Electromagnetics:
Electromagnetic Field Theory

Scattering From an Interface: Normal Incidence

Lecture Outline

- Review of Electromagnetic Boundary Conditions
- Problem at an Interface
- Reflection and Transmission
- Examples

Review of Electromagnetic Boundary Conditions

Physical Boundary Conditions

μ_1 and ε_1

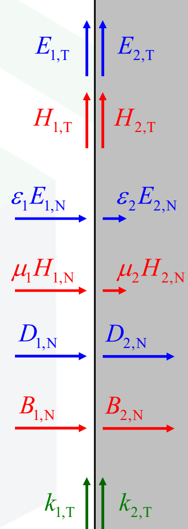
μ_2 and ε_2

Tangential components of E and H are continuous across an interface.

E and H fields normal to the interface are discontinuous across an interface.

Note: Normal components of D and B are continuous across the interface.

Tangential components of the wave vector are continuous across an interface.



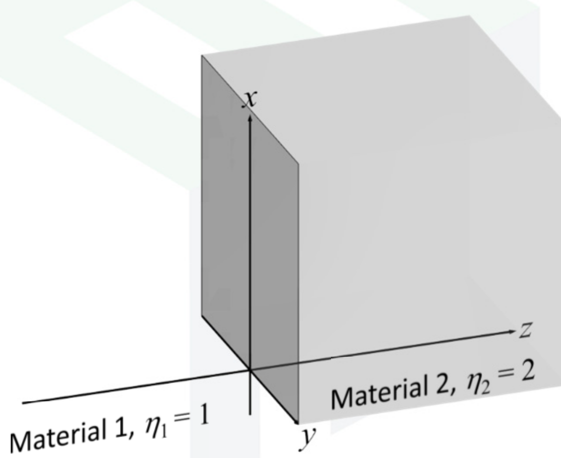
These are more complicated boundary conditions, physically and analytically.

EMPossible

Problem at an Interface

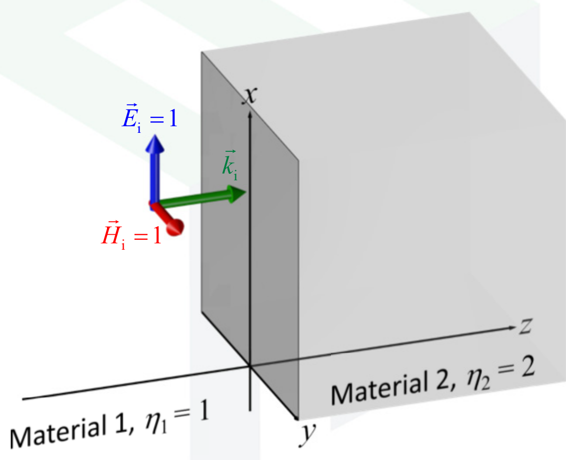
Problem at an Interface (1 of 7)

We start with a planar interface between two semi-infinite half-spaces. Each medium has a different impedance.



Problem at an Interface (2 of 7)

A wave is incident onto the interface at normal incidence.

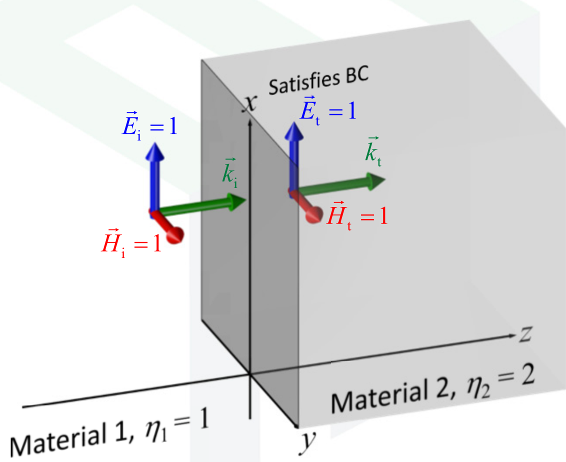


EMPossible

Slide 7

Problem at an Interface (3 of 7)

Boundary conditions require the tangential components of E and H to be continuous across the interface.

