Advanced Computation:
Computational Electromagnetics

Maxwell's Equations in Fourier Space

Outline

- What is Fourier space?
- Complex Fourier series in terms of the reciprocal lattice vectors
- Maxwell's equations in Fourier space
- Visualizing the plane wave expansion

What is Fourier Space?

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Fourier-Space Vs. Frequency-Domain

$$\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t}$$

$$\nabla \times \vec{H} = \varepsilon \frac{\partial \vec{E}}{\partial t}$$

Fourier transform x, y, and z to k_x, k_y, and k_z.

Fourier transform t to ω.

$$j\vec{k} \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t}$$

$$j\vec{k} \times \vec{H} = \varepsilon \frac{\partial \vec{E}}{\partial t}$$

Fourier Space

$$\nabla \times \vec{E} = -j\omega\mu\vec{H}$$

$$\nabla \times \vec{H} = j\omega\varepsilon\vec{E}$$

Frequency Domain

	Real-Space	Fourier-Space
Time-Domain	FDTD, Discontinuous Galerkin	Pseudo-spectral FDTD
Frequency-Domain	FDFD, FEM, MoM, MoL	RCWA, SAM, PWEM, spectral domain

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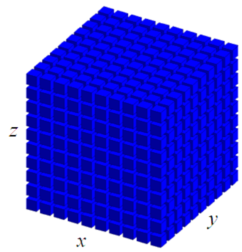
What is Fourier Space?

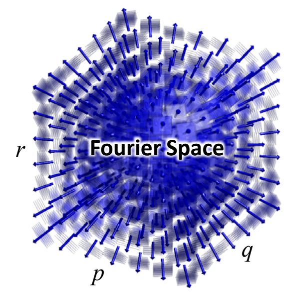


<u>Real Space</u>

So far, fields and devices were represented on an x-y-z grid where field values and material properties are defined at discrete points.

<u>Fourier Space</u>

In Fourier-space, fields are represented as a sum of plane waves at different angles and different wavelengths called *spatial harmonics*. Devices are also represented as the sum of sinusoidal gratings at different angles and periods.



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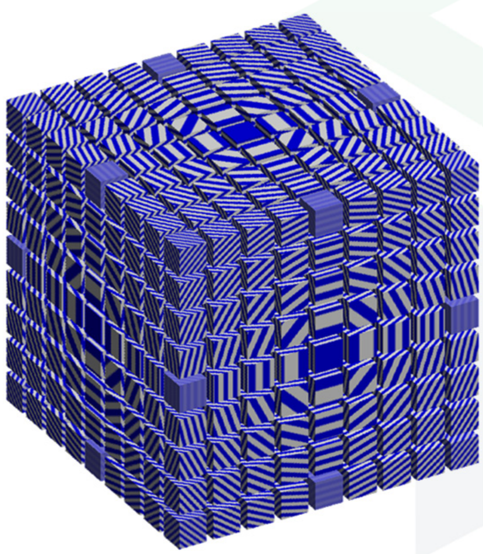
Visualizing the Spatial Harmonics



$$\vec{k}\left(p,q,r\right) = p\vec{T}_1 + q\vec{T}_2 + r\vec{T}_3$$

p, q, and r are the indices of the spatial harmonics.

$$p \equiv \text{integer} \qquad -\infty,\ldots,-2,-1,0,1,2,\ldots,\infty$$
$$q \equiv \text{integer} \qquad -\infty,\ldots,-2,-1,0,1,2,\ldots,\infty$$
$$r \equiv \text{integer} \qquad -\infty,\ldots,-2,-1,0,1,2,\ldots,\infty$$

$\vec{T}_1$, $\vec{T}_2$, and $\vec{T}_3$ are the reciprocal lattice vectors.

Each of these plane waves will be assigned its own complex amplitude to convey its magnitude and phase.

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Complex Fourier Series in Terms of the Reciprocal Lattice Vectors

Conventional Complex Fourier Series

Periodic functions can be expanded into a Fourier series.

