



Electromagnetics:  
Electromagnetic Field Theory

# Scattering From an Interface: Normal Incidence

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## Lecture Outline

- Review of Electromagnetic Boundary Conditions
- Problem at an Interface
- Reflection and Transmission
- Examples

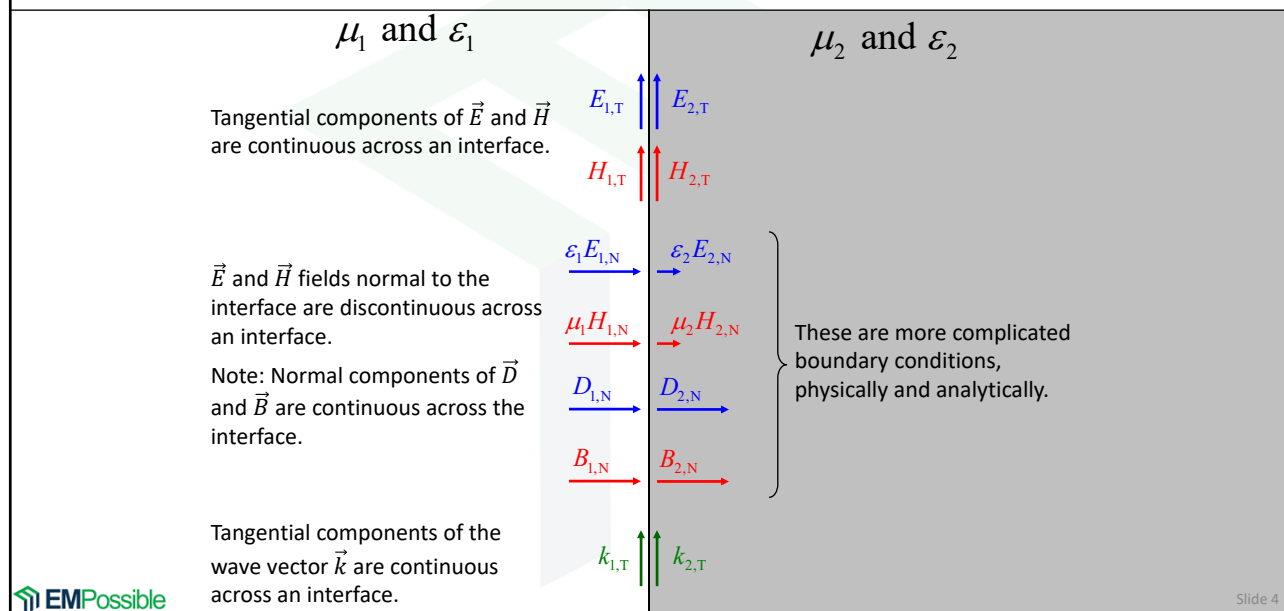
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# Review of Electromagnetic Boundary Conditions

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## Physical Boundary Conditions



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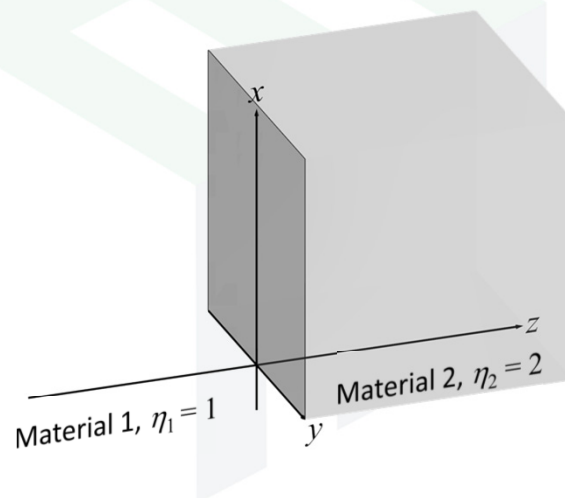
# Problem at an Interface

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## Problem at an Interface (1 of 7)

Start with a planar interface between two semi-infinite half-spaces. Each medium has a different impedance  $\eta$ .



