Electromagnetics:
Electromagnetic Field Theory

Scattering From an Interface: Oblique Incidence

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Lecture Outline

- Geometry of a Plane Wave at Oblique Incidence
- Boundary condition for $\vec{k}$
- Angle of reflection & refraction
- Fresnel equations
- Reflectance and Transmittance
- Example – Plot of Reflectance and Transmittance

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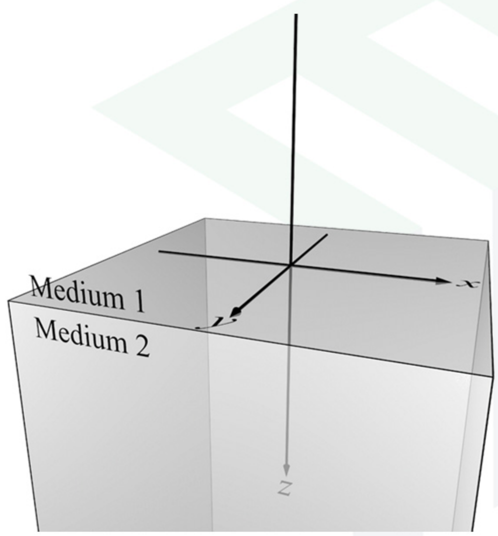
Geometry of a Plane Wave at Oblique Incidence

Geometry for Oblique Incidence (1 of 6)



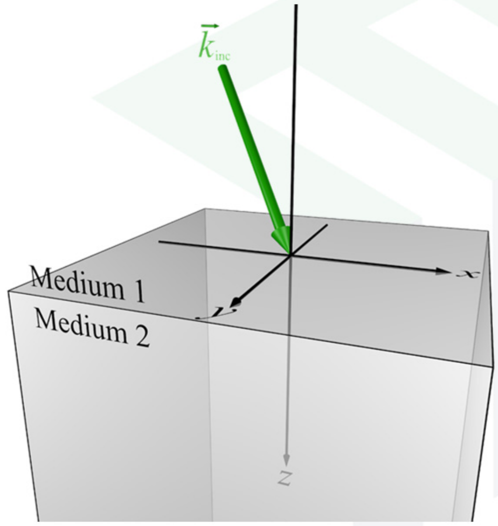
Start with a perfectly flat interface between two materials.

For mathematical convenience, let the interface lie exactly in the xy plane.

The coordinates must be drawn so they form a right-handed system.

$$\hat{a}_x \times \hat{a}_y = \hat{a}_z$$

Geometry for Oblique Incidence (2 of 6)



Let there be a wave described by the wave vector $\vec{k}_{\text{inc}}$ be incident onto the interface from above.

Recall the incident wave vector is:

$$\vec{k}_{\text{inc}} = k_x \hat{a}_x + k_y \hat{a}_y + k_z \hat{a}_z$$

$$\left|\vec{k}_{\text{inc}}\right| = \sqrt{k_x^2 + k_y^2 + k_z^2} = \frac{2\pi}{\lambda} = \omega\sqrt{\mu\varepsilon}$$

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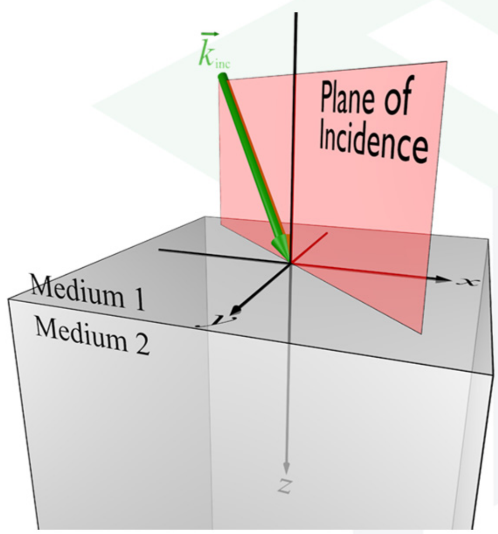
Geometry for Oblique Incidence (3 of 6)



The incident wave vector $\vec{k}_{\text{inc}}$ and the surface normal $\hat{a}_z$ define a plane. Both of these vectors lie within this plane.

This is called the *plane of incidence*.

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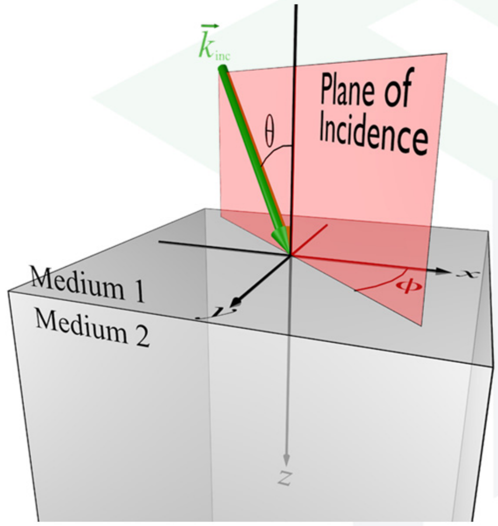
Geometry for Oblique Incidence (4 of 6)



The incident wave vector is defined by two angles, θ and ϕ.

$\theta \equiv$ elevation angle

$\phi \equiv$ azimuthal angle

The components of the incident wave vector can be calculated from these angles according to

$$\vec{k}_{\text{inc}} = k_x \hat{a}_x + k_y \hat{a}_y + k_z \hat{a}_z$$

$$k_x = k_0 n_1 \cos\phi \sin\theta$$
$$k_y = k_0 n_1 \sin\phi \sin\theta$$
$$k_z = k_0 n_1 \cos\theta$$

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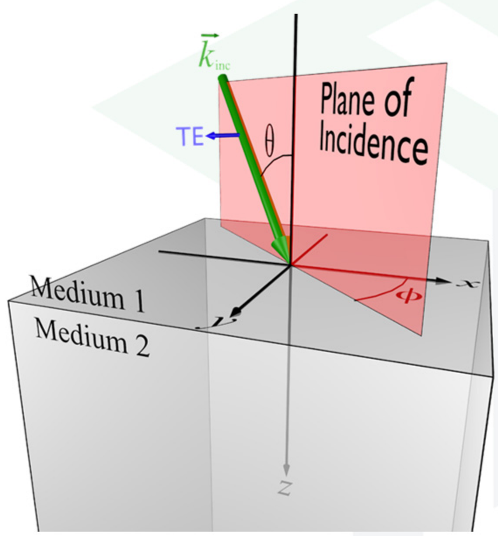
Geometry for Oblique Incidence (5 of 6)



The choice of directions for polarization becomes important when a device is involved.

The TE polarization is defined to be perpendicular to the plane of incidence.

$$\hat{a}_{\text{TE}} = \frac{\hat{a}_z \times \vec{k}_{\text{inc}}}{\left|\hat{a}_z \times \vec{k}_{\text{inc}}\right|}$$

TE polarization
s polarization
$\perp$ polarization

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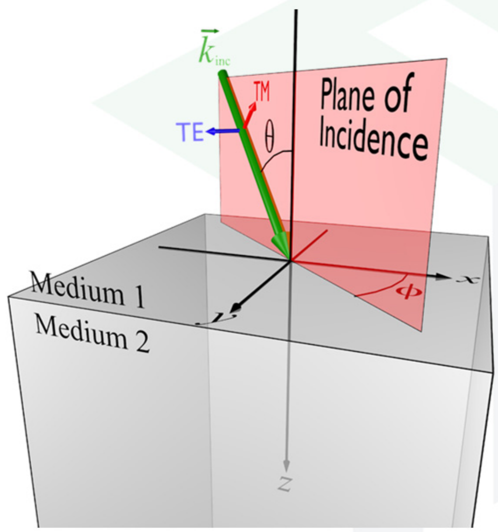
Geometry for Oblique Incidence (6 of 6)



The TM polarization is defined to be parallel to the plane of incidence.

$$\hat{a}_{\text{TM}} = \frac{\hat{a}_{\text{TE}} \times \vec{k}_{\text{inc}}}{\left|\hat{a}_{\text{TE}} \times \vec{k}_{\text{inc}}\right|}$$

TM polarization
p polarization
|| polarization

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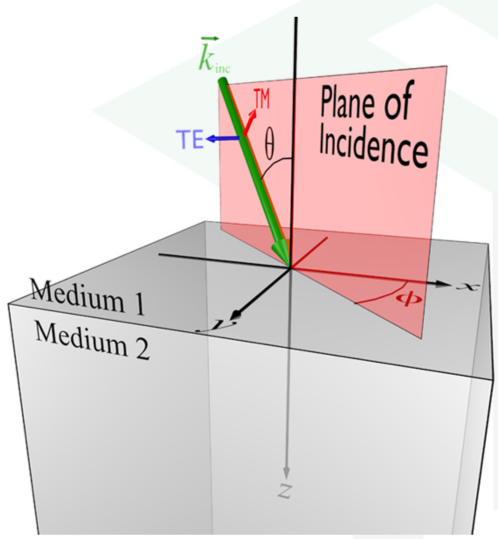
Convenient Choice for Plane of Incidence (1 of 2)



The incident wave, reflected wave, and transmitted wave all lie within the plane of incidence.