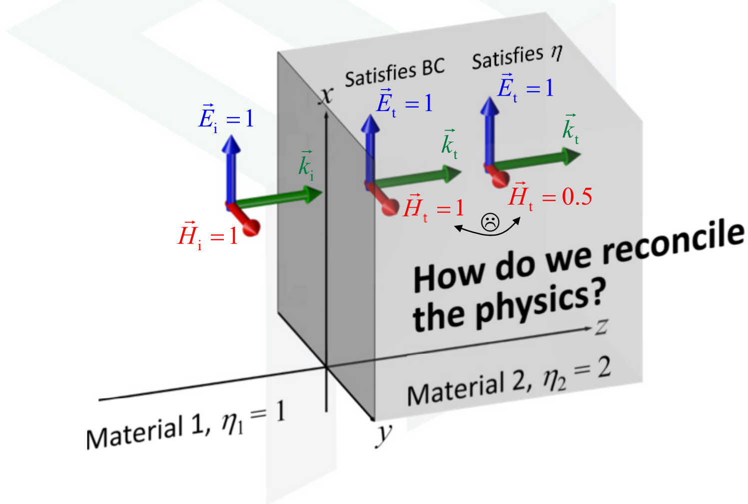


EMPossible

Slide 8

Problem at an Interface (4 of 7)

The impedance of Medium 2 requires that $E_t/H_t = 2$.

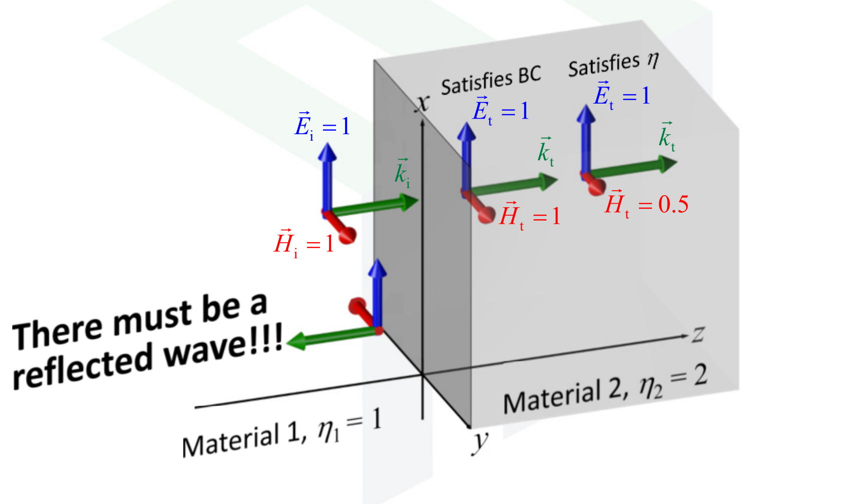


EMPossible

Slide 9

Problem at an Interface (5 of 7)

There must be a reflected wave in Medium 1 in order to reconcile the physics in Medium 2.

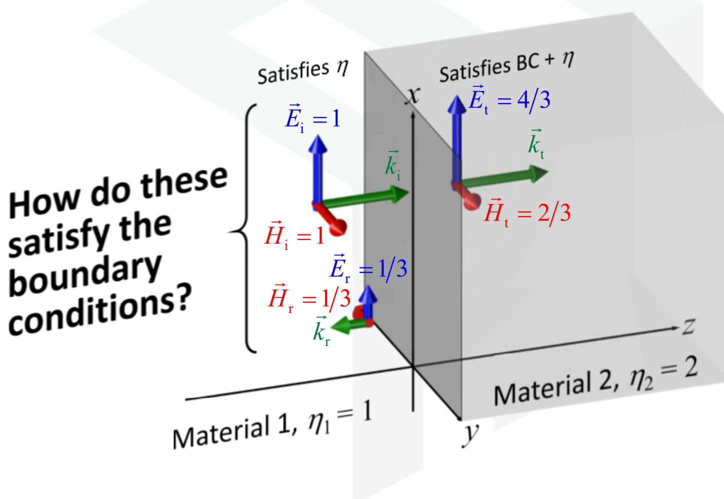


EMPossible

Slide 10

Problem at an Interface (6 of 7)

We express the problem and solve simultaneously for the transmitted and reflected wave.