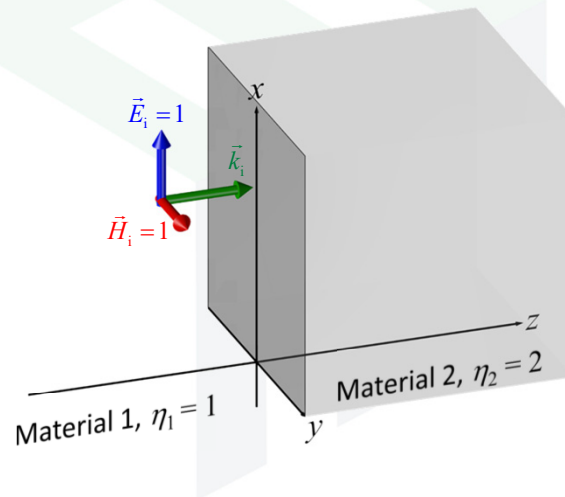
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## Problem at an Interface (2 of 7)

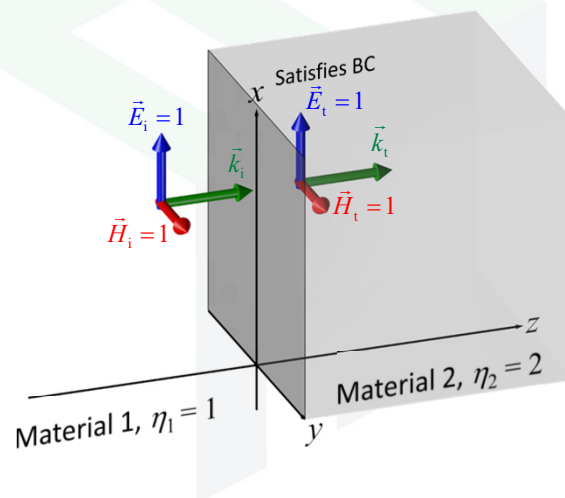
A wave is incident onto the interface at normal incidence.



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## Problem at an Interface (3 of 7)

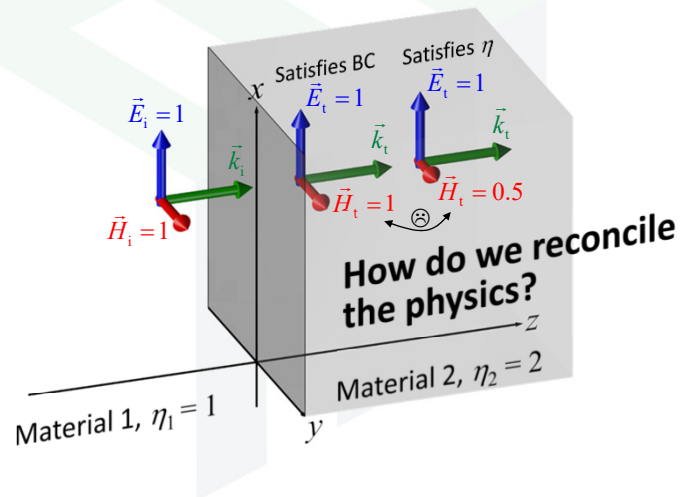
Boundary conditions require the tangential components of  $\vec{E}$  and  $\vec{H}$  to be continuous across the interface.



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## Problem at an Interface (4 of 7)

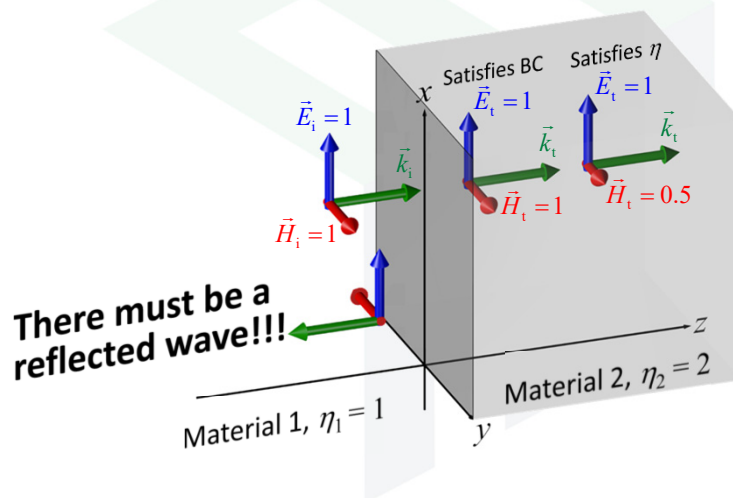
The impedance of Medium 2 requires that  $E_t/H_t = 2$ .