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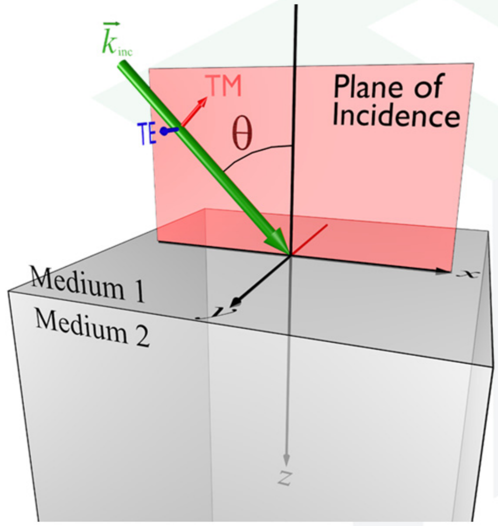
Convenient Choice for Plane of Incidence (2 of 2)



Since everything happens within the plane of incidence, the plane of incidence can be rotated to something more convenient to analyze.

Let the plane of incidence lie in the xz plane.

$$\hat{a}_{\text{TE}} \rightarrow \hat{a}_y,\ k_y \rightarrow 0$$

This rotation is valid for calculating angle of reflection, angle of refraction, and amplitude of the reflected and transmitted waves. However, vector quantities like $\vec{k}_{\text{inc}}$, $\hat{a}_{\text{TE}}$, $\hat{a}_{\text{TM}}$, and $\vec{P}$ will be different in the rotated system.

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Example #1 – TE and TM Directions

In spherical coordinates, a wave is incident at $\theta = 30°$ and $\phi = 120°$ from air into water. Calculate unit vectors in the TE and TM directions.

Solution

First, the incident wave vector is needed. Since only direction is of concern, the magnitude $k_0 n_1$ can be ignored.

$$\hat{k}_x = \cancel{k_0 n_1} \cos\phi \sin\theta = \cos(120°)\sin(30°) = -0.25$$
$$\hat{k}_y = \cancel{k_0 n_1} \sin\phi \sin\theta = \sin(120°)\sin(30°) = 0.4330$$
$$\hat{k}_z = \cancel{k_0 n_1} \cos\theta = \cos(30°) = 0.8660$$

The TE direction is

$$\hat{a}_{\text{TE}} = \frac{\hat{a}_z \times \vec{k}_{\text{inc}}}{\left|\hat{a}_z \times \vec{k}_{\text{inc}}\right|} = \frac{(0,0,1)\times(-0.25, 0.4330, 0.8660)}{\left|(0,0,1)\times(-0.25, 0.4330, 0.8660)\right|} = \boxed{(-0.8660, -0.5, 0)}$$

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Example #1 – TE and TM Directions

In spherical coordinates, a wave is incident at θ = 30° and ϕ = 120° from air into water. Calculate unit vectors in the TE and TM directions.

Solution cont'd

The TM direction is

$$\hat{a}_{\text{TM}} = \frac{\hat{a}_{\text{TE}} \times \vec{k}_{\text{inc}}}{\left|\hat{a}_{\text{TE}} \times \vec{k}_{\text{inc}}\right|} = \frac{(-0.8660,-0.5,0)\times(-0.25,0.4330,0.8660)}{\left|(-0.8660,-0.5,0)\times(-0.25,0.4330,0.8660)\right|}$$

$$= \boxed{(-0.4330,0.75,-0.5)}$$

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Boundary Condition for the Wave Vector $\vec{k}$

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Boundary Condition for $\vec{k}$ (1 of 3)

Without making any assumptions, the waves can be written as

$$\vec{E}_{\mathrm{i}}(\vec{r}) = \vec{E}_{0,\mathrm{i}} e^{-j\vec{k}_{\mathrm{i}} \bullet \vec{r}} = \vec{E}_{0,\mathrm{i}} e^{-jk_{x,\mathrm{i}}x} e^{-jk_{z,\mathrm{i}}z}$$

$$\vec{E}_{\mathrm{r}}(\vec{r}) = \vec{E}_{0,\mathrm{r}} e^{-j\vec{k}_{\mathrm{r}} \bullet \vec{r}} = \vec{E}_{0,\mathrm{r}} e^{-jk_{x,\mathrm{r}}x} e^{+jk_{z,\mathrm{r}}z}$$

$$\vec{E}_{\mathrm{t}}(\vec{r}) = \vec{E}_{0,\mathrm{t}} e^{-j\vec{k}_{\mathrm{t}} \bullet \vec{r}} = \vec{E}_{0,\mathrm{t}} e^{-jk_{x,\mathrm{t}}x} e^{-jk_{z,\mathrm{t}}z}$$

Remember $k_y = 0$ for this analysis.

However, the only thing of concern here is what is happening exactly on the interface. Therefore, let $z = 0$.

$$\vec{E}_{\mathrm{i}}(z=0) = \vec{E}_{0,\mathrm{i}} e^{-jk_{x,\mathrm{i}}x} e^{0} = \vec{E}_{0,\mathrm{i}} e^{-jk_{x,\mathrm{i}}x}$$

$$\vec{E}_{\mathrm{r}}(z=0) = \vec{E}_{0,\mathrm{r}} e^{-jk_{x,\mathrm{r}}x} e^{0} = \vec{E}_{0,\mathrm{r}} e^{-jk_{x,\mathrm{r}}x}$$

$$\vec{E}_{\mathrm{t}}(z=0) = \vec{E}_{0,\mathrm{t}} e^{-jk_{x,\mathrm{t}}x} e^{0} = \vec{E}_{0,\mathrm{t}} e^{-jk_{x,\mathrm{t}}x}$$

Boundary Condition for $\vec{k}$ (2 of 3)

Boundary conditions require the tangential components of the electric field $\vec{E}$ to be continuous across the interface.

$$E_{x,1} = E_{x,2}$$

$$E_{x,\mathrm{i}} + E_{x,\mathrm{r}} = E_{x,\mathrm{t}}$$

$$E_{x,\mathrm{i}} e^{-jk_{x,\mathrm{i}}x} + E_{x,\mathrm{r}} e^{-jk_{x,\mathrm{r}}x} = E_{x,\mathrm{t}} e^{-jk_{x,\mathrm{t}}x}$$

Boundary Condition for $\vec{k}$ (3 of 3)

$$E_{x,\mathrm{i}} e^{-jk_{x,\mathrm{i}}x} + E_{x,\mathrm{r}} e^{-jk_{x,\mathrm{r}}x} = E_{x,\mathrm{t}} e^{-jk_{x,\mathrm{t}}x}$$
$$E_{y,\mathrm{i}} e^{-jk_{x,\mathrm{i}}x} + E_{y,\mathrm{r}} e^{-jk_{x,\mathrm{r}}x} = E_{y,\mathrm{t}} e^{-jk_{x,\mathrm{t}}x}$$

The only possible way these equations can be satisfied is if

$$k_{x,\mathrm{i}} = k_{x,\mathrm{r}} = k_{x,\mathrm{t}}$$

k_x is the tangential component of the wave vector. This is generalized to any orientation of the plane of incidence as

$$\boxed{\vec{k}_{1,\tan} = \vec{k}_{2,\tan}}$$

The tangential components of $\vec{k}$ are continuous across the interface.

It can also be concluded from this that the incident wave, reflected wave, and transmitted wave all lie within the plane the incidence.

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What About $k_{z,\mathrm{r}}$ and $k_{z,\mathrm{t}}$?

The vector components of a plane wave must satisfy the dispersion relation of the medium the wave is in.