For 1D periodic functions, this is

$$f(x)=\sum_{p=-\infty}^{\infty}a(p)e^{j\frac{2\pi px}{\Lambda}} \qquad a(p)=\frac{1}{\Lambda}\int_{-\Lambda/2}^{\Lambda/2}f(x)e^{-j\frac{2\pi px}{\Lambda}}dx$$

For 2D periodic functions, this is

$$f(x,y)=\sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}a(p,q)e^{j\left(\frac{2\pi px}{\Lambda_x}+\frac{2\pi qy}{\Lambda_y}\right)} \qquad a(p,q)=\frac{1}{A}\iint_A f(x,y)e^{-j\left(\frac{2\pi px}{\Lambda_x}+\frac{2\pi qy}{\Lambda_y}\right)}dA$$

For 3D periodic functions, this is

$$f(x,y,z)=\sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\sum_{r=-\infty}^{\infty}a(p,q,r)e^{j\left(\frac{2\pi px}{\Lambda_x}+\frac{2\pi qy}{\Lambda_y}+\frac{2\pi rz}{\Lambda_z}\right)} \qquad a(p,q,r)=\frac{1}{V}\iiint_V f(x,y,z)e^{-j\left(\frac{2\pi px}{\Lambda_x}+\frac{2\pi qy}{\Lambda_y}+\frac{2\pi rz}{\Lambda_z}\right)}dV$$

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Generalized Complex Fourier Series

Fourier series can be written in terms of the reciprocal lattice vectors.

For 1D periodic functions, this is

$$f(x)=\sum_{p=-\infty}^{\infty} a(p)e^{jpTx} \qquad a(p)=\frac{1}{\Lambda}\int_{-\Lambda/2}^{\Lambda/2} f(x)e^{-jpTx}dx \qquad T=\frac{2\pi}{\Lambda}$$

For 2D periodic functions, this is

$$f(x,y)=\sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty} a(p,q)e^{j(p\vec{T}_1+q\vec{T}_2)\bullet\vec{r}} \qquad a(p,q)=\frac{1}{A}\iint_A f(x,y)e^{-j(p\vec{T}_1+q\vec{T}_2)\bullet\vec{r}}dA$$

For 3D periodic functions, this is

$$f(\vec{r})=\sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\sum_{r=-\infty}^{\infty} a(p,q,r)e^{j(p\vec{T}_1+q\vec{T}_2+r\vec{T}_3)\bullet\vec{r}} \qquad a(p,q,r)=\frac{1}{V}\iiint_V f(\vec{r})e^{-j(p\vec{T}_1+q\vec{T}_2+r\vec{T}_3)\bullet\vec{r}}dV$$

For rectangular, tetrahedral, or orthorhombic geometries, the reciprocal lattice vectors are:

$$\vec{T}_1=\frac{2\pi}{\Lambda_x}\hat{x} \qquad \vec{T}_2=\frac{2\pi}{\Lambda_y}\hat{y} \qquad \vec{T}_3=\frac{2\pi}{\Lambda_z}\hat{z}$$

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Maxwell’s Equations in Fourier Space

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Starting Point

Start with Maxwell's equations in the following form…

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = k_0 \mu_r \tilde{H}_x \qquad \frac{\partial \tilde{H}_z}{\partial y} - \frac{\partial \tilde{H}_y}{\partial z} = k_0 \varepsilon_r E_x$$

$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = k_0 \mu_r \tilde{H}_y \qquad \frac{\partial \tilde{H}_x}{\partial z} - \frac{\partial \tilde{H}_z}{\partial x} = k_0 \varepsilon_r E_y$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = k_0 \mu_r \tilde{H}_z \qquad \frac{\partial \tilde{H}_y}{\partial x} - \frac{\partial \tilde{H}_x}{\partial y} = k_0 \varepsilon_r E_z$$

Recall that the magnetic field was normalized according to

$$\vec{\tilde{H}} = -j\sqrt{\frac{\mu_0}{\varepsilon_0}}\vec{H}$$

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Fourier Expansion of the Materials

Assuming the device is infinitely periodic in all directions, the permittivity and permeability functions can be expanded into a generalized Fourier Series.