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## Problem at an Interface (5 of 7)

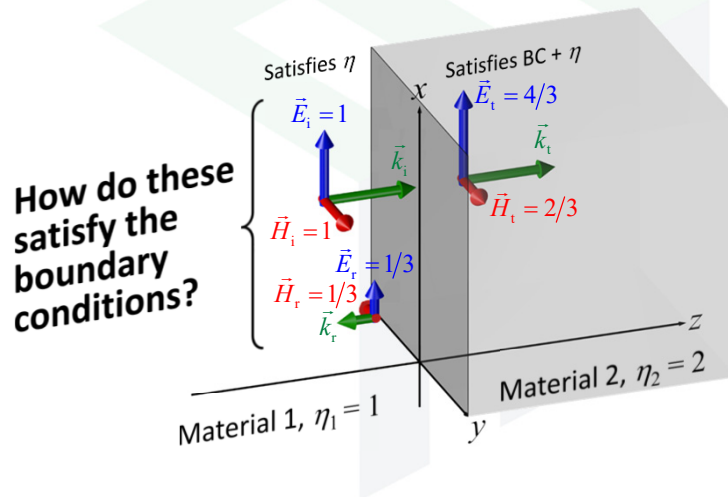
There must be a reflected wave in Medium 1 in order to reconcile the physics in Medium 2.



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## Problem at an Interface (6 of 7)

Express the problem and solve simultaneously for the transmitted and reflected wave.



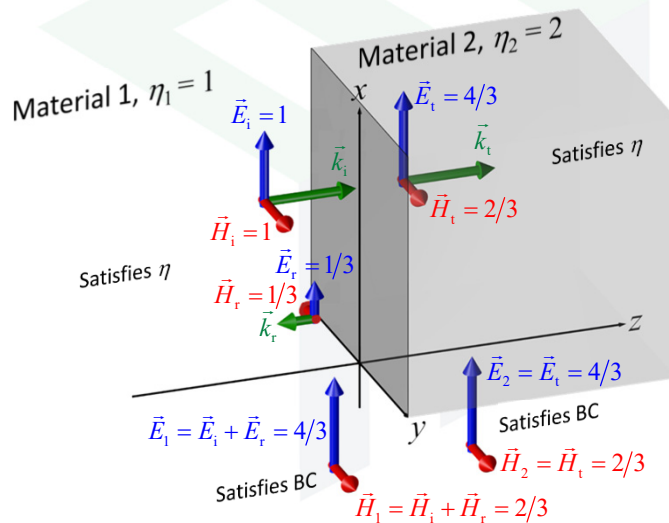
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## Problem at an Interface (7 of 7)

It is the total electric and magnetic fields that must be continuous across the interface. The individual waves must satisfy impedance.



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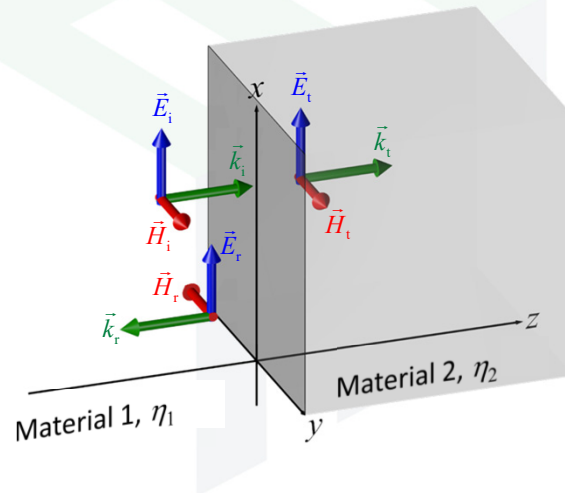
## Notes About the Interface

- Some of an incident wave may reflect from the interface.
- Some of an incident wave may transmit through the interface.
- The angle of transmitted wave may be different than the incident wave.

## Plane Wave at Normal Incidence

## Problem Setup

In general, there is an incident wave, a reflected wave, and a transmitted wave.



## Deriving Reflection & Transmission Coefficients

Step 1 – Write expression for the applied wave, reflected wave, and transmitted wave.