The dispersion relations for the incident, reflected, and transmitted waves are

$$\left|\vec{k}_\mathrm{i}\right|^2 = \left(k_0 n_1\right)^2 = k_{x,\mathrm{i}}^2 + k_{z,\mathrm{i}}^2$$
$$\left|\vec{k}_\mathrm{r}\right|^2 = \left(k_0 n_1\right)^2 = k_{x,\mathrm{r}}^2 + k_{z,\mathrm{r}}^2$$
$$\left|\vec{k}_\mathrm{t}\right|^2 = \left(k_0 n_2\right)^2 = k_{x,\mathrm{t}}^2 + k_{z,\mathrm{t}}^2$$

However it is known that $k_{x,\mathrm{i}} = k_{x,\mathrm{r}} = k_{x,\mathrm{t}}$ so all of these are called k_x.

$$\left|\vec{k}_\mathrm{i}\right|^2 = \left(k_0 n_1\right)^2 = k_x^2 + k_{z,\mathrm{i}}^2 \quad \rightarrow \quad k_{z,\mathrm{i}}^2 = \left(k_0 n_1\right)^2 - k_x^2$$
$$\left|\vec{k}_\mathrm{r}\right|^2 = \left(k_0 n_1\right)^2 = k_x^2 + k_{z,\mathrm{r}}^2 \quad \rightarrow \quad k_{z,\mathrm{r}}^2 = \left(k_0 n_1\right)^2 - k_x^2$$
$$\left|\vec{k}_\mathrm{t}\right|^2 = \left(k_0 n_2\right)^2 = k_x^2 + k_{z,\mathrm{t}}^2 \quad \rightarrow \quad k_{z,\mathrm{t}}^2 = \left(k_0 n_2\right)^2 - k_x^2$$

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Reflected Wave Vector Component $k_{z,\mathrm{r}}$

From the previous slide, the dispersion relation for the incident and reflected waves were

$$k_{z,\mathrm{i}}^2 = (k_0 n_1)^2 - k_x^2$$
$$k_{z,\mathrm{r}}^2 = (k_0 n_1)^2 - k_x^2$$

From these, it is observed that $k_{z,\mathrm{r}}^2 = k_{z,\mathrm{i}}^2$.

Therefore $k_{z,\mathrm{r}} = \pm k_{z,\mathrm{i}}$

This sign is resolved by recognizing that the reflected wave is propagating in the $-z$ direction.

It is concluded that

$$\boxed{k_{z,\mathrm{r}} = -k_{z,\mathrm{i}}}$$

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Example #2 – Transmitted Wave Vector

In spherical coordinates, a wave is incident at $\theta = 30°$ and $\phi = 120°$ from air into water. Calculate the wave vector of the transmitted wave if the wavelength λ_1 in air is 6.2832 m and the refractive index of water is 1.33.

Solution

The incident wave vector is

$$k_{1x} = k_0 n_1 \cos\phi \sin\theta = k \cos\phi \sin\theta = (2\pi/\lambda)\cos\phi \sin\theta$$
$$k_{1y} = k_0 n_1 \sin\phi \sin\theta = k \sin\phi \sin\theta = (2\pi/\lambda)\sin\phi \sin\theta$$
$$k_{1z} = k_0 n_1 \cos\theta = k \cos\theta = (2\pi/\lambda)\cos\theta$$

$$= (2\pi/6.2832)\cos(120°)\sin(30°) = -0.25\ \mathrm{m}^{-1}$$
$$= (2\pi/6.2832)\sin(120°)\sin(30°) = 0.4330\ \mathrm{m}^{-1}$$
$$= (2\pi/6.2832)\cos(30°) = 0.8660\ \mathrm{m}^{-1}$$

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Example #2 – Transmitted Wave Vector

In spherical coordinates, a wave is incident at $\theta = 30°$ and $\phi = 120°$ from air into water. Calculate the wave vector of the transmitted wave if the wavelength λ_1 in air is 6.2832 m and the refractive index of water is 1.33.

Solution cont'd

Tangential components of the wave vector are continuous across the interface.

$$k_{1x} = -0.25 \longrightarrow k_{2x} = k_{1x} = -0.25 \text{ m}^{-1}$$
$$k_{1y} = 0.4330 \longrightarrow k_{2y} = k_{1y} = 0.4330 \text{ m}^{-1}$$
$$k_{1z} = 0.8660$$

The longitudinal component of the wave vector is calculated from the dispersion relation.

$$k_{2x}^2 + k_{2y}^2 + k_{2z}^2 = \left|\vec{k}_2\right|^2 \quad \rightarrow k_{2z} = \sqrt{\left|\vec{k}_2\right|^2 - k_{2x}^2 - k_{2y}^2} = \sqrt{\left(\frac{2\pi}{\lambda_2}\right)^2 - k_{2x}^2 - k_{2y}^2}$$

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Example #2 – Transmitted Wave Vector

In spherical coordinates, a wave is incident at $\theta = 30°$ and $\phi = 120°$ from air into water. Calculate the wave vector of the transmitted wave if the wavelength λ_1 in air is 6.2832 m and the refractive index of water is 1.33.

Solution cont'd

The longitudinal component is

$$k_{2z} = \sqrt{\left(\frac{2\pi n}{\lambda_1}\right)^2 - k_{2x}^2 - k_{2y}^2} = \sqrt{\left(\frac{2\pi(1.33)}{6.2832}\right)^2 - (-0.25)^2 - (0.4330)^2}$$
$$= 1.2324 \text{ m}^{-1}$$

Altogether, the transmitted wave vector is

$$\boxed{\vec{k}_t = -0.25\hat{a}_x + 0.4330\hat{a}_y + 1.2324\hat{a}_z \text{ m}^{-1}}$$

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Angle of Reflection & Refraction

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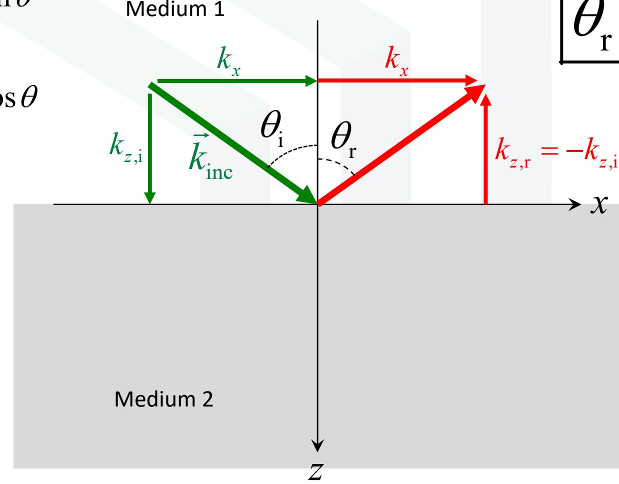
Law of Reflection

The angle of the incident and reflected wave can be related based on what is already known about the wave vector components.

$$k_x = k_0 n_1 \sin\theta$$
$$k_y = 0$$
$$k_{z,\text{i}} = k_0 n_1 \cos\theta$$



$$\boxed{\theta_\text{r} = \theta_\text{i}}$$

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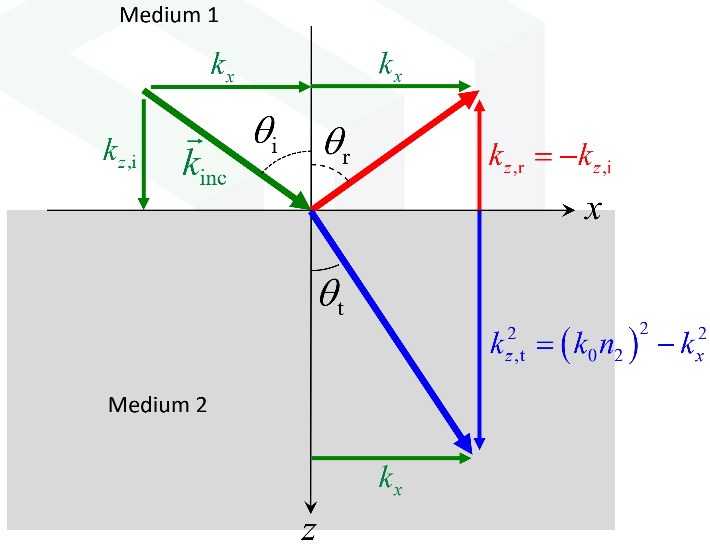
Geometry of Reflection and Refraction



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Snell's Law

Recall the dispersion relations for the incident and transmitted waves.

$$(k_0 n_1)^2 = k_x^2 + k_{z,\mathrm{i}}^2 \qquad (k_0 n_2)^2 = k_x^2 + k_{z,\mathrm{t}}^2$$

Solving both of these equations for k_x^2 gives

$$k_x^2 = (k_0 n_1)^2 - k_{z,\mathrm{i}}^2 \qquad k_x^2 = (k_0 n_2)^2 - k_{z,\mathrm{t}}^2$$

