EMPossible

Advanced Computation:

Computational Electromagnetics

# Matrix Form of Maxwell’s Equations in Fourier Space

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## Outline

- Matrix form of Maxwell’s equations in Fourier space
- Fast Fourier factorization
- Consequences of Fourier-space representation

# Matrix Form of Maxwell's Equations in Fourier Space

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# Maxwell's Equations in Fourier Space

Real-Space

Fourier-Space

$$\begin{aligned}\frac{\partial \tilde{H}_z}{\partial y}-\frac{\partial \tilde{H}_y}{\partial z}&=k_0\varepsilon_\mathrm{r}E_x\\ \frac{\partial \tilde{H}_x}{\partial z}-\frac{\partial \tilde{H}_z}{\partial x}&=k_0\varepsilon_\mathrm{r}E_y\\ \frac{\partial \tilde{H}_y}{\partial x}-\frac{\partial \tilde{H}_x}{\partial y}&=k_0\varepsilon_\mathrm{r}E_z\end{aligned} \quad\Rightarrow\quad \boxed{\begin{aligned}k_y(p,q,r)U_z(p,q,r)-k_z(p,q,r)U_y(p,q,r)&=jk_0a(p,q,r)*S_x(p,q,r)\\ k_z(p,q,r)U_x(p,q,r)-k_x(p,q,r)U_z(p,q,r)&=jk_0a(p,q,r)*S_y(p,q,r)\\ k_x(p,q,r)U_y(p,q,r)-k_y(p,q,r)U_x(p,q,r)&=jk_0a(p,q,r)*S_z(p,q,r)\end{aligned}}$$

$$\vec{k}(p,q,r)=k_x(p,q,r)\hat{a}_x+k_y(p,q,r)\hat{a}_y+k_z(p,q,r)\hat{a}_z=\vec{\beta}-p\vec{T}_1-q\vec{T}_2-r\vec{T}_3$$

$$p=-\infty,\cdots,-2,-1,0,1,2,\cdots,\infty$$

$$q=-\infty,\cdots,-2,-1,0,1,2,\cdots,\infty$$

$$r=-\infty,\cdots,-2,-1,0,1,2,\cdots,\infty$$

$$\begin{aligned}\frac{\partial E_z}{\partial y}-\frac{\partial E_y}{\partial z}&=k_0\mu_\mathrm{r}\tilde{H}_x\\ \frac{\partial E_x}{\partial z}-\frac{\partial E_z}{\partial x}&=k_0\mu_\mathrm{r}\tilde{H}_y\\ \frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}&=k_0\mu_\mathrm{r}\tilde{H}_z\end{aligned} \quad\Rightarrow\quad \boxed{\begin{aligned}k_y(p,q,r)S_z(p,q,r)-k_z(p,q,r)S_y(p,q,r)&=jk_0b(p,q,r)*U_x(p,q,r)\\ k_z(p,q,r)S_x(p,q,r)-k_x(p,q,r)S_z(p,q,r)&=jk_0b(p,q,r)*U_y(p,q,r)\\ k_x(p,q,r)S_y(p,q,r)-k_y(p,q,r)S_x(p,q,r)&=jk_0b(p,q,r)*U_z(p,q,r)\end{aligned}}$$

# Matrix Form

Take the first equation from the previous slide

$$k_y(p,q,r)U_z(p,q,r)-k_z(p,q,r)U_y(p,q,r)=jk_0\sum_{p'=-\infty}^{\infty}\sum_{q'=-\infty}^{\infty}\sum_{r'=-\infty}^{\infty}a(p-p',q-q',r-r')S_x(p',q',r')$$

This equation is written once for every combination of $p$, $q$ and $r$. For $P=Q=R=3$, this set of equations is