Step 2 – Enforce the boundary conditions at the interface.

Step 3 – Solve boundary condition equations for  $r$  and  $t$ .

Step 4 – Derive a relationship between  $r$  and  $t$ .

Step 5 – Inspect final equations and make conclusions.

## General Expression for the Incident Wave

Without loss of generality, the incident wave can be linearly polarized along the  $x$  direction.

$$\vec{E}_i(z) = E_{0,i} e^{-\gamma_1 z} \hat{a}_x$$

$$\vec{H}_i(z) = \frac{E_{0,i}}{\eta_1} e^{-\gamma_1 z} \hat{a}_y$$

## General Expression for the Reflected Wave

Assuming the interface is perfectly flat and both media are isotropic, the reflected wave will have the same polarization as the incident wave.

Further, it is travelling in the same medium as the incident wave so it has the same propagation constant, but opposite sign because it is travelling backwards.

$$\vec{E}_r(z) = E_{0,r} e^{\gamma_1 z} \hat{a}_x$$

$$\vec{H}_r(z) = -\frac{E_{0,r}}{\eta_1} e^{\gamma_1 z} \hat{a}_y$$

This sign enforces the handedness of the problem.

$\vec{E} \times \vec{H}$  in direction of wave

## General Expression for the Transmitted Wave

Assuming the interface is perfectly flat and both mediums are isotropic, the transmitted wave will have the same polarization as the incident wave.

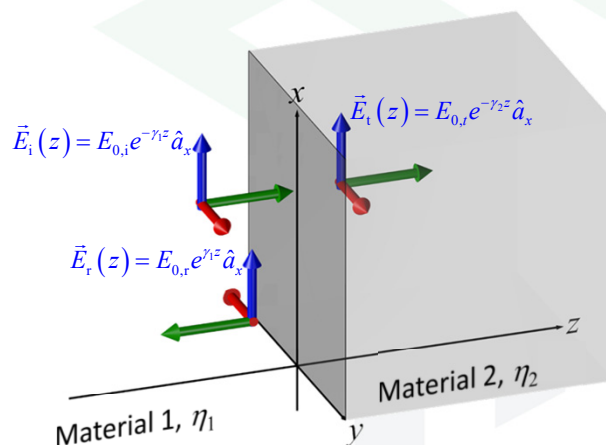
However, it is travelling in a different medium as the incident wave so it will have a different propagation constant.

$$\vec{E}_t(z) = E_{0,t} e^{-\gamma_2 z} \hat{a}_x$$

$$\vec{H}_t(z) = \frac{E_{0,t}}{\eta_2} e^{-\gamma_2 z} \hat{a}_y$$

## Quantifying Reflection and Transmission

Reflection and transmission is quantified by relating the various field amplitudes through the reflection coefficient  $r$  and transmission coefficient  $t$ .



Definition of  
Reflection Coefficient

$$r = \frac{E_{0,r}}{E_{0,i}}$$

Definition of  
Transmission Coefficient

$$t = \frac{E_{0,t}}{E_{0,i}}$$

## Enforce the Boundary Conditions (1 of 2)

Boundary conditions require that the tangential component of the total  $\vec{E}$  field must be continuous across the interface.

$$\vec{E}_1(0) = \vec{E}_2(0)$$

The total electric field in Medium 1 is the sum of the incident  $E_i$  and reflected  $E_r$  waves.

$$\vec{E}_i(0) + \vec{E}_r(0) = \vec{E}_t(0)$$

Substitute in the general expressions for the electric field component of the incident  $E_i$ , reflected  $E_r$ , and transmitted waves  $E_t$ .

$$E_{0,i}e^{0}\hat{a}_x + E_{0,r}e^{0}\hat{a}_x = E_{0,t}e^{0}\hat{a}_x$$

$$\downarrow$$

$$E_{0,i} + E_{0,r} = E_{0,t}$$

## Enforce the Boundary Conditions (2 of 2)

Boundary conditions require that the tangential component of the total  $\vec{H}$  field must be continuous across the interface.

$$\vec{H}_1(0) = \vec{H}_2(0)$$

The total magnetic field in Medium 1 is the sum of the incident  $H_i$  and reflected  $H_r$  waves.