The right-hand side of these equations must be equal.

$$(k_0 n_1)^2 - k_{z,\mathrm{i}}^2 = (k_0 n_2)^2 - k_{z,\mathrm{t}}^2$$

$$(k_0 n_1 \cos\theta_{\mathrm{i}})^2 \qquad (k_0 n_2 \cos\theta_{\mathrm{t}})^2$$

$$(k_0 n_1)^2 - (k_0 n_1 \cos\theta_{\mathrm{i}})^2 = (k_0 n_2)^2 - (k_0 n_2 \cos\theta_{\mathrm{t}})^2$$

$$n_1^2 (1 - \cos^2\theta_{\mathrm{i}}) = n_2^2 (1 - \cos^2\theta_{\mathrm{t}})$$

$$n_1^2 \sin^2\theta_{\mathrm{i}} = n_2^2 \sin^2\theta_{\mathrm{t}} \quad \rightarrow \quad \boxed{n_1 \sin\theta_{\mathrm{i}} = n_2 \sin\theta_{\mathrm{t}}}$$

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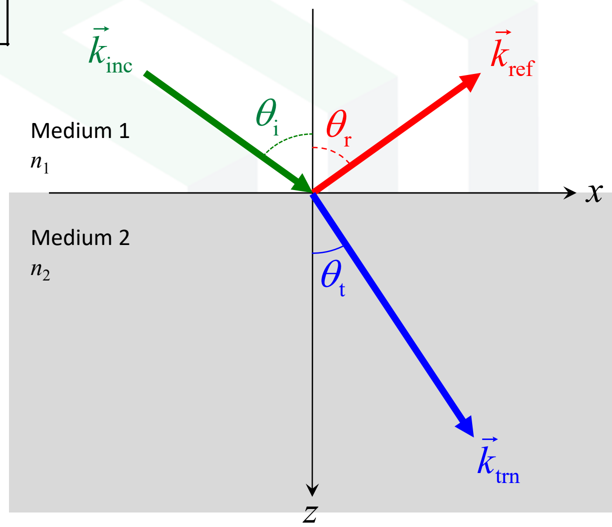
Summary of Scattering Angles

Snell's Law

$$n_1 \sin\theta_i = n_2 \sin\theta_t$$

Law of Reflection

$$\theta_r = \theta_i$$



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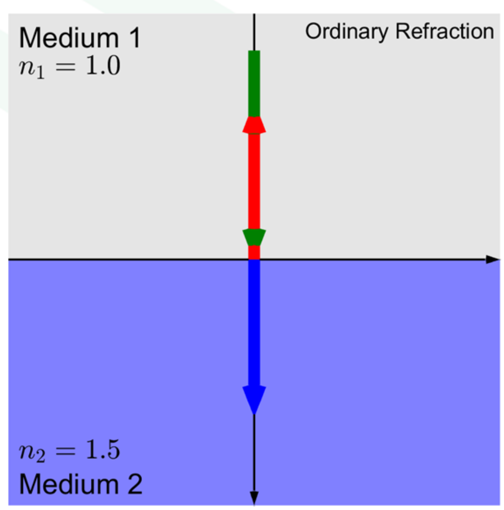
Animation of Reflection & Refraction

Law of Reflection

$$\theta_i = \theta_r$$

Snell's Law

$$n_1 \sin\theta_i = n_2 \sin\theta_t$$



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Example #3 – Scattering Angles

In spherical coordinates, a wave is incident at $\theta = 30°$ and $\phi = 120°$ from air into water. Determine the angle of incidence θ_i, angle of reflection θ_r, and angle of transmission θ_t.

Solution

The angle of incidence θ_i was given in the problem to be

$$\theta_i = \theta = \boxed{30°}$$

From the law of reflection, the angle of reflection θ_r is

$$\theta_r = \theta_i = \boxed{30°}$$

From Snell's law, the angle of transmission θ_t is

$$n_1 \sin\theta_i = n_2 \sin\theta_t \quad \rightarrow \theta_t = \sin^{-1}\left(\frac{n_1}{n_2}\sin\theta_i\right) = \sin^{-1}\left(\frac{1.0}{1.33}\sin 30°\right) = \boxed{22°}$$

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The Fresnel Equations

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Fresnel Equations

TE, s, ⊥ Polarization

$$r_{\text{TE}} = \frac{\eta_2 \cos\theta_{\text{i}} - \eta_1 \cos\theta_{\text{t}}}{\eta_2 \cos\theta_{\text{i}} + \eta_1 \cos\theta_{\text{t}}}$$

$$t_{\text{TE}} = \frac{2\eta_2 \cos\theta_{\text{i}}}{\eta_2 \cos\theta_{\text{i}} + \eta_1 \cos\theta_{\text{t}}}$$

$$1 + r_{\text{TE}} = t_{\text{TE}}$$

TM, p, || Polarization

$$r_{\text{TM}} = \frac{\eta_2 \cos\theta_{\text{t}} - \eta_1 \cos\theta_{\text{i}}}{\eta_2 \cos\theta_{\text{t}} + \eta_1 \cos\theta_{\text{i}}}$$

$$t_{\text{TM}} = \frac{2\eta_2 \cos\theta_{\text{i}}}{\eta_2 \cos\theta_{\text{t}} + \eta_1 \cos\theta_{\text{i}}}$$

$$1 + r_{\text{TM}} = \frac{\cos\theta_{\text{t}}}{\cos\theta_{\text{i}}} t_{\text{TM}}$$

Law of reflection and Snell's law gives the directions of the reflected and transmitted waves relative to the incident wave.

The Fresnel equations tell how much gets reflected and transmitted.

These equations are valid for lossy media as long as the impedances and angles are allowed to be complex numbers.

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Derivation of TE Fresnel Equations (1 of 5)

Start with the general expressions for the incident, reflected, and transmitted waves for the TE polarization.

$$\vec{E}_{\text{i}} = \left(E_{0,\text{i}}\hat{a}_y\right) e^{-jk_0 n_1\left(\sin\theta_{\text{i}} x + \cos\theta_{\text{i}} z\right)}$$

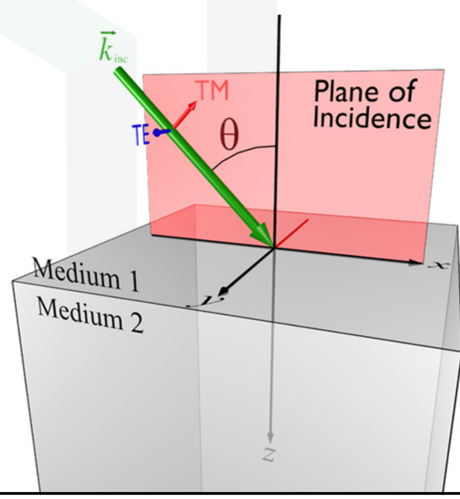
$$\vec{E}_{\text{r}} = \left(E_{0,\text{r}}\hat{a}_y\right) e^{-jk_0 n_1\left(\sin\theta_{\text{r}} x - \cos\theta_{\text{r}} z\right)}$$

$$\vec{E}_{\text{t}} = \left(E_{0,\text{t}}\hat{a}_y\right) e^{-jk_0 n_2\left(\sin\theta_{\text{t}} x + \cos\theta_{\text{t}} z\right)}$$

$$\vec{H}_{\text{i}} = \frac{E_{0,\text{i}}}{\eta_1}\left(-\cos\theta_{\text{i}}\hat{a}_x + \sin\theta_{\text{i}}\hat{a}_z\right) e^{-jk_0 n_1\left(\sin\theta_{\text{i}} x + \cos\theta_{\text{i}} z\right)}$$

$$\vec{H}_{\text{r}} = \frac{E_{0,\text{r}}}{\eta_1}\left(\cos\theta_{\text{r}}\hat{a}_x + \sin\theta_{\text{r}}\hat{a}_z\right) e^{-jk_0 n_1\left(\sin\theta_{\text{r}} x - \cos\theta_{\text{r}} z\right)}$$

$$\vec{H}_{\text{t}} = \frac{E_{0,\text{t}}}{\eta_2}\left(-\cos\theta_{\text{t}}\hat{a}_x + \sin\theta_{\text{t}}\hat{a}_z\right) e^{-jk_0 n_2\left(\sin\theta_{\text{t}} x + \cos\theta_{\text{t}} z\right)}$$



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Derivation of TE Fresnel Equations (2 of 5)