$$\left.\begin{aligned}
&k_y(-1,-1,-1)U_z(-1,-1,-1)-k_z(-1,-1,-1)U_y(-1,-1,-1)=jk_0\sum_{p'=-1}^{1}\sum_{q'=-1}^{1}\sum_{r'=-1}^{1}a(-1-p',-1-q',-1-r')S_x(p',q',r')\\
&k_y(0,-1,-1)U_z(0,-1,-1)-k_z(0,-1,-1)U_y(0,-1,-1)=jk_0\sum_{p'=-1}^{1}\sum_{q'=-1}^{1}\sum_{r'=-1}^{1}a(0-p',-1-q',-1-r')S_x(p',q',r')\\
&k_y(1,-1,-1)U_z(1,-1,-1)-k_z(1,-1,-1)U_y(1,-1,-1)=jk_0\sum_{p'=-1}^{1}\sum_{q'=-1}^{1}\sum_{r'=-1}^{1}a(1-p',-1-q',-1-r')S_x(p',q',r')\\
&k_y(-1,0,-1)U_z(-1,0,-1)-k_z(-1,0,-1)U_y(-1,0,-1)=jk_0\sum_{p'=-1}^{1}\sum_{q'=-1}^{1}\sum_{r'=-1}^{1}a(-1-p',0-q',-1-r')S_x(p',q',r')\\
&k_y(0,0,-1)U_z(0,0,-1)-k_z(0,0,-1)U_y(0,0,-1)=jk_0\sum_{p'=-1}^{1}\sum_{q'=-1}^{1}\sum_{r'=-1}^{1}a(0-p',0-q',-1-r')S_x(p',q',r')\\
&k_y(1,0,-1)U_z(1,0,-1)-k_z(1,0,-1)U_y(1,0,-1)=jk_0\sum_{p'=-1}^{1}\sum_{q'=-1}^{1}\sum_{r'=-1}^{1}a(1-p',0-q',-1-r')S_x(p',q',r')\\
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&k_y(1,1,-1)U_z(1,1,-1)-k_z(1,1,-1)U_y(1,1,-1)=jk_0\sum_{p'=-1}^{1}\sum_{q'=-1}^{1}\sum_{r'=-1}^{1}a(1-p',1-q',-1-r')S_x(p',q',r')\\
&k_y(-1,-1,0)U_z(-1,-1,0)-k_z(-1,-1,0)U_y(-1,-1,0)=jk_0\sum_{p'=-1}^{1}\sum_{q'=-1}^{1}\sum_{r'=-1}^{1}a(-1-p',-1-q',0-r')S_x(p',q',r')\\
&k_y(0,-1,0)U_z(0,-1,0)-k_z(0,-1,0)U_y(0,-1,0)=jk_0\sum_{p'=-1}^{1}\sum_{q'=-1}^{1}\sum_{r'=-1}^{1}a(0-p',-1-q',0-r')S_x(p',q',r')\\
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&k_y(1,0,0)U_z(1,0,0)-k_z(1,0,0)U_y(1,0,0)=jk_0\sum_{p'=-1}^{1}\sum_{q'=-1}^{1}\sum_{r'=-1}^{1}a(1-p',0-q',0-r')S_x(p',q',r')\\
&k_y(-1,1,0)U_z(-1,1,0)-k_z(-1,1,0)U_y(-1,1,0)=jk_0\sum_{p'=-1}^{1}\sum_{q'=-1}^{1}\sum_{r'=-1}^{1}a(-1-p',1-q',0-r')S_x(p',q',r')\\
&k_y(0,1,0)U_z(0,1,0)-k_z(0,1,0)U_y(0,1,0)=jk_0\sum_{p'=-1}^{1}\sum_{q'=-1}^{1}\sum_{r'=-1}^{1}a(0-p',1-q',0-r')S_x(p',q',r')\\
&k_y(1,1,0)U_z(1,1,0)-k_z(1,1,0)U_y(1,1,0)=jk_0\sum_{p'=-1}^{1}\sum_{q'=-1}^{1}\sum_{r'=-1}^{1}a(1-p',1-q',0-r')S_x(p',q',r')\\
&k_y(-1,-1,1)U_z(-1,-1,1)-k_z(-1,-1,1)U_y(-1,-1,1)=jk_0\sum_{p'=-1}^{1}\sum_{q'=-1}^{1}\sum_{r'=-1}^{1}a(-1-p',-1-q',1-r')S_x(p',q',r')\\
&k_y(0,-1,1)U_z(0,-1,1)-k_z(0,-1,1)U_y(0,-1,1)=jk_0\sum_{p'=-1}^{1}\sum_{q'=-1}^{1}\sum_{r'=-1}^{1}a(0-p',-1-q',1-r')S_x(p',q',r')\\
&k_y(1,-1,1)U_z(1,-1,1)-k_z(1,-1,1)U_y(1,-1,1)=jk_0\sum_{p'=-1}^{1}\sum_{q'=-1}^{1}\sum_{r'=-1}^{1}a(1-p',-1-q',1-r')S_x(p',q',r')\\
&k_y(-1,0,1)U_z(-1,0,1)-k_z(-1,0,1)U_y(-1,0,1)=jk_0\sum_{p'=-1}^{1}\sum_{q'=-1}^{1}\sum_{r'=-1}^{1}a(-1-p',0-q',1-r')S_x(p',q',r')\\
&k_y(0,0,1)U_z(0,0,1)-k_z(0,0,1)U_y(0,0,1)=jk_0\sum_{p'=-1}^{1}\sum_{q'=-1}^{1}\sum_{r'=-1}^{1}a(0-p',0-q',1-r')S_x(p',q',r')\\
&k_y(1,0,1)U_z(1,0,1)-k_z(1,0,1)U_y(1,0,1)=jk_0\sum_{p'=-1}^{1}\sum_{q'=-1}^{1}\sum_{r'=-1}^{1}a(1-p',0-q',1-r')S_x(p',q',r')\\
&k_y(-1,1,1)U_z(-1,1,1)-k_z(-1,1,1)U_y(-1,1,1)=jk_0\sum_{p'=-1}^{1}\sum_{q'=-1}^{1}\sum_{r'=-1}^{1}a(-1-p',1-q',1-r')S_x(p',q',r')\\
&k_y(0,1,1)U_z(0,1,1)-k_z(0,1,1)U_y(0,1,1)=jk_0\sum_{p'=-1}^{1}\sum_{q'=-1}^{1}\sum_{r'=-1}^{1}a(0-p',1-q',1-r')S_x(p',q',r')\\
&k_y(1,1,1)U_z(1,1,1)-k_z(1,1,1)U_y(1,1,1)=jk_0\sum_{p'=-1}^{1}\sum_{q'=-1}^{1}\sum_{r'=-1}^{1}a(1-p',1-q',1-r')S_x(p',q',r')
\end{aligned}\right\}$$