$$\varepsilon_r(\vec{r}) = \sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\sum_{r=-\infty}^{\infty} a(p,q,r)\, e^{j(p\vec{T}_1 + q\vec{T}_2 + r\vec{T}_3)\bullet\vec{r}}$$

$$a(p,q,r) = \frac{1}{V}\iiint_V \varepsilon_r(\vec{r})\, e^{-j(p\vec{T}_1 + q\vec{T}_2 + r\vec{T}_3)\bullet\vec{r}}\, dV$$

$$\mu_r(\vec{r}) = \sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\sum_{r=-\infty}^{\infty} b(p,q,r)\, e^{j(p\vec{T}_1 + q\vec{T}_2 + r\vec{T}_3)\bullet\vec{r}}$$

$$b(p,q,r) = \frac{1}{V}\iiint_V \mu_r(\vec{r})\, e^{-j(p\vec{T}_1 + q\vec{T}_2 + r\vec{T}_3)\bullet\vec{r}}\, dV$$

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Fourier Expansion of the Fields

The expansions are slightly different for fields because a wave could be travelling in any direction $\vec{\beta}$. The field must obey Bloch's theorem.

$$\vec{E}(\vec{r}) = e^{-j\vec{\beta}\bullet\vec{r}} \sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\sum_{r=-\infty}^{\infty} \vec{S}(p,q,r)\, e^{j\left(p\vec{T}_1+q\vec{T}_2+r\vec{T}_3\right)\bullet\vec{r}}$$

Think of $\vec{\beta}$ as $\vec{k}_{\text{inc}}$, but be careful with that analogy.

$$= \sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\sum_{r=-\infty}^{\infty} \vec{S}(p,q,r)\, e^{-j\left(\vec{\beta}-p\vec{T}_1-q\vec{T}_2-r\vec{T}_3\right)\bullet\vec{r}}$$

Let this be $\vec{k}(p,q,r)$

$e^{-j\vec{\beta}\bullet\vec{r}}$ was brought inside summation and combined with second exponential.

$$= \sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\sum_{r=-\infty}^{\infty} \vec{S}(p,q,r)\, e^{-j\vec{k}(p,q,r)\bullet\vec{r}}$$

This is clearly a set of plane waves with amplitudes $\vec{S}(p,q,r)$.

$$= \sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\sum_{r=-\infty}^{\infty} \vec{S}(p,q,r)\, e^{-j\left[k_x(p,q,r)x+k_y(p,q,r)y+k_z(p,q,r)z\right]}$$

$$\vec{k}(p,q,r) = \vec{\beta} - p\vec{T}_1 - q\vec{T}_2 - r\vec{T}_3$$

$$k_x(p,q,r) = \beta_x - pT_{1,x} - qT_{2,x} - rT_{3,x}$$
$$k_y(p,q,r) = \beta_y - pT_{1,y} - qT_{2,y} - rT_{3,y}$$
$$k_z(p,q,r) = \beta_z - pT_{1,z} - qT_{2,z} - rT_{3,z}$$

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Substitute Expansions into Maxwell's Equations

$$\tilde{H}_y(\vec{r}) = \sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\sum_{r=-\infty}^{\infty} U_y(p,q,r)\, e^{-j\left[k_x(p,q,r)x+k_y(p,q,r)y+k_z(p,q,r)z\right]}$$

$$\varepsilon_r(\vec{r}) = \sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\sum_{r=-\infty}^{\infty} a(p,q,r)\, e^{j\left(p\vec{T}_1+q\vec{T}_2+r\vec{T}_3\right)\bullet\vec{r}}$$

$$\tilde{H}_z(\vec{r}) = \sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\sum_{r=-\infty}^{\infty} U_z(p,q,r)\, e^{-j\left[k_x(p,q,r)x+k_y(p,q,r)y+k_z(p,q,r)z\right]}$$

$$E_x(\vec{r}) = \sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\sum_{r=-\infty}^{\infty} S_x(p,q,r)\, e^{-j\left[k_x(p,q,r)x+k_y(p,q,r)y+k_z(p,q,r)z\right]}$$

$$\frac{\partial \tilde{H}_z}{\partial y} - \frac{\partial \tilde{H}_y}{\partial z} = k_0 \varepsilon_r E_x$$

↓

$$\frac{\partial}{\partial y}\left[\sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\sum_{r=-\infty}^{\infty} U_z(p,q,r)\, e^{-j\left[k_x(p,q,r)x+k_y(p,q,r)y+k_z(p,q,r)z\right]}\right] - \frac{\partial}{\partial z}\left[\sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\sum_{r=-\infty}^{\infty} U_y(p,q,r)\, e^{-j\left[k_x(p,q,r)x+k_y(p,q,r)y+k_z(p,q,r)z\right]}\right]$$
$$= k_0\left[\sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\sum_{r=-\infty}^{\infty} a(p,q,r)\, e^{j\left(p\vec{T}_1+q\vec{T}_2+r\vec{T}_3\right)\bullet\vec{r}}\right]\left[\sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\sum_{r=-\infty}^{\infty} S_x(p,q,r)\, e^{-j\left[k_x(p,q,r)x+k_y(p,q,r)y+k_z(p,q,r)z\right]}\right]$$