The boundary conditions at the interface are

$$\vec{E}_{\mathrm{i}}(0)\Big|_{xy} + \vec{E}_{\mathrm{r}}(0)\Big|_{xy} = \vec{E}_{\mathrm{t}}(0)\Big|_{xy}$$

$$E_{0,\mathrm{i}} e^{-jk_0 n_1 (\sin\theta_{\mathrm{i}} x)} + E_{0,\mathrm{r}} e^{-jk_0 n_1 (\sin\theta_{\mathrm{r}} x)} = E_{0,\mathrm{t}} e^{-jk_0 n_2 (\sin\theta_{\mathrm{t}} x)}$$

Snell's Law

$$E_{0,\mathrm{i}} e^{-jk_0 n_1 (\sin\theta_{\mathrm{i}} x)} + E_{0,\mathrm{r}} e^{-jk_0 n_1 (\sin\theta_{\mathrm{i}} x)} = E_{0,\mathrm{t}} e^{-jk_0 n_1 (\sin\theta_{\mathrm{i}} x)}$$

$$E_{0,\mathrm{i}} + E_{0,\mathrm{r}} = E_{0,\mathrm{t}}$$

$$\vec{H}_{\mathrm{i}}(0)\Big|_{xy} + \vec{H}_{\mathrm{r}}(0)\Big|_{xy} = \vec{H}_{\mathrm{t}}(0)\Big|_{xy}$$

$$\frac{E_{0,\mathrm{i}}}{\eta_1}(-\cos\theta_{\mathrm{i}}) e^{-jk_0 n_1 (\sin\theta_{\mathrm{i}} x)} + \frac{E_{0,\mathrm{r}}}{\eta_1}(\cos\theta_{\mathrm{r}}) e^{-jk_0 n_1 (\sin\theta_{\mathrm{r}} x)} = \frac{E_{0,\mathrm{t}}}{\eta_2}(-\cos\theta_{\mathrm{t}}) e^{-jk_0 n_2 (\sin\theta_{\mathrm{t}} x)}$$

Snell's Law

$$\frac{E_{0,\mathrm{i}}}{\eta_1}(-\cos\theta_{\mathrm{i}}) e^{-jk_0 n_1 (\sin\theta_{\mathrm{i}} x)} + \frac{E_{0,\mathrm{r}}}{\eta_1}(\cos\theta_{\mathrm{r}}) e^{-jk_0 n_1 (\sin\theta_{\mathrm{i}} x)} = \frac{E_{0,\mathrm{t}}}{\eta_2}(-\cos\theta_{\mathrm{t}}) e^{-jk_0 n_1 (\sin\theta_{\mathrm{i}} x)}$$

$$\frac{E_{0,\mathrm{i}}\cos\theta_{\mathrm{i}}}{\eta_1} - \frac{E_{0,\mathrm{r}}\cos\theta_{\mathrm{r}}}{\eta_1} = \frac{E_{0,\mathrm{t}}\cos\theta_{\mathrm{t}}}{\eta_2}$$

$$\frac{E_{0,\mathrm{i}}\cos\theta_{\mathrm{i}}}{\eta_1} - \frac{E_{0,\mathrm{r}}\cos\theta_{\mathrm{i}}}{\eta_1} = \frac{E_{0,\mathrm{t}}\cos\theta_{\mathrm{t}}}{\eta_2}$$

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Derivation of TE Fresnel Equations (3 of 5)

$$E_{0,\mathrm{i}} + E_{0,\mathrm{r}} = E_{0,\mathrm{t}} \quad \text{Eq. (1a)}$$

$$\frac{E_{0,\mathrm{i}}\cos\theta_{\mathrm{i}}}{\eta_1} - \frac{E_{0,\mathrm{r}}\cos\theta_{\mathrm{i}}}{\eta_1} = \frac{E_{0,\mathrm{t}}\cos\theta_{\mathrm{t}}}{\eta_2} \quad \text{Eq. (1b)}$$

Substituting Eq. (1a) into Eq. (1b) to eliminate $E_{0,\mathrm{t}}$ gives

$$\frac{E_{0,\mathrm{i}}\cos\theta_{\mathrm{i}}}{\eta_1} - \frac{E_{0,\mathrm{r}}\cos\theta_{\mathrm{i}}}{\eta_1} = \frac{E_{0,\mathrm{t}}\cos\theta_{\mathrm{t}}}{\eta_2}$$

$$\frac{E_{0,\mathrm{i}}\cos\theta_{\mathrm{i}}}{\eta_1} - \frac{E_{0,\mathrm{r}}\cos\theta_{\mathrm{i}}}{\eta_1} = \frac{(E_{0,\mathrm{i}} + E_{0,\mathrm{r}})\cos\theta_{\mathrm{t}}}{\eta_2}$$

$$\frac{E_{0,\mathrm{r}}\cos\theta_{\mathrm{i}}}{\eta_1} + \frac{E_{0,\mathrm{r}}\cos\theta_{\mathrm{t}}}{\eta_2} = \frac{E_{0,\mathrm{i}}\cos\theta_{\mathrm{i}}}{\eta_1} - \frac{E_{0,\mathrm{i}}\cos\theta_{\mathrm{t}}}{\eta_2}$$

$$E_{0,\mathrm{r}}\left(\frac{\cos\theta_{\mathrm{i}}}{\eta_1} + \frac{\cos\theta_{\mathrm{t}}}{\eta_2}\right) = E_{0,\mathrm{i}}\left(\frac{\cos\theta_{\mathrm{i}}}{\eta_1} - \frac{\cos\theta_{\mathrm{t}}}{\eta_2}\right)$$

$$\frac{E_{0,\mathrm{r}}}{E_{0,\mathrm{i}}} = \left(\frac{\cos\theta_{\mathrm{i}}}{\eta_1} - \frac{\cos\theta_{\mathrm{t}}}{\eta_2}\right) \Big/ \left(\frac{\cos\theta_{\mathrm{i}}}{\eta_1} + \frac{\cos\theta_{\mathrm{t}}}{\eta_2}\right)$$

$$r_{\mathrm{TE}} = \frac{E_{0,\mathrm{r}}}{E_{0,\mathrm{i}}} = \frac{\eta_2\cos\theta_{\mathrm{i}} - \eta_1\cos\theta_{\mathrm{t}}}{\eta_2\cos\theta_{\mathrm{i}} + \eta_1\cos\theta_{\mathrm{t}}}$$

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Derivation of TE Fresnel Equations (4 of 5)

$$E_{0,\mathrm{i}} + E_{0,\mathrm{r}} = E_{0,\mathrm{t}} \quad \text{Eq. (1a)}$$

$$r_{\mathrm{TE}} = \frac{E_{0,\mathrm{r}}}{E_{0,\mathrm{i}}} = \frac{\eta_2 \cos\theta_\mathrm{i} - \eta_1 \cos\theta_\mathrm{t}}{\eta_2 \cos\theta_\mathrm{i} + \eta_1 \cos\theta_\mathrm{t}}$$

Substituting the expression for r into Eq. (1a) to eliminate $E_{0,\mathrm{r}}$ gives

$$E_{0,\mathrm{i}} + E_{0,\mathrm{r}} = E_{0,\mathrm{t}}$$

$$E_{0,\mathrm{i}} + E_{0,\mathrm{i}} \frac{\eta_2 \cos\theta_\mathrm{i} - \eta_1 \cos\theta_\mathrm{t}}{\eta_2 \cos\theta_\mathrm{i} + \eta_1 \cos\theta_\mathrm{t}} = E_{0,\mathrm{t}}$$

$$E_{0,\mathrm{i}} \left(1 + \frac{\eta_2 \cos\theta_\mathrm{i} - \eta_1 \cos\theta_\mathrm{t}}{\eta_2 \cos\theta_\mathrm{i} + \eta_1 \cos\theta_\mathrm{t}} \right) = E_{0,\mathrm{t}}$$

$$\frac{E_{0,\mathrm{t}}}{E_{0,\mathrm{i}}} = 1 + \frac{\eta_2 \cos\theta_\mathrm{i} - \eta_1 \cos\theta_\mathrm{t}}{\eta_2 \cos\theta_\mathrm{i} + \eta_1 \cos\theta_\mathrm{t}}$$

$$\frac{E_{0,\mathrm{t}}}{E_{0,\mathrm{i}}} = \frac{\eta_2 \cos\theta_\mathrm{i} + \eta_1 \cos\theta_\mathrm{t}}{\eta_2 \cos\theta_\mathrm{i} + \eta_1 \cos\theta_\mathrm{t}} + \frac{\eta_2 \cos\theta_\mathrm{i} - \eta_1 \cos\theta_\mathrm{t}}{\eta_2 \cos\theta_\mathrm{i} + \eta_1 \cos\theta_\mathrm{t}}$$

$$\boxed{t_{\mathrm{TE}} = \frac{E_{0,\mathrm{t}}}{E_{0,\mathrm{i}}} = \frac{2\eta_2 \cos\theta_\mathrm{i}}{\eta_2 \cos\theta_\mathrm{i} + \eta_1 \cos\theta_\mathrm{t}}}$$