This large set of equations can be written in matrix form as

$$\mathbf{K}_y\mathbf{u}_z-\mathbf{K}_z\mathbf{u}_y=jk_0[\![\varepsilon_\mathrm{r}]\!]\mathbf{s}_x$$

# Explanation of the Terms in the Matrix Equation

$$\mathbf{K}_y\mathbf{u}_z-\mathbf{K}_z\mathbf{u}_y=jk_0[\![\varepsilon_\mathrm{r}]\!]\mathbf{s}_x$$

$$\underbrace{k_y(p,q,r)}\underbrace{U_z(p,q,r)}-\underbrace{k_z(p,q,r)}\underbrace{U_y(p,q,r)}=jk_0\underbrace{\sum_{p'=-\infty}^{\infty}\sum_{q'=-\infty}^{\infty}\sum_{r'=-\infty}^{\infty}a(p-p',q-q',r-r')}\underbrace{S_x(p',q',r')}$$

$k_y(p,q,r)$ and $k_z(p,q,r)$ are performing point-by-point multiplications. These are represented by diagonal matrices containing all of the wave vector components along the center diagonal.

This is a convolution operation. It is represented by a Toeplitz matrix.

$U_y(p,q,r)$, $U_z(p,q,r)$ and $S_x(p,q,r)$ are the unknown functions. They are represented by column vectors containing the amplitudes of each spatial harmonic in the expansion.

$$\mathbf{K}_i=\begin{bmatrix} k_i\left(-\frac{P}{2},-\frac{Q}{2},-\frac{R}{2}\right) & & 0 \\ & \ddots & \\ 0 & & k_i\left(+\frac{P}{2},+\frac{Q}{2},+\frac{R}{2}\right)\end{bmatrix}$$

Only Toeplitz for 1D

$$\downarrow$$

$$[\![\varepsilon_\mathrm{r}]\!]=\begin{bmatrix}\text{Toeplitz}\end{bmatrix}$$

$$\mathbf{u}_i=\begin{bmatrix}U_i\left(-\frac{P}{2},-\frac{Q}{2},-\frac{R}{2}\right)\\ \vdots \\ \vdots \\ U_i\left(+\frac{P}{2},+\frac{Q}{2},+\frac{R}{2}\right)\end{bmatrix}\qquad \mathbf{s}_i=\begin{bmatrix}S_i\left(-\frac{P}{2},-\frac{Q}{2},-\frac{R}{2}\right)\\ \vdots \\ \vdots \\ S_i\left(+\frac{P}{2},+\frac{Q}{2},+\frac{R}{2}\right)\end{bmatrix}$$

# Matrix Form of Maxwell's Equations in Fourier Space

Analytical Equations

$$k_y(p,q,r)U_z(p,q,r)-k_z(p,q,r)U_y(p,q,r)=jk_0a(p,q,r)*S_x(p,q,r)$$
$$k_z(p,q,r)U_x(p,q,r)-k_x(p,q,r)U_z(p,q,r)=jk_0a(p,q,r)*S_y(p,q,r)$$
$$k_x(p,q,r)U_y(p,q,r)-k_y(p,q,r)U_x(p,q,r)=jk_0a(p,q,r)*S_z(p,q,r)$$

$$k_y(p,q,r)S_z(p,q,r)-k_z(p,q,r)S_y(p,q,r)=jk_0b(p,q,r)*U_x(p,q,r)$$
$$k_z(p,q,r)S_x(p,q,r)-k_x(p,q,r)S_z(p,q,r)=jk_0b(p,q,r)*U_y(p,q,r)$$
$$k_x(p,q,r)S_y(p,q,r)-k_y(p,q,r)S_x(p,q,r)=jk_0b(p,q,r)*U_z(p,q,r)$$

