Visualizing the Data

$$\mathbf{Ax}=k_0^2\mathbf{x}$$



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Consequences of k_0^2 Being the Eigen-Value

The quantity k_0^2 is really just frequency scaled by the speed of light.

$$k_0 = \frac{\omega}{c_0}$$

In the present formulation, k_0 is the eigen value so it is the unknown quantity.

k_0 is not known when constructing the eigen-value problem.

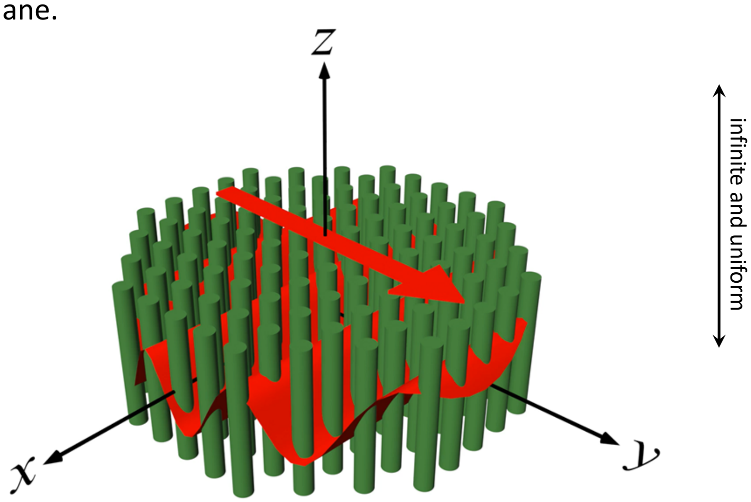
Since frequency is not known, it is not possible to build frequency-dependent material properties (i.e. dispersion) into the simulation without modifying the basic PWEM algorithm.

The basic PWEM cannot incorporate material dispersion.
It must be modified to account for this. See PWEM Extras.

Formulation of Efficient 2D Plane Wave Expansion Method

Two-Dimensional Lattices

For 2D problems, the device is uniform and infinite in the z-direction and wave propagation is restricted to the x-y plane.



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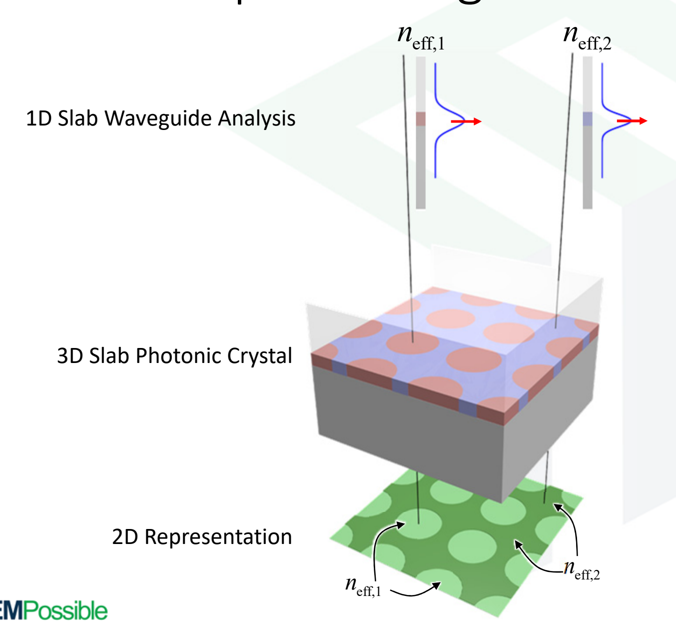
Representing Slab Photonic Crystals



Step 1 – Analyze vertical cross sections of photonic crystal slab as slab waveguides. Calculate the effective refractive index of each cross section.

Step 2 – Build a 2D representation of the photonic crystal slab using just the effective refractive indices from Step 1.

Be careful to consider the polarization. The alignment of the electric field must be consistent.