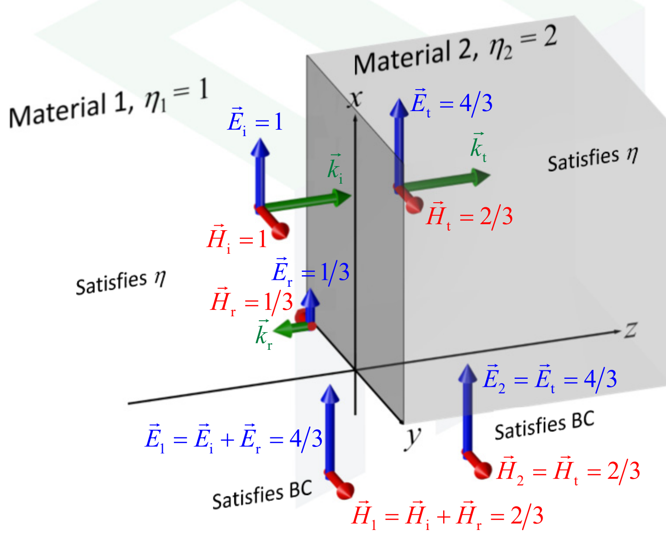


EMPossible

Slide 11

Problem at an Interface (7 of 7)

It is the total electric and magnetic fields that must be continuous across the interface.
The individual waves must satisfy impedance.



EMPossible

Slide 12

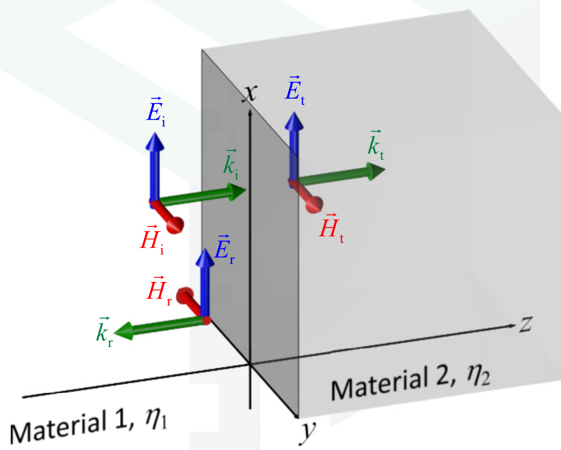
Notes About the Interface

- Some of an incident wave may reflect from the interface.
- Some of an incident wave may transmit through the interface.
- While not discussed here, some of an incident wave may be absorbed at the interface.
- The angle of transmitted wave may be different than the incident wave.

Plane Wave at Normal Incidence

Problem Setup

In general, we have an incident wave, a reflected wave, and a transmitted wave.



EMPossible

Slide 15

Deriving Reflection & Transmission Coefficients

Step 1 – Write expression for the applied wave, reflected wave, and transmitted wave.

Step 2 – Enforce the boundary conditions at the interface.

Step 3 – Solve boundary condition equations for r and t.

Step 4 – Derive a relationship between r and t.

Step 5 – Inspect final equations and make conclusions.

EMPossible

Slide 16

General Expression for the Incident Wave

Without loss of generality, we can let the incident wave be linearly polarized along the x direction.

$$\vec{E}_{\mathrm{i}}(z) = E_{0,\mathrm{i}} e^{-\gamma_1 z} \hat{a}_x$$

$$\vec{H}_{\mathrm{i}}(z) = \frac{E_{0,\mathrm{i}}}{\eta_1} e^{-\gamma_1 z} \hat{a}_y$$

EMPossible

Slide 17

General Expression for the Reflected Wave

Assuming the interface is perfectly flat and both mediums are isotropic, the reflected wave will have the same polarization as the incident wave.

Further, it is travelling in the same medium as the incident wave so it has the same propagation constant, but opposite sign because it is travelling backwards.

$$\vec{E}_{\mathrm{r}}(z) = E_{0,\mathrm{r}} e^{\gamma_1 z} \hat{a}_x$$

$$\vec{H}_{\mathrm{r}}(z) = -\frac{E_{0,\mathrm{r}}}{\eta_1} e^{\gamma_1 z} \hat{a}_y$$

This sign enforces the handedness of the problem.