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Derivation of TE Fresnel Equations (5 of 5)

$$E_{0,\mathrm{i}} + E_{0,\mathrm{r}} = E_{0,\mathrm{t}} \quad \text{Eq. (1a)}$$

Divide Eq. (1a) by $E_{0,\mathrm{i}}$ to get

$$E_{0,\mathrm{i}} + E_{0,\mathrm{r}} = E_{0,\mathrm{t}}$$

$$\frac{E_{0,\mathrm{i}} + E_{0,\mathrm{r}}}{E_{0,\mathrm{i}}} = \frac{E_{0,\mathrm{t}}}{E_{0,\mathrm{i}}}$$

$$\frac{E_{0,\mathrm{i}}}{E_{0,\mathrm{i}}} + \frac{E_{0,\mathrm{r}}}{E_{0,\mathrm{i}}} = \frac{E_{0,\mathrm{t}}}{E_{0,\mathrm{i}}}$$

$$\boxed{1 + r_{\mathrm{TE}} = t_{\mathrm{TE}}}$$

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Derivation of TM Fresnel Equations (1 of 6)

Start with the general expressions for the incident, reflected, and transmitted waves for the TM polarization.

$$\vec{E}_{\mathrm{i}} = E_{0,\mathrm{i}}\left(\cos\theta_{\mathrm{i}}\hat{a}_x - \sin\theta_{\mathrm{i}}\hat{a}_z\right)e^{-jk_0n_1\left(\sin\theta_{\mathrm{t}}x+\cos\theta_{\mathrm{t}}z\right)}$$

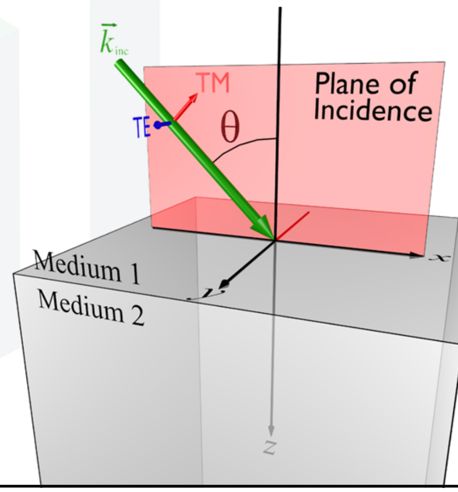
$$\vec{E}_{\mathrm{r}} = E_{0,\mathrm{r}}\left(\cos\theta_{\mathrm{r}}\hat{a}_x + \sin\theta_{\mathrm{r}}\hat{a}_z\right)e^{-jk_0n_1\left(\sin\theta_{\mathrm{t}}x-\cos\theta_{\mathrm{t}}z\right)}$$

$$\vec{E}_{\mathrm{t}} = E_{0,\mathrm{t}}\left(\cos\theta_{\mathrm{t}}\hat{a}_x - \sin\theta_{\mathrm{t}}\hat{a}_z\right)e^{-jk_0n_2\left(\sin\theta_{\mathrm{t}}x+\cos\theta_{\mathrm{t}}z\right)}$$

$$\vec{H}_{\mathrm{i}} = \left(\frac{E_{0,\mathrm{i}}}{\eta_1}\hat{a}_y\right)e^{-jk_0n_1\left(\sin\theta_{\mathrm{i}}x+\cos\theta_{\mathrm{i}}z\right)}$$

$$\vec{H}_{\mathrm{r}} = \left(-\frac{E_{0,\mathrm{r}}}{\eta_1}\hat{a}_y\right)e^{-jk_0n_1\left(\sin\theta_{\mathrm{r}}x-\cos\theta_{\mathrm{r}}z\right)}$$

$$\vec{H}_{\mathrm{t}} = \left(\frac{E_{0,\mathrm{t}}}{\eta_2}\hat{a}_y\right)e^{-jk_0n_2\left(\sin\theta_{\mathrm{t}}x+\cos\theta_{\mathrm{t}}z\right)}$$



Derivation of TM Fresnel Equations (2 of 6)

The boundary conditions for the electric field at the interface are

$$\vec{E}_{\mathrm{i}}(0)\Big|_{xy} + \vec{E}_{\mathrm{r}}(0)\Big|_{xy} = \vec{E}_{\mathrm{t}}(0)\Big|_{xy}$$

$$E_{0,\mathrm{i}}\left(\cos\theta_{\mathrm{i}}\hat{a}_x\right)e^{-jk_0n_1\left(\sin\theta_{\mathrm{t}}x\right)} + E_{0,\mathrm{r}}\left(\cos\theta_{\mathrm{r}}\hat{a}_x\right)e^{-jk_0n_1\left(\sin\theta_{\mathrm{t}}x\right)} = E_{0,\mathrm{t}}\left(\cos\theta_{\mathrm{t}}\hat{a}_x\right)e^{-jk_0n_2\left(\sin\theta_{\mathrm{t}}x\right)}$$

$$E_{0,\mathrm{i}}\left(\cos\theta_{\mathrm{i}}\hat{a}_x\right)e^{-jk_0n_1\left(\sin\theta_{\mathrm{t}}x\right)} + E_{0,\mathrm{r}}\left(\cos\theta_{\mathrm{r}}\hat{a}_x\right)e^{-jk_0n_1\left(\sin\theta_{\mathrm{t}}x\right)} = E_{0,\mathrm{t}}\left(\cos\theta_{\mathrm{t}}\hat{a}_x\right)e^{-jk_0n_1\left(\sin\theta_{\mathrm{i}}x\right)}$$

$$E_{0,\mathrm{i}}\cos\theta_{\mathrm{i}} + E_{0,\mathrm{r}}\cos\theta_{\mathrm{r}} = E_{0,\mathrm{t}}\cos\theta_{\mathrm{t}}$$

$$E_{0,\mathrm{i}}\cos\theta_{\mathrm{i}} + E_{0,\mathrm{r}}\cos\theta_{\mathrm{i}} = E_{0,\mathrm{t}}\cos\theta_{\mathrm{t}}$$

$$E_{0,\mathrm{i}} + E_{0,\mathrm{r}} = E_{0,\mathrm{t}}\frac{\cos\theta_{\mathrm{t}}}{\cos\theta_{\mathrm{i}}}$$

Derivation of TM Fresnel Equations (3 of 6)

The boundary conditions for the magnetic field at the interface are

$$\vec{H}_{\mathrm{i}}(0)\Big|_{xy}+\vec{H}_{\mathrm{r}}(0)\Big|_{xy}=\vec{H}_{\mathrm{t}}(0)\Big|_{xy}$$

$$\left(\frac{E_{0,\mathrm{i}}}{\eta_1}\hat{a}_y\right)e^{-jk_0n_1(\sin\theta_{\mathrm{i}}x)}+\left(-\frac{E_{0,\mathrm{r}}}{\eta_1}\hat{a}_y\right)e^{-jk_0n_1(\sin\theta_{\mathrm{r}}x)}=\left(\frac{E_{0,\mathrm{t}}}{\eta_2}\hat{a}_y\right)e^{-jk_0n_2(\sin\theta_{\mathrm{t}}x)}$$

$$\left(\frac{E_{0,\mathrm{i}}}{\eta_1}\hat{a}_y\right)e^{-jk_0n_1(\sin\theta_{\mathrm{i}}x)}+\left(-\frac{E_{0,\mathrm{r}}}{\eta_1}\hat{a}_y\right)e^{-jk_0n_1(\sin\theta_{\mathrm{i}}x)}=\left(\frac{E_{0,\mathrm{t}}}{\eta_2}\hat{a}_y\right)e^{-jk_0n_1(\sin\theta_{\mathrm{i}}x)}$$

$$\frac{E_{0,\mathrm{i}}}{\eta_1}-\frac{E_{0,\mathrm{r}}}{\eta_1}=\frac{E_{0,\mathrm{t}}}{\eta_2}$$