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Algebra for the Left Side Terms

First ugly term…

$$\frac{\partial}{\partial y}\left[\sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\sum_{r=-\infty}^{\infty}U_z(p,q,r)e^{-j\left[k_x(p,q,r)x+k_y(p,q,r)y+k_z(p,q,r)z\right]}\right]=\sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\sum_{r=-\infty}^{\infty}U_z(p,q,r)\frac{\partial}{\partial y}e^{-j\left[k_x(p,q,r)x+k_y(p,q,r)y+k_z(p,q,r)z\right]}$$

$$=\sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\sum_{r=-\infty}^{\infty}U_z(p,q,r)\left[-jk_y(p,q,r)\right]e^{-j\left[k_x(p,q,r)x+k_y(p,q,r)y+k_z(p,q,r)z\right]}$$

$$=\sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\sum_{r=-\infty}^{\infty}-jk_y(p,q,r)U_z(p,q,r)e^{-j\left[k_x(p,q,r)x+k_y(p,q,r)y+k_z(p,q,r)z\right]}$$

Second ugly term…

$$\frac{\partial}{\partial z}\left[\sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\sum_{r=-\infty}^{\infty}U_y(p,q,r)e^{-j\left[k_x(p,q,r)x+k_y(p,q,r)y+k_z(p,q,r)z\right]}\right]=\sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\sum_{r=-\infty}^{\infty}U_y(p,q,r)\frac{\partial}{\partial z}e^{-j\left[k_x(p,q,r)x+k_y(p,q,r)y+k_z(p,q,r)z\right]}$$

$$=\sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\sum_{r=-\infty}^{\infty}U_y(p,q,r)\left[-jk_z(p,q,r)\right]e^{-j\left[k_x(p,q,r)x+k_y(p,q,r)y+k_z(p,q,r)z\right]}$$

$$=\sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\sum_{r=-\infty}^{\infty}-jk_z(p,q,r)U_y(p,q,r)e^{-j\left[k_x(p,q,r)x+k_y(p,q,r)y+k_z(p,q,r)z\right]}$$

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Algebra for the Right-Side Term

Third ugly term…

This term as the product of two triple summations.

$$\left[\sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\sum_{r=-\infty}^{\infty}a(p,q,r)e^{j\left(p\vec{T}_1+q\vec{T}_2+r\vec{T}_3\right)\bullet\vec{r}}\right]\left[\sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\sum_{r=-\infty}^{\infty}S_x(p,q,r)e^{-j\left[k_x(p,q,r)x+k_y(p,q,r)y+k_z(p,q,r)z\right]}\right]$$

This is called a Cauchy product and is handled as follows.

$$\left(\sum_{n=0}^{\infty}a_n\right)\cdot\left(\sum_{n=0}^{\infty}b_n\right)=\sum_{n=0}^{\infty}c_n \qquad c_n=\sum_{m=0}^{n}a_m b_{n-m}$$

Applying this rule to the triple summations, gives

$$\sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\sum_{r=-\infty}^{\infty}\left\{e^{-j\left[k_x(p,q,r)x+k_y(p,q,r)y+k_z(p,q,r)z\right]}\sum_{p'=-\infty}^{\infty}\sum_{q'=-\infty}^{\infty}\sum_{r'=-\infty}^{\infty}a(p-p',q-q',r-r')S_x(p',q',r')\right\}$$