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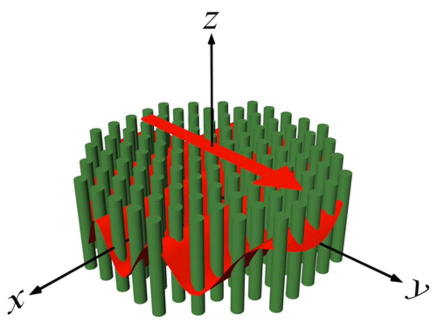
Reduction to Two Dimensions

For the 2D devices described on the previous slide where the waves are restricted to the plane of the device, the wave has no vector components in the z-direction.

$$\mathbf{K}_z = \mathbf{0}$$



Maxwell's equations reduce to

$$\mathbf{K}_y\mathbf{u}_z - \cancel{\mathbf{K}_z\mathbf{u}_y} = jk_0[\![\varepsilon_r]\!]\mathbf{s}_x \qquad \mathbf{K}_y\mathbf{s}_z - \cancel{\mathbf{K}_z\mathbf{s}_y} = jk_0[\![\mu_r]\!]\mathbf{u}_x$$
$$\cancel{\mathbf{K}_z\mathbf{u}_x} - \mathbf{K}_x\mathbf{u}_z = jk_0[\![\varepsilon_r]\!]\mathbf{s}_y \qquad \cancel{\mathbf{K}_z\mathbf{s}_x} - \mathbf{K}_x\mathbf{s}_z = jk_0[\![\mu_r]\!]\mathbf{u}_y$$
$$\mathbf{K}_x\mathbf{u}_y - \mathbf{K}_y\mathbf{u}_x = jk_0[\![\varepsilon_r]\!]\mathbf{s}_z \qquad \mathbf{K}_x\mathbf{s}_y - \mathbf{K}_y\mathbf{s}_x = jk_0[\![\mu_r]\!]\mathbf{u}_z$$

↓

$$\mathbf{K}_y\mathbf{u}_z = jk_0[\![\varepsilon_r]\!]\mathbf{s}_x \qquad \mathbf{K}_y\mathbf{s}_z = jk_0[\![\mu_r]\!]\mathbf{u}_x$$
$$-\mathbf{K}_x\mathbf{u}_z = jk_0[\![\varepsilon_r]\!]\mathbf{s}_y \qquad -\mathbf{K}_x\mathbf{s}_z = jk_0[\![\mu_r]\!]\mathbf{u}_y$$
$$\mathbf{K}_x\mathbf{u}_y - \mathbf{K}_y\mathbf{u}_x = jk_0[\![\varepsilon_r]\!]\mathbf{s}_z \qquad \mathbf{K}_x\mathbf{s}_y - \mathbf{K}_y\mathbf{s}_x = jk_0[\![\mu_r]\!]\mathbf{u}_z$$

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Two Distinct Modes

After inspecting the remaining equations, it can be seen that Maxwell's equations have split into two independent sets of equations. These correspond to two possible electromagnetic modes.

$$\mathbf{K}_y\mathbf{u}_z = jk_0[\![\varepsilon_r]\!]\mathbf{s}_x$$
$$-\mathbf{K}_x\mathbf{u}_z = jk_0[\![\varepsilon_r]\!]\mathbf{s}_y$$
$$\mathbf{K}_x\mathbf{u}_y - \mathbf{K}_y\mathbf{u}_x = jk_0[\![\varepsilon_r]\!]\mathbf{s}_z$$

E-Mode

$$\mathbf{K}_y\mathbf{s}_z = jk_0[\![\mu_r]\!]\mathbf{u}_x$$
$$-\mathbf{K}_x\mathbf{s}_z = jk_0[\![\mu_r]\!]\mathbf{u}_y$$
$$\mathbf{K}_x\mathbf{s}_y - \mathbf{K}_y\mathbf{s}_x = jk_0[\![\mu_r]\!]\mathbf{u}_z$$

H-Mode

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Two Distinct Modes

After inspecting the remaining equations, it can be seen that Maxwell's equations have split into two independent sets of equations. These correspond to two possible electromagnetic modes.