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Derivation of TM Fresnel Equations (4 of 6)

$$E_{0,\mathrm{i}}+E_{0,\mathrm{r}}=E_{0,\mathrm{t}}\frac{\cos\theta_{\mathrm{t}}}{\cos\theta_{\mathrm{i}}}\quad\text{Eq. (1a)}$$

$$\frac{E_{0,\mathrm{i}}}{\eta_1}-\frac{E_{0,\mathrm{r}}}{\eta_1}=\frac{E_{0,\mathrm{t}}}{\eta_2}\quad\text{Eq. (1b)}$$

Substituting Eq. (1a) into Eq. (1b) to eliminate $E_{0,\mathrm{t}}$ gives

$$\frac{E_{0,\mathrm{i}}}{\eta_1}-\frac{E_{0,\mathrm{r}}}{\eta_1}=\frac{E_{0,\mathrm{t}}}{\eta_2}$$

$$\frac{E_{0,\mathrm{i}}}{\eta_1}-\frac{E_{0,\mathrm{r}}}{\eta_1}=\frac{\frac{\cos\theta_{\mathrm{i}}}{\cos\theta_{\mathrm{t}}}\left(E_{0,\mathrm{i}}+E_{0,\mathrm{r}}\right)}{\eta_2}$$

$$\frac{E_{0,\mathrm{i}}}{\eta_1}-\frac{E_{0,\mathrm{r}}}{\eta_1}=\frac{E_{0,\mathrm{i}}}{\eta_2}\frac{\cos\theta_{\mathrm{i}}}{\cos\theta_{\mathrm{t}}}+\frac{E_{0,\mathrm{r}}}{\eta_2}\frac{\cos\theta_{\mathrm{i}}}{\cos\theta_{\mathrm{t}}}$$

$$\frac{E_{0,\mathrm{r}}}{\eta_2}\frac{\cos\theta_{\mathrm{i}}}{\cos\theta_{\mathrm{t}}}+\frac{E_{0,\mathrm{r}}}{\eta_1}=\frac{E_{0,\mathrm{i}}}{\eta_1}-\frac{E_{0,\mathrm{i}}}{\eta_2}\frac{\cos\theta_{\mathrm{i}}}{\cos\theta_{\mathrm{t}}}$$

$$E_{0,\mathrm{r}}\left(\frac{1}{\eta_2}\frac{\cos\theta_{\mathrm{i}}}{\cos\theta_{\mathrm{t}}}+\frac{1}{\eta_1}\right)=E_{0,\mathrm{i}}\left(\frac{1}{\eta_1}-\frac{1}{\eta_2}\frac{\cos\theta_{\mathrm{i}}}{\cos\theta_{\mathrm{t}}}\right)$$

$$\frac{E_{0,\mathrm{r}}}{E_{0,\mathrm{i}}}=\frac{\frac{1}{\eta_1}-\frac{1}{\eta_2}\frac{\cos\theta_{\mathrm{i}}}{\cos\theta_{\mathrm{t}}}}{\frac{1}{\eta_2}\frac{\cos\theta_{\mathrm{i}}}{\cos\theta_{\mathrm{t}}}+\frac{1}{\eta_1}}$$

$$\boxed{r_{\mathrm{TM}}=\frac{E_{0,\mathrm{r}}}{E_{0,\mathrm{i}}}=\frac{\eta_2\cos\theta_{\mathrm{t}}-\eta_1\cos\theta_{\mathrm{i}}}{\eta_2\cos\theta_{\mathrm{t}}+\eta_1\cos\theta_{\mathrm{i}}}}$$

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Derivation of TM Fresnel Equations (5 of 6)

$$E_{0,\mathrm{i}} + E_{0,\mathrm{r}} = E_{0,\mathrm{t}} \frac{\cos\theta_\mathrm{t}}{\cos\theta_\mathrm{i}} \quad \text{Eq. (1a)}$$

$$r_\mathrm{TM} = \frac{E_{0,\mathrm{r}}}{E_{0,\mathrm{i}}} = \frac{\eta_2 \cos\theta_\mathrm{t} - \eta_1 \cos\theta_\mathrm{i}}{\eta_2 \cos\theta_\mathrm{t} + \eta_1 \cos\theta_\mathrm{i}}$$

Substituting the expression for r into Eq. (1a) to eliminate $E_{0,\mathrm{r}}$ gives

$$E_{0,\mathrm{i}} + E_{0,\mathrm{r}} = E_{0,\mathrm{t}} \frac{\cos\theta_\mathrm{t}}{\cos\theta_\mathrm{i}}$$

$$E_{0,\mathrm{i}} + E_{0,\mathrm{i}} \frac{\eta_2 \cos\theta_\mathrm{t} - \eta_1 \cos\theta_\mathrm{i}}{\eta_2 \cos\theta_\mathrm{t} + \eta_1 \cos\theta_\mathrm{i}} = E_{0,\mathrm{t}} \frac{\cos\theta_\mathrm{t}}{\cos\theta_\mathrm{i}}$$

$$E_{0,\mathrm{i}} \left(1 + \frac{\eta_2 \cos\theta_\mathrm{t} - \eta_1 \cos\theta_\mathrm{i}}{\eta_2 \cos\theta_\mathrm{t} + \eta_1 \cos\theta_\mathrm{i}} \right) = E_{0,\mathrm{t}} \frac{\cos\theta_\mathrm{t}}{\cos\theta_\mathrm{i}}$$

$$\frac{E_{0,\mathrm{t}}}{E_{0,\mathrm{i}}} = \frac{1 + \frac{\eta_2 \cos\theta_\mathrm{t} - \eta_1 \cos\theta_\mathrm{i}}{\eta_2 \cos\theta_\mathrm{t} + \eta_1 \cos\theta_\mathrm{i}}}{\frac{\cos\theta_\mathrm{t}}{\cos\theta_\mathrm{i}}}$$

$$\boxed{t_\mathrm{TM} = \frac{E_{0,\mathrm{t}}}{E_{0,\mathrm{i}}} = \frac{2\eta_2 \cos\theta_\mathrm{i}}{\eta_2 \cos\theta_\mathrm{t} + \eta_1 \cos\theta_\mathrm{i}}}$$

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Derivation of TM Fresnel Equations (6 of 6)

$$E_{0,\mathrm{i}} + E_{0,\mathrm{r}} = E_{0,\mathrm{t}} \frac{\cos\theta_\mathrm{t}}{\cos\theta_\mathrm{i}} \quad \text{Eq. (1a)}$$

Divide Eq. (1a) by $E_{0,\mathrm{i}}$ to get

$$E_{0,\mathrm{i}} + E_{0,\mathrm{r}} = E_{0,\mathrm{t}} \frac{\cos\theta_\mathrm{t}}{\cos\theta_\mathrm{i}}$$

$$\frac{E_{0,\mathrm{i}} + E_{0,\mathrm{r}}}{E_{0,\mathrm{i}}} = \frac{E_{0,\mathrm{t}} \frac{\cos\theta_\mathrm{t}}{\cos\theta_\mathrm{i}}}{E_{0,\mathrm{i}}}$$

$$\frac{E_{0,\mathrm{i}}}{E_{0,\mathrm{i}}} + \frac{E_{0,\mathrm{r}}}{E_{0,\mathrm{i}}} = \frac{E_{0,\mathrm{t}}}{E_{0,\mathrm{i}}} \frac{\cos\theta_\mathrm{t}}{\cos\theta_\mathrm{i}}$$

$$\boxed{1 + r_\mathrm{TM} = \frac{\cos\theta_\mathrm{t}}{\cos\theta_\mathrm{i}} t_\mathrm{TM}}$$

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Example #4 – Fresnel Equations

In spherical coordinates, a wave is incident at $\theta = 30°$ and $\phi = 120°$ from air into water. Calculate the reflection and transmission coefficients for both TE and TM polarizations.

Solution

The material impedances are

$$\eta_1 = \frac{\eta_0}{n_1} = \frac{376.73\ \Omega}{1.0} = 376.73\ \Omega \qquad \eta_2 = \frac{\eta_0}{n_2} = \frac{376.73\ \Omega}{1.33} = 283.26\ \Omega$$

The scattering angles were previously found to be

$$\theta_i = \theta_r = 30° \qquad \theta_t = 22°$$

The reflection coefficient for the TE polarization is then

$$r_{TE} = \frac{\eta_2 \cos\theta_i - \eta_1 \cos\theta_t}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t} = \frac{(283.26\ \Omega)\cos 30° - (376.73\ \Omega)\cos 22°}{(283.26\ \Omega)\cos 30° + (376.73\ \Omega)\cos 22°} = \boxed{-0.1749}$$

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Example #4 – Fresnel Equations

In spherical coordinates, a wave is incident at $\theta = 30°$ and $\phi = 120°$ from air into water. Calculate the reflection and transmission coefficients for both TE and TM polarizations.