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Combine the Terms Inside Summation

$$\frac{\partial}{\partial y}\left[\sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\sum_{r=-\infty}^{\infty}U_z(p,q,r)e^{-j\left[k_x(p,q,r)x+k_y(p,q,r)y+k_z(p,q,r)z\right]}\right]-\frac{\partial}{\partial z}\left[\sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\sum_{r=-\infty}^{\infty}U_y(p,q,r)e^{-j\left[k_x(p,q,r)x+k_y(p,q,r)y+k_z(p,q,r)z\right]}\right]$$
$$=k_0\left[\sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\sum_{r=-\infty}^{\infty}a(p,q,r)e^{j(p\vec{T}_1+q\vec{T}_2+r\vec{T}_3)\bullet\vec{r}}\right]\left[\sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\sum_{r=-\infty}^{\infty}S_x(p,q,r)e^{-j\left[k_x(p,q,r)x+k_y(p,q,r)y+k_z(p,q,r)z\right]}\right]$$

$$\Downarrow$$

$$\sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\sum_{r=-\infty}^{\infty}-jk_y(p,q,r)U_z(p,q,r)e^{-j\left[k_x(p,q,r)x+k_y(p,q,r)y+k_z(p,q,r)z\right]}-\sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\sum_{r=-\infty}^{\infty}-jk_z(p,q,r)U_y(p,q,r)e^{-j\left[k_x(p,q,r)x+k_y(p,q,r)y+k_z(p,q,r)z\right]}$$
$$=k_0\sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\sum_{r=-\infty}^{\infty}\left\{e^{-j\left[k_x(p,q,r)x+k_y(p,q,r)y+k_z(p,q,r)z\right]}\sum_{p'=-\infty}^{\infty}\sum_{q'=-\infty}^{\infty}\sum_{r'=-\infty}^{\infty}a(p-p',q-q',r-r')S_x(p',q',r')\right\}$$

The equation can now be brought inside a single triple summation.

$$\sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\sum_{r=-\infty}^{\infty}\left\{\begin{array}{c}-jk_y(p,q,r)U_z(p,q,r)e^{-j\left[k_x(p,q,r)x+k_y(p,q,r)y+k_z(p,q,r)z\right]}+jk_z(p,q,r)U_y(p,q,r)e^{-j\left[k_x(p,q,r)x+k_y(p,q,r)y+k_z(p,q,r)z\right]}\\=k_0e^{-j\left[k_x(p,q,r)x+k_y(p,q,r)y+k_z(p,q,r)z\right]}\sum_{p'=-\infty}^{\infty}\sum_{q'=-\infty}^{\infty}\sum_{r'=-\infty}^{\infty}a(p-p',q-q',r-r')S_x(p',q',r')\end{array}\right\}$$

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Final Equation for $(p,q,r)^{\text{th}}$ Harmonic

$$\sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\sum_{r=-\infty}^{\infty}\left\{\begin{array}{c}-jk_y(p,q,r)U_z(p,q,r)e^{-j\left[k_x(p,q,r)x+k_y(p,q,r)y+k_z(p,q,r)z\right]}+jk_z(p,q,r)U_y(p,q,r)e^{-j\left[k_x(p,q,r)x+k_y(p,q,r)y+k_z(p,q,r)z\right]}\\=k_0e^{-j\left[k_x(p,q,r)x+k_y(p,q,r)y+k_z(p,q,r)z\right]}\sum_{p'=-\infty}^{\infty}\sum_{q'=-\infty}^{\infty}\sum_{r'=-\infty}^{\infty}a(p-p',q-q',r-r')S_x(p',q',r')\end{array}\right\}$$

The equation inside the braces much be satisfied for each combination of (p,q,r).

$$-jk_y(p,q,r)U_z(p,q,r)e^{-j\left[k_x(p,q,r)x+k_y(p,q,r)y+k_z(p,q,r)z\right]}+jk_z(p,q,r)U_y(p,q,r)e^{-j\left[k_x(p,q,r)x+k_y(p,q,r)y+k_z(p,q,r)z\right]}$$
$$=k_0e^{-j\left[k_x(p,q,r)x+k_y(p,q,r)y+k_z(p,q,r)z\right]}\sum_{p'=-\infty}^{\infty}\sum_{q'=-\infty}^{\infty}\sum_{r'=-\infty}^{\infty}a(p-p',q-q',r-r')S_x(p',q',r')$$

Last, divide both sides by the common exponential term and move the j to the right-hand side.

$$k_y(p,q,r)U_z(p,q,r)-k_z(p,q,r)U_y(p,q,r)=jk_0\sum_{p'=-\infty}^{\infty}\sum_{q'=-\infty}^{\infty}\sum_{r'=-\infty}^{\infty}a(p-p',q-q',r-r')S_x(p',q',r')$$