→

Matrix Equations

$$\mathbf{K}_y\mathbf{u}_z-\mathbf{K}_z\mathbf{u}_y=jk_0\llbracket\varepsilon_\mathrm{r}\rrbracket\mathbf{s}_x$$
$$\mathbf{K}_z\mathbf{u}_x-\mathbf{K}_x\mathbf{u}_z=jk_0\llbracket\varepsilon_\mathrm{r}\rrbracket\mathbf{s}_y$$
$$\mathbf{K}_x\mathbf{u}_y-\mathbf{K}_y\mathbf{u}_x=jk_0\llbracket\varepsilon_\mathrm{r}\rrbracket\mathbf{s}_z$$

$$\mathbf{K}_y\mathbf{s}_z-\mathbf{K}_z\mathbf{s}_y=jk_0\llbracket\mu_\mathrm{r}\rrbracket\mathbf{u}_x$$
$$\mathbf{K}_z\mathbf{s}_x-\mathbf{K}_x\mathbf{s}_z=jk_0\llbracket\mu_\mathrm{r}\rrbracket\mathbf{u}_y$$
$$\mathbf{K}_x\mathbf{s}_y-\mathbf{K}_y\mathbf{s}_x=jk_0\llbracket\mu_\mathrm{r}\rrbracket\mathbf{u}_z$$

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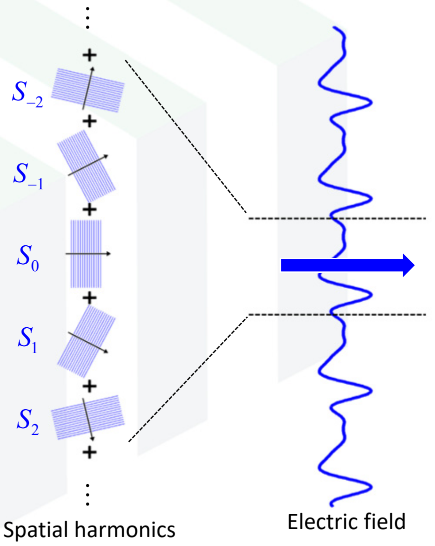
# Interpreting the Column Vectors

Each element of the column vector $\mathbf{s}_i$ is the complex amplitude of a spatial harmonic.

$$\mathbf{s}_i=\begin{bmatrix}\vdots\\S_{-2}\\S_{-1}\\S_0\\S_1\\S_2\\\vdots\end{bmatrix}$$

Column vector



# Fast Fourier Factorization (FFF)

## Product of Two Functions

Consider the product of two periodic functions that have the same period:

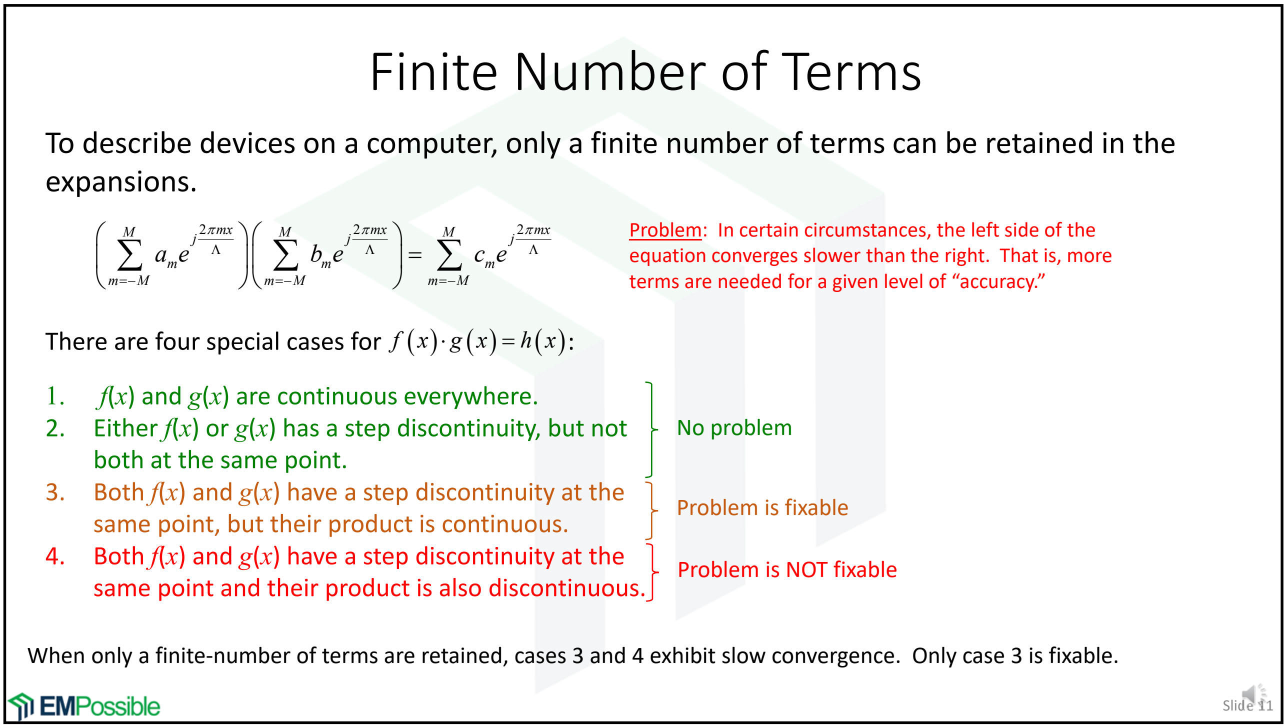
$$f(x)\cdot g(x)=h(x)$$

Expand each function into its own Fourier series.

$$\left(\sum_{m=-\infty}^{\infty} a_m e^{j\frac{2\pi mx}{\Lambda}}\right)\left(\sum_{m=-\infty}^{\infty} b_m e^{j\frac{2\pi mx}{\Lambda}}\right)=\sum_{m=-\infty}^{\infty} c_m e^{j\frac{2\pi mx}{\Lambda}}$$

This is exact, as long as an infinite number of terms is used.

Obviously, only a finite number of terms can be retained in the expansion if it is to be solved on a computer.



# Finite Number of Terms

To describe devices on a computer, only a finite number of terms can be retained in the expansions.

$$\left(\sum_{m=-M}^{M} a_m e^{j\frac{2\pi m x}{\Lambda}}\right)\left(\sum_{m=-M}^{M} b_m e^{j\frac{2\pi m x}{\Lambda}}\right)=\sum_{m=-M}^{M} c_m e^{j\frac{2\pi m x}{\Lambda}}$$

Problem: In certain circumstances, the left side of the equation converges slower than the right. That is, more terms are needed for a given level of "accuracy."

There are four special cases for $f(x)\cdot g(x)=h(x)$:

1. $f(x)$ and $g(x)$ are continuous everywhere.
2. Either $f(x)$ or $g(x)$ has a step discontinuity, but not both at the same point.
3. Both $f(x)$ and $g(x)$ have a step discontinuity at the same point, but their product is continuous.
4. Both $f(x)$ and $g(x)$ have a step discontinuity at the same point and their product is also discontinuous.

Cases 1–2: No problem

Case 3: Problem is fixable

Case 4: Problem is NOT fixable

When only a finite-number of terms are retained, cases 3 and 4 exhibit slow convergence. Only case 3 is fixable.

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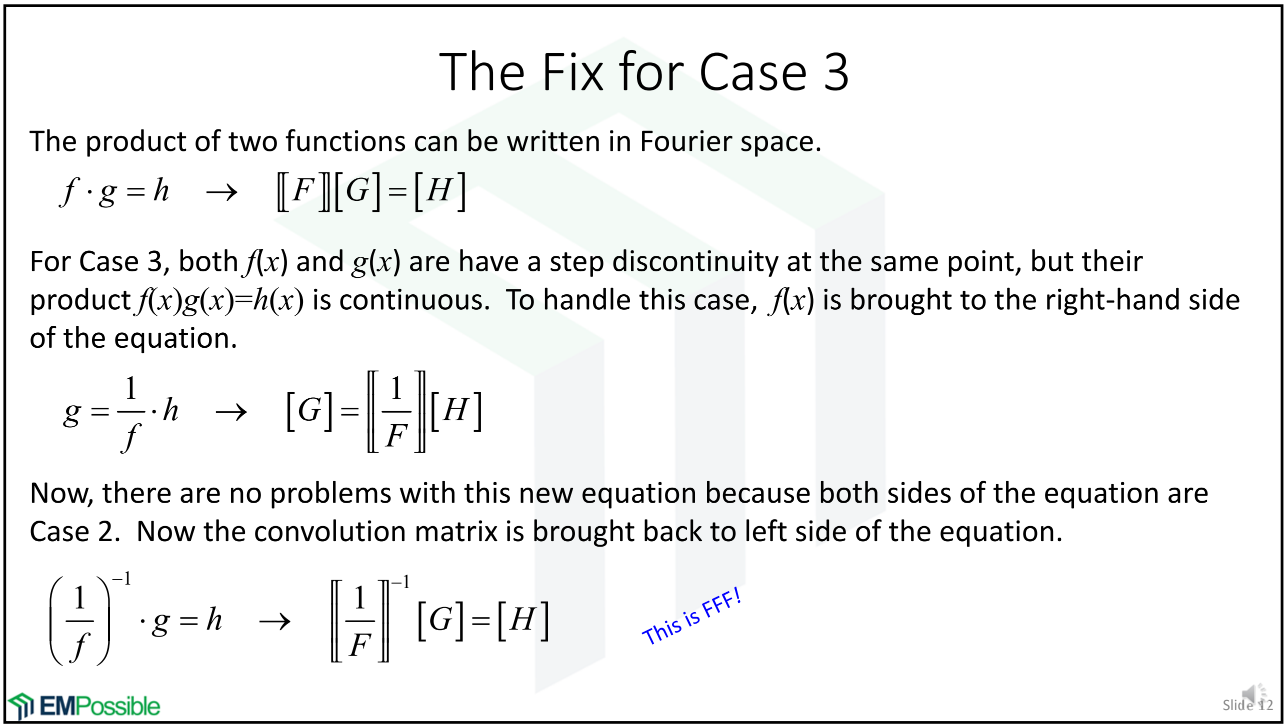