$$\mathbf{K}_y\mathbf{u}_z = jk_0[\![\varepsilon_r]\!]\mathbf{s}_x$$
$$-\mathbf{K}_x\mathbf{u}_z = jk_0[\![\varepsilon_r]\!]\mathbf{s}_y$$
$$\mathbf{K}_x\mathbf{u}_y - \mathbf{K}_y\mathbf{u}_x = jk_0[\![\varepsilon_r]\!]\mathbf{s}_z$$

$$\mathbf{K}_y\mathbf{s}_z = jk_0[\![\mu_r]\!]\mathbf{u}_x$$
$$-\mathbf{K}_x\mathbf{s}_z = jk_0[\![\mu_r]\!]\mathbf{u}_y$$
$$\mathbf{K}_x\mathbf{s}_y - \mathbf{K}_y\mathbf{s}_x = jk_0[\![\mu_r]\!]\mathbf{u}_z$$

E-Mode

$$\mathbf{K}_x\mathbf{u}_y - \mathbf{K}_y\mathbf{u}_x = jk_0[\![\varepsilon_r]\!]\mathbf{s}_z$$
$$\mathbf{K}_y\mathbf{s}_z = jk_0[\![\mu_r]\!]\mathbf{u}_x$$
$$-\mathbf{K}_x\mathbf{s}_z = jk_0[\![\mu_r]\!]\mathbf{u}_y$$

H-Mode

$$\mathbf{K}_x\mathbf{s}_y - \mathbf{K}_y\mathbf{s}_x = jk_0[\![\mu_r]\!]\mathbf{u}_z$$
$$\mathbf{K}_y\mathbf{u}_z = jk_0[\![\varepsilon_r]\!]\mathbf{s}_x$$
$$-\mathbf{K}_x\mathbf{u}_z = jk_0[\![\varepsilon_r]\!]\mathbf{s}_y$$

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2D Eigen-Value Problems

Two eigen-value problems can now be derived for two dimensional photonic crystals.

E-Mode

$$\mathbf{K}_x\mathbf{u}_y - \mathbf{K}_y\mathbf{u}_x = jk_0[\![\varepsilon_r]\!]\mathbf{s}_z \rightarrow \mathbf{K}_x\left(\frac{j}{k_0}[\![\mu_r]\!]^{-1}\mathbf{K}_x\mathbf{s}_z\right) - \mathbf{K}_y\left(-\frac{j}{k_0}[\![\mu_r]\!]^{-1}\mathbf{K}_y\mathbf{s}_z\right) = jk_0[\![\varepsilon_r]\!]\mathbf{s}_z$$
$$\mathbf{K}_y\mathbf{s}_z = jk_0[\![\mu_r]\!]\mathbf{u}_x \rightarrow \mathbf{u}_x = -\frac{j}{k_0}[\![\mu_r]\!]^{-1}\mathbf{K}_y\mathbf{s}_z$$
$$-\mathbf{K}_x\mathbf{s}_z = jk_0[\![\mu_r]\!]\mathbf{u}_y \rightarrow \mathbf{u}_y = \frac{j}{k_0}[\![\mu_r]\!]^{-1}\mathbf{K}_x\mathbf{s}_z$$
$$\boxed{\left(\mathbf{K}_x[\![\mu_r]\!]^{-1}\mathbf{K}_x + \mathbf{K}_y[\![\mu_r]\!]^{-1}\mathbf{K}_y\right)\mathbf{s}_z = k_0^2[\![\varepsilon_r]\!]\mathbf{s}_z}$$