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Alternate Derivation

Start with

$$\frac{\partial \tilde{H}_z}{\partial y} - \frac{\partial \tilde{H}_y}{\partial z} = k_0 \varepsilon_r E_x$$

Point-by-point multiplication in real-space…

Fourier-transform this equation in x, y, and z resulting in

$$k_y(p,q,r)U_z(p,q,r) - k_z(p,q,r)U_y(p,q,r) = jk_0 a * S_x$$

…becomes a convolution in Fourier-space.

$$a = \text{FT}\{\varepsilon_r\}$$
$$S_x = \text{FT}\{E_x\}$$

It can now be seen that the strange triple summation remaining in the Fourier-space equation is actually a 3D convolution in Fourier space!

$$a * S_x \quad \rightarrow \quad \sum_{p'=-\infty}^{\infty} \sum_{q'=-\infty}^{\infty} \sum_{r'=-\infty}^{\infty} a(p-p', q-q', r-r') S_x(p', q', r')$$

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Summary of Maxwell's Equations in Fourier Space

Real-Space

$$\frac{\partial \tilde{H}_z}{\partial y} - \frac{\partial \tilde{H}_y}{\partial z} = k_0 \varepsilon_r E_x$$
$$\frac{\partial \tilde{H}_x}{\partial z} - \frac{\partial \tilde{H}_z}{\partial x} = k_0 \varepsilon_r E_y$$
$$\frac{\partial \tilde{H}_y}{\partial x} - \frac{\partial \tilde{H}_x}{\partial y} = k_0 \varepsilon_r E_z$$

Fourier-Space

$$k_y(p,q,r)U_z(p,q,r) - k_z(p,q,r)U_y(p,q,r) = jk_0 a(p,q,r) * S_x(p,q,r)$$
$$k_z(p,q,r)U_x(p,q,r) - k_x(p,q,r)U_z(p,q,r) = jk_0 a(p,q,r) * S_y(p,q,r)$$
$$k_x(p,q,r)U_y(p,q,r) - k_y(p,q,r)U_x(p,q,r) = jk_0 a(p,q,r) * S_z(p,q,r)$$

$$\vec{k}(p,q,r) = k_x(p,q,r)\hat{a}_x + k_y(p,q,r)\hat{a}_y + k_z(p,q,r)\hat{a}_z = \vec{\beta} - p\vec{T}_1 - q\vec{T}_2 - r\vec{T}_3$$
$$p = -\infty, \cdots, -2, -1, 0, 1, 2, \cdots, \infty$$
$$q = -\infty, \cdots, -2, -1, 0, 1, 2, \cdots, \infty$$
$$r = -\infty, \cdots, -2, -1, 0, 1, 2, \cdots, \infty$$

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = k_0 \mu_r \tilde{H}_x$$
$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = k_0 \mu_r \tilde{H}_y$$
$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = k_0 \mu_r \tilde{H}_z$$

$$k_y(p,q,r)S_z(p,q,r) - k_z(p,q,r)S_y(p,q,r) = jk_0 b(p,q,r) * U_x(p,q,r)$$
$$k_z(p,q,r)S_x(p,q,r) - k_x(p,q,r)S_z(p,q,r) = jk_0 b(p,q,r) * U_y(p,q,r)$$
$$k_x(p,q,r)S_y(p,q,r) - k_y(p,q,r)S_x(p,q,r) = jk_0 b(p,q,r) * U_z(p,q,r)$$

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Vector Form of Maxwell's Equations in Fourier Space

$$k_y(p,q,r)U_z(p,q,r)-k_z(p,q,r)U_y(p,q,r)=jk_0a(p,q,r)*S_x(p,q,r)$$
$$k_z(p,q,r)U_x(p,q,r)-k_x(p,q,r)U_z(p,q,r)=jk_0a(p,q,r)*S_y(p,q,r)$$
$$k_x(p,q,r)U_y(p,q,r)-k_y(p,q,r)U_x(p,q,r)=jk_0a(p,q,r)*S_z(p,q,r)$$

$$\vec{k}(p,q,r)\times\vec{U}(p,q,r)=jk_0\varepsilon_{\mathrm{r}}(p,q,r)*\vec{S}(p,q,r)$$

$$k_y(p,q,r)S_z(p,q,r)-k_z(p,q,r)S_y(p,q,r)=jk_0b(p,q,r)*U_x(p,q,r)$$
$$k_z(p,q,r)S_x(p,q,r)-k_x(p,q,r)S_z(p,q,r)=jk_0b(p,q,r)*U_y(p,q,r)$$
$$k_x(p,q,r)S_y(p,q,r)-k_y(p,q,r)S_x(p,q,r)=jk_0b(p,q,r)*U_z(p,q,r)$$

$$\vec{k}(p,q,r)\times\vec{S}(p,q,r)=jk_0\mu_{\mathrm{r}}(p,q,r)*\vec{U}(p,q,r)$$