$$\vec{H}_i(0) + \vec{H}_r(0) = \vec{H}_t(0)$$

Substitute in the general expressions for the magnetic field component of the incident  $H_i$ , reflected  $H_r$ , and transmitted waves  $H_t$ .

$$\frac{E_{0,i}}{\eta_1}e^{0}\hat{a}_y - \frac{E_{0,r}}{\eta_1}e^{0}\hat{a}_y = \frac{E_{0,t}}{\eta_2}e^{0}\hat{a}_y$$

$$\downarrow$$

$$\frac{E_{0,i}}{\eta_1} - \frac{E_{0,r}}{\eta_1} = \frac{E_{0,t}}{\eta_2}$$

## Reflection Coefficient, $r$

$$E_{0,i} + E_{0,r} = E_{0,t} \quad \text{Eq. (1a)}$$

$$\frac{E_{0,i}}{\eta_1} - \frac{E_{0,r}}{\eta_1} = \frac{E_{0,t}}{\eta_2} \quad \text{Eq. (1b)}$$

Substitute Eq. (1a) into Eq. (1b) to eliminate  $E_{0,t}$ .

$$\frac{E_{0,i}}{\eta_1} - \frac{E_{0,r}}{\eta_1} = \frac{E_{0,i} + E_{0,r}}{\eta_2} \quad \text{Eq. (2)}$$

Solve Eq. (2) for  $E_{0,r}/E_{0,i}$  because this is the definition of the reflection coefficient  $r$ .

$$\frac{1}{\eta_1} E_{0,i} - \frac{1}{\eta_1} E_{0,r} = \frac{1}{\eta_2} E_{0,i} + \frac{1}{\eta_2} E_{0,r}$$

$$\left( \frac{1}{\eta_1} + \frac{1}{\eta_2} \right) E_{0,r} = \left( \frac{1}{\eta_1} - \frac{1}{\eta_2} \right) E_{0,i}$$

$$(\eta_2 + \eta_1) E_{0,r} = (\eta_2 - \eta_1) E_{0,i}$$

$$r = \frac{E_{0,r}}{E_{0,i}} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$$

## Transmission Coefficient, $t$

Solve the new equation for  $r$  for  $E_{0,r}$ .

$$\frac{E_{0,r}}{E_{0,i}} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} \rightarrow E_{0,r} = E_{0,i} \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} \quad \text{Eq. (3)}$$

Substitute Eq. (3) into Eq. (1a) and solve for  $E_{0,t}/E_{0,i}$  because this is the definition of the transmission coefficient  $t$ .

$$E_{0,t} = E_{0,i} + E_{0,r}$$

$$E_{0,t} = E_{0,i} + E_{0,i} \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \left( 1 + \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} \right) E_{0,i}$$

$$t = \frac{E_{0,t}}{E_{0,i}} = 1 + \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{\eta_2 + \eta_1}{\eta_2 + \eta_1} + \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$$

$$t = \frac{E_{0,t}}{E_{0,i}} = \frac{2\eta_2}{\eta_2 + \eta_1}$$

## Relation Between $r$ and $t$

Divide Eq. (1a) by  $E_{0,i}$ .

$$E_{0,i} + E_{0,r} = E_{0,t} \quad \text{Eq. (1a)}$$

$$\frac{E_{0,i}}{E_{0,i}} + \frac{E_{0,r}}{E_{0,i}} = \frac{E_{0,t}}{E_{0,i}}$$

This equals 1

This is the definition of  $t$

This is the definition of  $r$

$$\boxed{1 + r = t}$$

CAUTION: This equation looks like conservation of power, but it is not!  $r$  and  $t$  are associated with field amplitudes, not power quantities.

## $r$ & $t$ in Terms of Refractive Index (1 of 2)

When materials are specified via the refractive index, there is almost always a built-in assumption that the material has no magnetic response.

$$\mu_r = 1.0$$

Impedance in terms of refractive is then

$$\eta = \sqrt{\frac{\mu}{\epsilon}} = \sqrt{\frac{\mu_0 \mu_r}{\epsilon_0 \epsilon_r}} = \sqrt{\frac{\mu_0}{\epsilon_0}} \sqrt{\frac{\mu_r}{\epsilon_r}} = \eta_0 \sqrt{\frac{\mu_r}{\epsilon_r}} = \frac{\eta_0}{\sqrt{\epsilon_r}} = \frac{\eta_0}{n}$$