H-Mode

$$\mathbf{K}_x\mathbf{s}_y - \mathbf{K}_y\mathbf{s}_x = jk_0[\![\mu_r]\!]\mathbf{u}_z \rightarrow \mathbf{K}_x\left(\frac{j}{k_0}[\![\varepsilon_r]\!]^{-1}\mathbf{K}_x\mathbf{u}_z\right) - \mathbf{K}_y\left(-\frac{j}{k_0}[\![\varepsilon_r]\!]^{-1}\mathbf{K}_y\mathbf{u}_z\right) = jk_0[\![\mu_r]\!]\mathbf{u}_z$$
$$\mathbf{K}_y\mathbf{u}_z = jk_0[\![\varepsilon_r]\!]\mathbf{s}_x \rightarrow \mathbf{s}_x = -\frac{j}{k_0}[\![\varepsilon_r]\!]^{-1}\mathbf{K}_y\mathbf{u}_z$$
$$-\mathbf{K}_x\mathbf{u}_z = jk_0[\![\varepsilon_r]\!]\mathbf{s}_y \rightarrow \mathbf{s}_y = \frac{j}{k_0}[\![\varepsilon_r]\!]^{-1}\mathbf{K}_x\mathbf{u}_z$$
$$\boxed{\left(\mathbf{K}_x[\![\varepsilon_r]\!]^{-1}\mathbf{K}_x + \mathbf{K}_y[\![\varepsilon_r]\!]^{-1}\mathbf{K}_y\right)\mathbf{u}_z = k_0^2[\![\mu_r]\!]\mathbf{u}_z}$$

Note: For non-magnetic materials, this mode calculates slower.

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Solution in Homogeneous Unit Cell

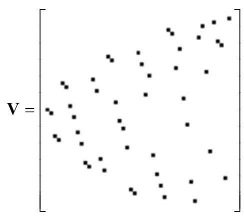
For homogeneous unit cells, the $\mathbf{A}$ and $\mathbf{B}$ matrices of the generalized eigen-value problem are diagonal. In this case, the modes have a closed form solution.

$$\mathbf{A}\mathbf{v} = \lambda \mathbf{B}\mathbf{v}$$

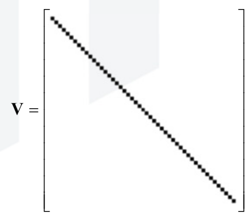
$$\mathbf{A} = \mathbf{K}_x^2 + \mathbf{K}_y^2 \quad \rightarrow \quad \mathbf{V} = \frac{1}{\sqrt{\mu_\mathrm{r}\varepsilon_\mathrm{r}}}\mathbf{I}$$

$$\mathbf{B} = \mu_\mathrm{r}\varepsilon_\mathrm{r}\mathbf{I} \qquad \boldsymbol{\lambda} = \mathbf{B}^{-1}\mathbf{A}$$

When solved numerically, the columns of the eigen-modes may be scrambled.



numerical



closed form

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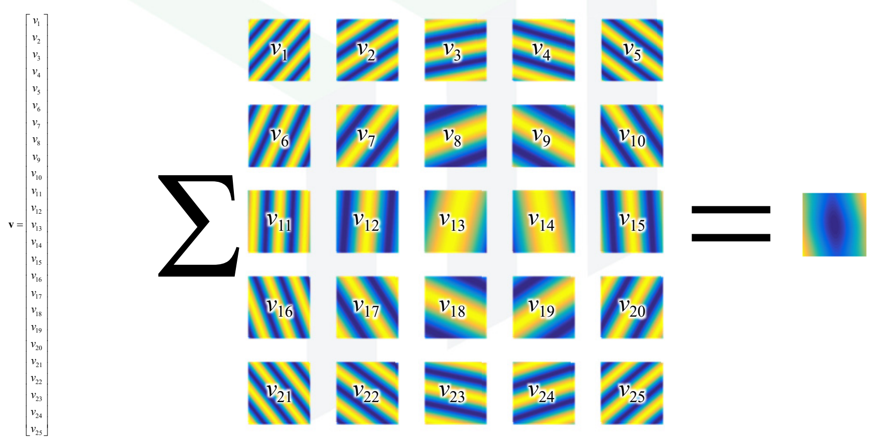
Interpreting the Eigen-Vectors

An eigen-vector contains the complex amplitudes of all the spatial harmonics (i.e. plane waves in the expansion) for that mode.

The numbers are complex to reflect both magnitude and phase of the spatial harmonics.



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