$\vec{E} \times \vec{H}$ in direction of wave

EMPossible

Slide 18

General Expression for the Transmitted Wave

Assuming the interface is perfectly flat and both mediums are isotropic, the transmitted wave will have the same polarization as the incident wave.

However, it is travelling in a different medium as the incident wave so it will have a different propagation constant.

$$\vec{E}_{\text{t}}(z) = E_{0,\text{t}} e^{-\gamma_2 z} \hat{a}_x$$

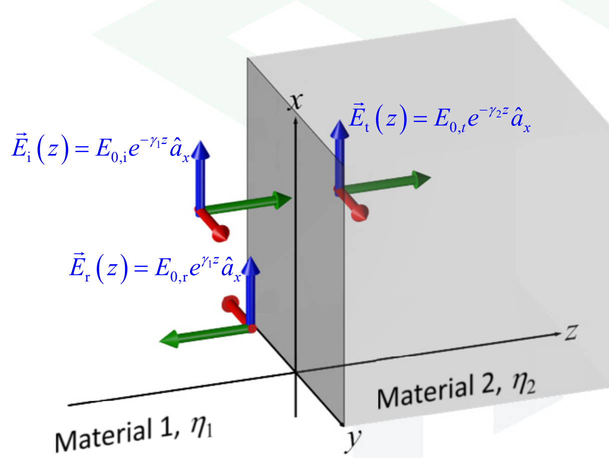
$$\vec{H}_{\text{t}}(z) = \frac{E_{0,\text{t}}}{\eta_2} e^{-\gamma_2 z} \hat{a}_y$$

EMPossible

Slide 19

Quantifying Reflection and Transmission

We quantify reflection and transmission by relating the various field amplitudes through the reflection coefficient r and transmission coefficient t.



Definition of Reflection Coefficient

$$r = \frac{E_{0,\text{r}}}{E_{0,\text{i}}}$$

Definition of Transmission Coefficient

$$t = \frac{E_{0,\text{t}}}{E_{0,\text{i}}}$$

EMPossible

Slide 20

Enforce the Boundary Conditions (1 of 2)

Boundary conditions require that the tangential component of the total E field must be continuous across the interface.

$$\vec{E}_1(0) = \vec{E}_2(0)$$

The total electric field in Medium 1 is the sum of the incident E_i and reflected E_r waves.

$$\vec{E}_\text{i}(0) + \vec{E}_\text{r}(0) = \vec{E}_\text{t}(0)$$

Substitute in our general expressions for the electric field component of the incident E_i, reflected E_r, and transmitted waves E_t.

$$E_{0,\text{i}} e^0 \hat{a}_x + E_{0,\text{r}} e^0 \hat{a}_x = E_{0,\text{t}} e^0 \hat{a}_x$$

$$\downarrow$$

$$E_{0,\text{i}} + E_{0,\text{r}} = E_{0,\text{t}}$$

EMPossible

Slide 21

Enforce the Boundary Conditions (2 of 2)

Boundary conditions require that the tangential component of the total H field must be continuous across the interface.

$$\vec{H}_1(0) = \vec{H}_2(0)$$

The total magnetic field in Medium 1 is the sum of the incident H_i and reflected H_r waves.

$$\vec{H}_\text{i}(0) + \vec{H}_\text{r}(0) = \vec{H}_\text{t}(0)$$

Substitute in our general expressions for the magnetic field component of the incident H_i, reflected H_r, and transmitted waves H_t.

$$\frac{E_{0,\text{i}}}{\eta_1} e^0 \hat{a}_y - \frac{E_{0,\text{r}}}{\eta_1} e^0 \hat{a}_y = \frac{E_{0,\text{t}}}{\eta_2} e^0 \hat{a}_y$$

$$\downarrow$$

$$\frac{E_{0,\text{i}}}{\eta_1} - \frac{E_{0,\text{r}}}{\eta_1} = \frac{E_{0,\text{t}}}{\eta_2}$$

EMPossible

Slide 22

Reflection Coefficient, r

$$E_{0,\mathrm{i}} + E_{0,\mathrm{r}} = E_{0,\mathrm{t}} \qquad \text{Eq. (1a)}$$

$$\frac{E_{0,\mathrm{i}}}{\eta_1} - \frac{E_{0,\mathrm{r}}}{\eta_1} = \frac{E_{0,\mathrm{t}}}{\eta_2} \qquad \text{Eq. (1b)}$$