# The Fix for Case 3

The product of two functions can be written in Fourier space.

$$f\cdot g=h \quad\rightarrow\quad [\![F]\!][G]=[H]$$

For Case 3, both $f(x)$ and $g(x)$ are have a step discontinuity at the same point, but their product $f(x)g(x)=h(x)$ is continuous. To handle this case, $f(x)$ is brought to the right-hand side of the equation.

$$g=\frac{1}{f}\cdot h \quad\rightarrow\quad [G]=\left[\!\left[\frac{1}{F}\right]\!\right][H]$$

Now, there are no problems with this new equation because both sides of the equation are Case 2. Now the convolution matrix is brought back to left side of the equation.

$$\left(\frac{1}{f}\right)^{-1}\cdot g=h \quad\rightarrow\quad \left[\!\left[\frac{1}{F}\right]\!\right]^{-1}[G]=[H]$$

This is FFF!

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# FFF and Maxwell’s Equations

In Maxwell’s equations, there exists a product of two functions…

$$\varepsilon_{\mathrm{r}}\left(\vec{r}\right)\cdot\vec{E}\left(\vec{r}\right)$$

The dielectric function is discontinuous at the interface between two materials. Boundary conditions require that

$$E_{1,\|} = E_{2,\|}$$

Tangential component is continuous across the interface

$$\varepsilon_1 E_{1,\perp} = \varepsilon_2 E_{2,\perp}$$

Normal component is discontinuous across the interface,
but the product of $\varepsilon E_{\perp}$ is continuous.

In conclusion, the convolution matrix must be handled differently for the tangential and normal components. This implies that the final convolution matrix will be a tensor.

# FFF for Maxwell’s Equations

First, the electric field is decomposed into tangential and normal components at all interfaces.

$$[\![\varepsilon_{\mathrm{r}}]\!]\mathbf{s} = [\![\varepsilon_{\mathrm{r}}]\!]\left[\mathbf{s}_{\|} + \mathbf{s}_{\perp}\right] = [\![\varepsilon_{\mathrm{r}}]\!]\mathbf{s}_{\|} + [\![\varepsilon_{\mathrm{r}}]\!]\mathbf{s}_{\perp}$$

This creates the opportunity to associate different convolution matrices with the different field components.

$$[\![\varepsilon_{\mathrm{r}}]\!]\mathbf{s} \rightarrow \underbrace{[\![\varepsilon_{\mathrm{r},\|}]\!]\mathbf{s}_{\|}}_{\substack{\text{Case 2.}\\ \text{No problems.}}} + \underbrace{[\![\varepsilon_{\mathrm{r},\perp}]\!]\mathbf{s}_{\perp}}_{\substack{\text{Case 3.}\\ \text{Fixable with FFF.}}}$$

$$[\![\varepsilon_{\mathrm{r}}]\!]_{\mathrm{FFF}}\,\mathbf{s} = [\![\varepsilon_{\mathrm{r},\|}]\!]\mathbf{s}_{\|} + [\![1/\varepsilon_{\mathrm{r},\perp}]\!]^{-1}\mathbf{s}_{\perp}$$

# Normal Vector Field

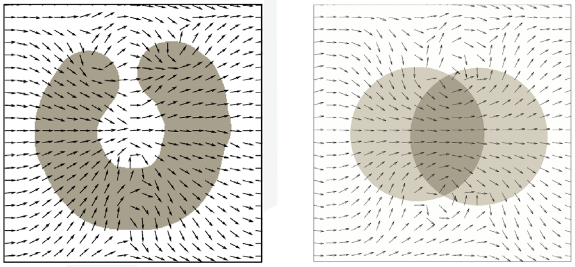
To implement FFF, which directions are parallel and perpendicular must be determined at each point in space.