The reflection coefficient  $r$  in terms of refractive index  $n$  is then

$$r = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{\frac{\eta_0}{n_2} - \frac{\eta_0}{n_1}}{\frac{\eta_0}{n_2} + \frac{\eta_0}{n_1}} = \frac{\frac{1}{n_2} - \frac{1}{n_1}}{\frac{1}{n_2} + \frac{1}{n_1}} = \frac{n_1 - n_2}{n_1 + n_2}$$

$$\boxed{r = \frac{n_1 - n_2}{n_1 + n_2}}$$

## $r$ & $t$ in Terms of Refractive Index (2 of 2)

The transmission coefficient  $t$  in terms of refractive index  $n$  is then

$$t = \frac{2\eta_2}{\eta_2 + \eta_1} = \frac{2\frac{\eta_0}{n_2}}{\frac{\eta_0}{n_2} + \frac{\eta_0}{n_1}} = \frac{\frac{2}{n_2}}{\frac{1}{n_2} + \frac{1}{n_1}} = \frac{2n_1}{n_1 + n_2}$$

$$t = \frac{2n_1}{n_1 + n_2}$$

## Notes About Scattering at Normal Incidence

- Many times  $\Gamma$  is used synonymously with  $r$ . In this course, use of  $\Gamma$  is reserved for reflection in transmission lines.
- The subtraction operation in the expression for reflection coefficient means that  $r$  can be positive or negative.  
If  $\eta_1 > \eta_2$ , the reflected wave will experience a  $180^\circ$  phase shift.
- $1 + r = t$  (This is not conservation of power)
- Both  $r$  and  $t$  are dimensionless and may be complex. They are complex because both the phase and amplitude can be affected at an interface.
- $0 \leq |r| \leq 1$  and  $0 \leq |t| \leq 1$

# Examples

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## Example #1 – Reflection and Transmission from Ceramic (1 of 2)

What is the reflection coefficient  $r$  and transmission coefficient  $t$  of a 10 GHz electromagnetic wave incident from water onto a ceramic material with dielectric constant 2.25?

### Solution

Before reflection and transmission can be calculated, the impedance of both mediums is needed. The first medium is water, which has a refractive index around 1.33.

$$\eta_1 = \frac{\eta_0}{n_1} = \frac{376.73 \Omega}{1.33} = 283.26 \Omega$$

The second medium is specified as  $\epsilon_r = 2.25$ . Since the permeability was not specified, it will be assumed to be  $\mu_r = 1.0$ .

$$\eta_2 = \sqrt{\frac{\mu_0 \mu_r}{\epsilon_0 \epsilon_r}} = \eta_0 \sqrt{\frac{\mu_r}{\epsilon_r}} = (376.73 \Omega) \sqrt{\frac{1.0}{2.25}} = 251.17 \Omega$$



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## Example #1 – Reflection and Transmission from Ceramic (2 of 2)

What is the reflection coefficient  $r$  and transmission coefficient  $t$  of a 10 GHz electromagnetic wave incident from water onto a ceramic material with dielectric constant 2.25?

### Solution cont'd

The reflection coefficient  $r$  is

$$r = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{(251.17 \Omega) - (283.26 \Omega)}{(251.17 \Omega) + (283.26 \Omega)} = \boxed{-0.06}$$

The transmission coefficient  $t$  is

$$t = \frac{2\eta_2}{\eta_2 + \eta_1} = \frac{2(251.17 \Omega)}{(251.17 \Omega) + (283.26 \Omega)} = \boxed{0.94}$$

## Example #2 – Reflection of Light from Glass

What is the reflection coefficient for light reflecting off of glass?

### Solution

Since this problem deals with optics, it will be assumed the magnetic response (i.e.  $\mu_r = 1.0$ ) is negligible and the problem will be solved in terms of refractive index  $n$ .

$$r = \frac{n_1 - n_2}{n_1 + n_2}$$

The problem implies light reflects from air to glass so

$$n_1 \approx 1.0 \quad \text{and} \quad n_2 \approx 1.5$$

The reflection coefficient  $r$  is then

$$r = \frac{1.0 - 1.5}{1.0 + 1.5} = \frac{-0.5}{2.5} = \boxed{-0.2}$$