Substitute Eq. (1a) into Eq. (1b) to eliminate $E_{0,\mathrm{t}}$.

$$\frac{E_{0,\mathrm{i}}}{\eta_1} - \frac{E_{0,\mathrm{r}}}{\eta_1} = \frac{E_{0,\mathrm{i}} + E_{0,\mathrm{r}}}{\eta_2} \qquad \text{Eq. (2)}$$

Solve Eq. (2) for $E_{0,\mathrm{r}}/E_{0,\mathrm{i}}$ because this is our definition of the reflection coefficient r.

$$\frac{1}{\eta_1}E_{0,\mathrm{i}} - \frac{1}{\eta_1}E_{0,\mathrm{r}} = \frac{1}{\eta_2}E_{0,\mathrm{i}} + \frac{1}{\eta_2}E_{0,\mathrm{r}}$$

$$\left(\frac{1}{\eta_1} + \frac{1}{\eta_2}\right)E_{0,\mathrm{r}} = \left(\frac{1}{\eta_1} - \frac{1}{\eta_2}\right)E_{0,\mathrm{i}}$$

$$\left(\eta_2 + \eta_1\right)E_{0,\mathrm{r}} = \left(\eta_2 - \eta_1\right)E_{0,\mathrm{i}}$$

$$r = \frac{E_{0,\mathrm{r}}}{E_{0,\mathrm{i}}} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$$

EMPossible

Slide 23

Transmission Coefficient, t

Solve our new equation for r for $E_{0,\mathrm{r}}$.

$$\frac{E_{0,\mathrm{r}}}{E_{0,\mathrm{i}}} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} \quad \rightarrow \quad E_{0,\mathrm{r}} = E_{0,\mathrm{i}}\frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} \qquad \text{Eq. (3)}$$

Substitute Eq. (3) into Eq. (1a) and solve for $E_{0,\mathrm{t}}/E_{0,\mathrm{i}}$ because this is our definition of the transmission coefficient t.

$$E_{0,\mathrm{t}} = E_{0,\mathrm{i}} + E_{0,\mathrm{r}}$$

$$E_{0,\mathrm{t}} = E_{0,\mathrm{i}} + E_{0,\mathrm{i}}\frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \left(1 + \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}\right)E_{0,\mathrm{i}}$$

$$t = \frac{E_{0,\mathrm{t}}}{E_{0,\mathrm{i}}} = 1 + \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{\eta_2 + \eta_1}{\eta_2 + \eta_1} + \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$$

$$t = \frac{E_{0,\mathrm{t}}}{E_{0,\mathrm{i}}} = \frac{2\eta_2}{\eta_2 + \eta_1}$$

EMPossible

Slide 24

Relation Between r and t

Divide Eq. (1a) by $E_{0,\mathrm{i}}$.

$$E_{0,\mathrm{i}} + E_{0,\mathrm{r}} = E_{0,\mathrm{t}} \qquad \text{Eq. (1a)}$$

$$\frac{E_{0,\mathrm{i}}}{E_{0,\mathrm{i}}} + \frac{E_{0,\mathrm{r}}}{E_{0,\mathrm{i}}} = \frac{E_{0,\mathrm{t}}}{E_{0,\mathrm{i}}}$$

This equals 1

This is our definition of t

This is our definition of r

$$\boxed{1 + r = t}$$

CAUTION: This equation looks like conservation of power, but it is not! r and t are associated with field amplitudes, not power quantities.

EMPossible

Slide 25

r & t in Terms of Refractive Index (1 of 2)

When materials are specified via the refractive index, there is almost always a built-in assumption that the material has no magnetic response.

$$\mu_\mathrm{r} = 1.0$$

Impedance in terms of refractive is then

$$\eta = \sqrt{\frac{\mu}{\varepsilon}} = \sqrt{\frac{\mu_0 \mu_\mathrm{r}}{\varepsilon_0 \varepsilon_\mathrm{r}}} = \sqrt{\frac{\mu_0}{\varepsilon_0}} \sqrt{\frac{\mu_\mathrm{r}}{\varepsilon_\mathrm{r}}} = \eta_0 \sqrt{\frac{\cancel{\mu_\mathrm{r}}}{\varepsilon_\mathrm{r}}} = \frac{\eta_0}{\sqrt{\varepsilon_\mathrm{r}}} = \frac{\eta_0}{n}$$