For arbitrarily shaped devices, this comes from knowledge of the materials within the layer.

A vector function must be constructed throughout the grid that is normal to all the interfaces. This called the “normal vector” field.

$$\hat{n}(x,y,z)$$

This can be very difficult to calculate!!



P. Gotz, T. Schuster, K. Frenner, S. Rafler, W. Osten, “Normal vector method for the RCWA with automated vector field generation,” Opt. Express 16(22), 17295-17301 (2008).

# Incorporating Normal Vector Function

Recall the FFF fix

$$[\![\varepsilon_{\mathrm{r}}]\!]_{\mathrm{FFF}}\,\mathbf{s} = [\![\varepsilon_{\mathrm{r}}]\!]\mathbf{s}_{\parallel} + [\![1/\varepsilon_{\mathrm{r}}]\!]^{-1}\mathbf{s}_{\perp} \qquad \text{Eq. (1)}$$

The parallel and perpendicular components of $\mathbf{s}$ can be calculated using the normal vector matrix $\mathbf{N}$.

$$\mathbf{s}_{\perp} = \mathbf{N}\mathbf{s} \qquad \text{Eq. (2)}$$

$$\mathbf{s}_{\parallel} = \mathbf{s} - \mathbf{N}\mathbf{s} = (\mathbf{I}-\mathbf{N})\mathbf{s} \qquad \text{Eq. (3)}$$

Substituting these into the FFF equation yields

$$[\![\varepsilon_{\mathrm{r}}]\!]_{\mathrm{FFF}}\,\mathbf{s} = [\![\varepsilon_{\mathrm{r}}]\!](\mathbf{I}-\mathbf{N})\mathbf{s} + [\![1/\varepsilon_{\mathrm{r}}]\!]^{-1}\mathbf{N}\mathbf{s}$$

Substitute Eq. (2) and (3) into Eq. (1).

$$= [\![\varepsilon_{\mathrm{r}}]\!]\mathbf{s} - [\![\varepsilon_{\mathrm{r}}]\!]\mathbf{N}\mathbf{s} + [\![1/\varepsilon_{\mathrm{r}}]\!]^{-1}\mathbf{N}\mathbf{s}$$

Expand the equation.

$$= \left([\![\varepsilon_{\mathrm{r}}]\!] - [\![\varepsilon_{\mathrm{r}}]\!]\mathbf{N} + [\![1/\varepsilon_{\mathrm{r}}]\!]^{-1}\mathbf{N}\right)\mathbf{s}$$

Factor out **s**.

This defines a revised convolution matrix that incorporates FFF.

# Revised Convolution Matrix

The convolution matrix incorporating FFF is then

$$[\![\varepsilon_{\mathrm{r}}]\!]_{\mathrm{FFF}} = [\![\varepsilon_{\mathrm{r}}]\!] - [\![\varepsilon_{\mathrm{r}}]\!]\mathbf{N} + [\![1/\varepsilon_{\mathrm{r}}]\!]^{-1}\mathbf{N}$$

This is often written as

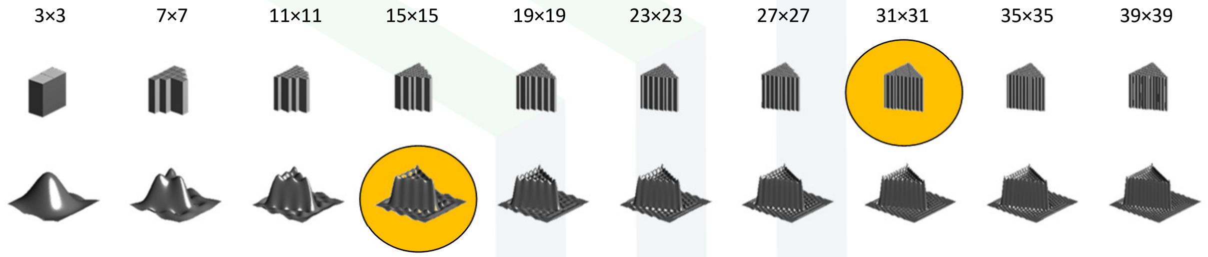
$$[\![\varepsilon_{\mathrm{r}}]\!]_{\mathrm{FFF}} = [\![\varepsilon_{\mathrm{r}}]\!] + \underbrace{[\![\Delta\varepsilon_{\mathrm{r}}]\!]\mathbf{N}}_{} \qquad [\![\Delta\varepsilon_{\mathrm{r}}]\!] = [\![1/\varepsilon_{\mathrm{r}}]\!]^{-1} - [\![\varepsilon_{\mathrm{r}}]\!]$$

This is interpreted as
a correction term
that incorporates FFF.