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Visualizing the Plane Wave Expansion

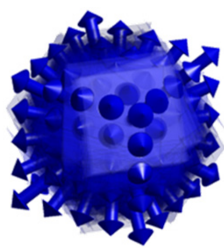
Slide 22

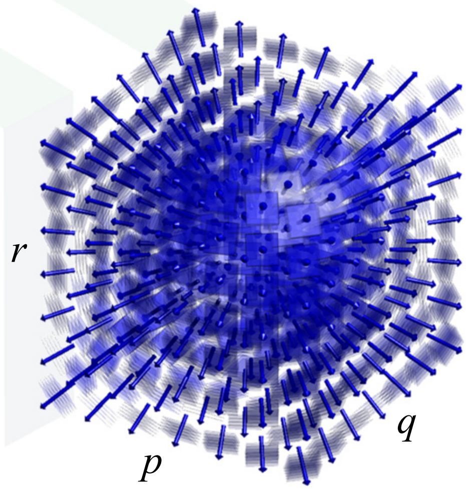
Visualizing Maxwell's Equations in Fourier Space

In real-space, the field values are known at discrete points.

In Fourier-space, amplitudes are known of discrete plane waves.

A less clear, but more accurate picture is when all the plane waves overlap.





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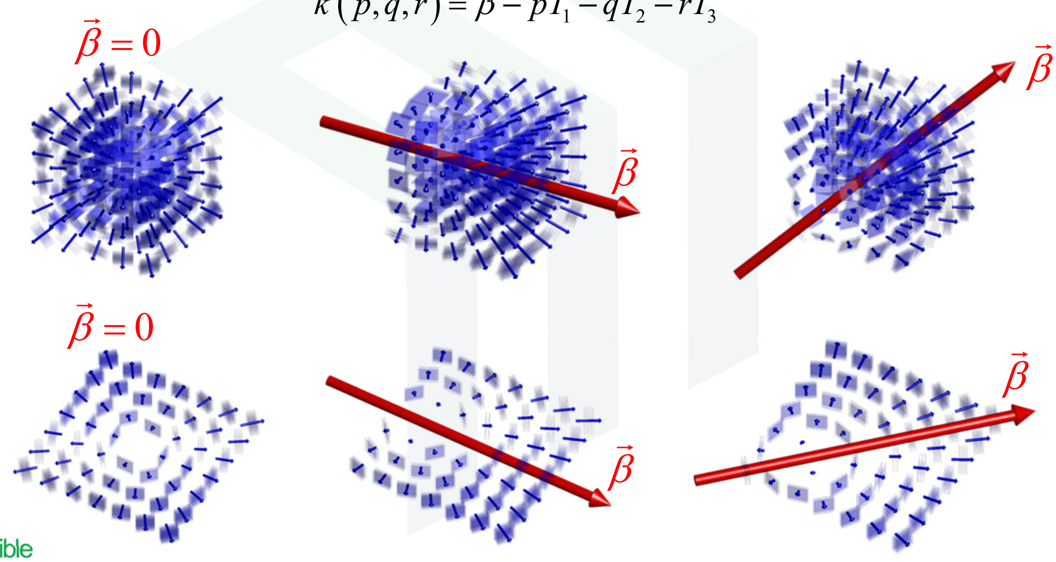
Visualizing Expansions for Cubic Symmetry

$$\vec{k}\left(p,q,r\right)=\vec{\beta}-p\vec{T}_1-q\vec{T}_2-r\vec{T}_3$$



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Visualizing Expansions with Different Symmetries

$$\vec{k}(p,q,r) = \vec{\beta} - p\vec{T}_1 - q\vec{T}_2 - r\vec{T}_3$$