The reflection coefficient r in terms of refractive index n is then

$$r = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{\frac{\eta_0}{n_2} - \frac{\eta_0}{n_1}}{\frac{\eta_0}{n_2} + \frac{\eta_0}{n_1}} = \frac{\frac{1}{n_2} - \frac{1}{n_1}}{\frac{1}{n_2} + \frac{1}{n_1}} = \frac{n_1 - n_2}{n_1 + n_2}$$

$$\boxed{r = \frac{n_1 - n_2}{n_1 + n_2}}$$

EMPossible

Slide 26

r & t in Terms of Refractive Index (2 of 2)

The transmission coefficient t in terms of refractive index n is then

$$t = \frac{2\eta_2}{\eta_2 + \eta_1} = \frac{2\frac{\eta_0}{n_2}}{\frac{\eta_0}{n_2} + \frac{\eta_0}{n_1}} = \frac{\frac{2}{n_2}}{\frac{1}{n_2} + \frac{1}{n_1}} = \frac{2n_1}{n_1 + n_2}$$

$$\boxed{t = \frac{2n_1}{n_1 + n_2}}$$

Notes About Scattering at Normal Incidence

- Many times Γ is used synonymously with r. We will reserve use of Γ for reflection in transmission lines.
- The subtraction operation in the expression for reflection coefficient means that r can be positive or negative. If $\eta_1 > \eta_2$, the reflected wave will experience a 180° phase shift.
- $1 + r = t$ (This is not conservation of power)
- Both r and t are dimensionless and may be complex. They are complex because both the phase and amplitude can be affected at an interface.
- $0 \leq |r| \leq 1$ and $0 \leq |t| \leq 1$

Examples

Example #1 – Reflection and Transmission from Ceramic (1 of 2)

What is the reflection coefficient r and transmission coefficient t of a 10 GHz electromagnetic wave incident from water onto a ceramic material with dielectric constant 2.25?

Solution

Before reflection and transmission can be calculated, the impedance of both mediums is needed. The first medium is water, which has a refractive index around 1.33.

$$\eta_1 = \frac{\eta_0}{n_1} = \frac{376.73\ \Omega}{1.33} = 283.26\ \Omega$$

The second medium is specified as ε_r = 2.25. Since the permeability was not specified, it will be assumed to be μ_r = 1.0.

$$\eta_2 = \sqrt{\frac{\mu_0 \mu_r}{\varepsilon_0 \varepsilon_r}} = \eta_0 \sqrt{\frac{\mu_r}{\varepsilon_r}} = (376.73\ \Omega)\sqrt{\frac{1.0}{2.25}} = 251.17\ \Omega$$

EMPossible

Example #1 – Reflection and Transmission from Ceramic (2 of 2)

What is the reflection coefficient r and transmission coefficient t of a 10 GHz electromagnetic wave incident from water onto a ceramic material with dielectric constant 2.25?

Solution cont'd

The reflection coefficient r is

$$r = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{(251.17\ \Omega) - (283.26\ \Omega)}{(251.17\ \Omega) + (283.26\ \Omega)} = \boxed{-0.06}$$

The transmission coefficient t is

$$t = \frac{2\eta_2}{\eta_2 + \eta_1} = \frac{2(251.17\ \Omega)}{(251.17\ \Omega) + (283.26\ \Omega)} = \boxed{0.94}$$

Example #2 – Reflection of Light from Glass

What is the reflection coefficient for light reflecting off of glass?

Solution

Since this problem deals with optics, it will be assumed the magnetic response (i.e. μ_r = 1.0) is negligible and the problem will be solved in terms of refractive index n.

$$r = \frac{n_1 - n_2}{n_1 + n_2}$$

The problem implies light reflects from air to glass so

$$n_1 \approx 1.0 \quad \text{and} \quad n_2 \approx 1.5$$

The reflection coefficient r is then

$$r = \frac{1.0 - 1.5}{1.0 + 1.5} = \frac{-0.5}{2.5} = \boxed{-0.2}$$