# Consequences of Fourier-Space

# Efficient Representation of Devices

Along a given direction, approximately half the number of the terms are needed in Fourier space than would be needed in real space.



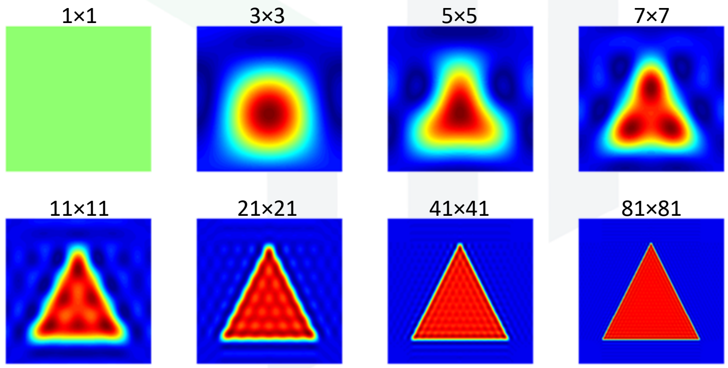
For 2D problems in real space, 4× more terms are needed making the matrices 16× larger.

For 3D problems in real space, 8× more terms are needed making the matrices 64× larger.

# Blurring from Too Few Harmonics

If too few harmonics are used, the geometry of the device is blurred.

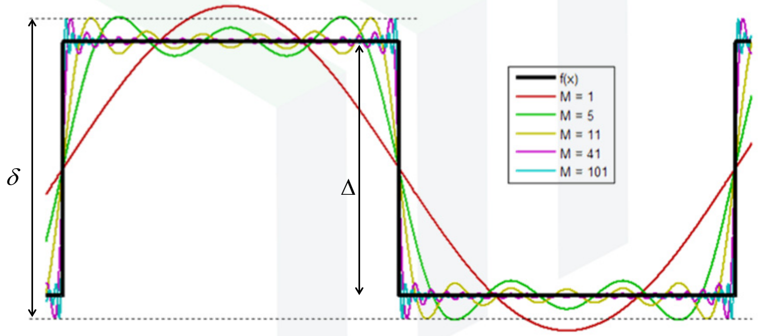
- Boundaries are artificially blurred.
- Reflections at boundaries are artificially reduced.
- It is difficult or impossible to resolve fine features or rapidly varying fields.



Rule of Thumb: # harmonics = 10 per $\lambda$

# Gibb’s Phenomena

A problem occurs when a discontinuous function (material interface) is represented by continuous basis functions (sin’s and cos’s). When the Fourier transform is used, “spikes” appear around each discontinuity. Fourier space methods act is if those spikes are actually present.



http://mathworld.wolfram.com/GibbsPhenomenon.html

$$\frac{\delta}{\Delta} = \frac{2}{\pi}\int_0^{\pi} \frac{\sin x}{x}\,dx \cong 1.1789797445$$

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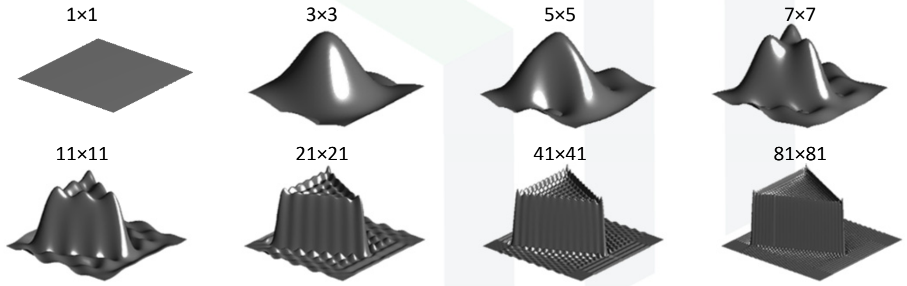
# Gibb’s Phenomena in Maxwell’s Equations

A Fourier-space numerical method treats the spikes as if they are real.

- The magnitude of the spikes remains constant no matter how many harmonics are used.
- The magnitude of the spikes is proportional to the severity of the discontinuity.
- The width of the spikes becomes more narrow with increasing number of harmonics.
- In Fourier-space, Maxwell’s equations really think the spikes are there.



Due to Gibb’s phenomenon, Fourier-space analysis is most efficient for structures with low to moderate index contrast, but many people have modeled metals effectively.

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