Solution cont'd

The reflection coefficient for the TM polarization is

$$r_{TM} = \frac{\eta_2 \cos\theta_t - \eta_1 \cos\theta_i}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_i} = \frac{(283.26\ \Omega)\cos 22° - (376.73\ \Omega)\cos 30°}{(283.26\ \Omega)\cos 22° + (376.73\ \Omega)\cos 30°} = \boxed{-0.1080}$$

The transmission coefficient for the TE polarization is

$$t_{TE} = 1 + r_{TE} = 1 + (-0.1749) = \boxed{0.8251}$$

The transmission coefficient for the TM polarization is

$$1 + r_{TM} = \frac{\cos\theta_t}{\cos\theta_i} t_{TM} \quad \rightarrow t_{TM} = \frac{\cos\theta_i}{\cos\theta_t}(1 + r_{TM}) = \frac{\cos 30°}{\cos 22°}\left[1 + (-0.1080)\right] = \boxed{0.8332}$$

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Reflectance & Transmittance

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Fresnel Equations for Complex Media

The Fresnel equations for complex media remain the same as lossless media as long as the impedance and angles are made complex.

TE, s, ⊥ Polarization

If $\text{Re}\left[\dfrac{\tilde{\eta}_{\text{trn}}\cos\tilde{\theta}_{\text{inc}}}{\tilde{\eta}_{\text{inc}}\cos\tilde{\theta}_{\text{trn}}}\right] < 0$, then change sign of either $\tilde{\eta}_{\text{inc}}$ or $\tilde{\eta}_{\text{trn}}$

$$\tilde{r}_{\text{TE}} = \frac{\tilde{E}_{0,\text{r}}}{\tilde{E}_{0,\text{i}}} = \frac{\tilde{\eta}_{\text{t}}\cos\tilde{\theta}_{\text{i}} - \tilde{\eta}_{\text{i}}\cos\tilde{\theta}_{\text{t}}}{\tilde{\eta}_{\text{t}}\cos\tilde{\theta}_{\text{i}} + \tilde{\eta}_{\text{i}}\cos\tilde{\theta}_{\text{t}}}$$

$$\tilde{t}_{\text{TE}} = \frac{\tilde{E}_{0,\text{t}}}{\tilde{E}_{0,\text{i}}} = \frac{2\tilde{\eta}_{\text{t}}\cos\tilde{\theta}_{\text{i}}}{\tilde{\eta}_{\text{t}}\cos\tilde{\theta}_{\text{i}} + \tilde{\eta}_{\text{i}}\cos\tilde{\theta}_{\text{t}}}$$

$$1 + \tilde{r}_{\text{TE}} = \tilde{t}_{\text{TE}}$$

TM, p, || Polarization

If $\text{Re}\left[\dfrac{\tilde{\eta}_{\text{trn}}\cos\tilde{\theta}_{\text{trn}}}{\tilde{\eta}_{\text{inc}}\cos\tilde{\theta}_{\text{inc}}}\right] < 0$, then change sign of either $\tilde{\eta}_{\text{inc}}$ or $\tilde{\eta}_{\text{trn}}$

$$\tilde{r}_{\text{TM}} = \frac{\tilde{E}_{0,\text{r}}}{\tilde{E}_{0,\text{i}}} = \frac{\tilde{\eta}_{\text{t}}\cos\tilde{\theta}_{\text{t}} - \tilde{\eta}_{\text{i}}\cos\tilde{\theta}_{\text{i}}}{\tilde{\eta}_{\text{t}}\cos\tilde{\theta}_{\text{t}} + \tilde{\eta}_{\text{i}}\cos\tilde{\theta}_{\text{i}}}$$

$$\tilde{t}_{\text{TM}} = \frac{\tilde{E}_{0,\text{t}}}{\tilde{E}_{0,\text{i}}} = \frac{2\tilde{\eta}_{\text{t}}\cos\tilde{\theta}_{\text{i}}}{\tilde{\eta}_{\text{t}}\cos\tilde{\theta}_{\text{t}} + \tilde{\eta}_{\text{i}}\cos\tilde{\theta}_{\text{i}}}$$

$$1 + \tilde{r}_{\text{TM}} = \frac{\cos\tilde{\theta}_{\text{t}}}{\cos\tilde{\theta}_{\text{i}}}\tilde{t}_{\text{TM}}$$

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Definition of R and T

The reflection and transmission coefficients relate the amplitudes of the reflected and transmitted waves relative to the incident wave.

$$\tilde{r} = \frac{\tilde{E}_{0,\mathrm{r}}}{\tilde{E}_{0,\mathrm{i}}} \qquad \tilde{t} = \frac{\tilde{E}_{0,\mathrm{t}}}{\tilde{E}_{0,\mathrm{i}}}$$

The reflectance and transmittance describes the fraction of power that is reflected or transmitted from the interface.

$$R = \frac{P_\mathrm{r}}{P_\mathrm{i}} \qquad T = \frac{P_\mathrm{t}}{P_\mathrm{i}}$$

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Power Flow

In the frequency-domain, the RMW Poynting vector describes power flow. It is calculated from the electric and magnetic fields as

$$\vec{\tilde{\wp}} = \frac{1}{2}\mathrm{Re}\left[\vec{\tilde{E}} \times \vec{\tilde{H}}^*\right]$$

In medium 1, there exists the incident wave and reflected wave.

$$\vec{\tilde{E}}_1 = \vec{\tilde{E}}_\mathrm{inc} + \vec{\tilde{E}}_\mathrm{ref} \qquad \vec{\tilde{H}}_1 = \vec{\tilde{H}}_\mathrm{inc} + \vec{\tilde{H}}_\mathrm{ref}$$

Substituting these into the definition of the complex Poynting vector gives

$$\vec{\tilde{\wp}} = \underbrace{\frac{1}{2}\mathrm{Re}\left(\vec{\tilde{E}}_\mathrm{inc} \times \vec{\tilde{H}}^*_\mathrm{inc}\right)}_{\text{Power of incident wave}} + \underbrace{\frac{1}{2}\mathrm{Re}\left(\vec{\tilde{E}}_\mathrm{ref} \times \vec{\tilde{H}}^*_\mathrm{ref}\right)}_{\text{Power of reflected wave}} + \underbrace{\frac{1}{2}\mathrm{Re}\left(\vec{\tilde{E}}_\mathrm{inc} \times \vec{\tilde{H}}^*_\mathrm{ref} + \vec{\tilde{E}}_\mathrm{ref} \times \vec{\tilde{H}}^*_\mathrm{inc}\right)}_{\text{Cross term}}$$

Power of incident wave

Power of reflected wave

Cross term – power flow due to interference between incident and reflected wave when there is loss.

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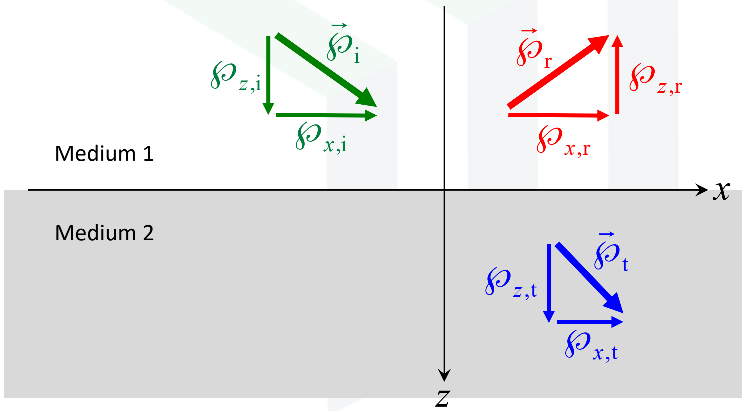
What Carries Power To and From the Interface?

The flow of power is described by the complex Poynting vector $\vec{\wp}$.

However, it is only the components of the Poynting vector that are perpendicular to the interface that carry power to and from the interface.



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z Components of Poynting Vectors (TE)

Incident Wave

$$\tilde{\wp}_{z,\text{inc}} = \text{Re}\left\{\frac{\left|\tilde{E}_{0,\text{inc}}\right|^2}{2\tilde{\eta}^*_{\text{inc}}}\cos\tilde{\theta}^*_{\text{inc}}\right\}$$

Reflected Wave

$$\tilde{\wp}_{z,\text{ref}} = \text{Re}\left\{-\frac{\left|\tilde{E}_{0,\text{ref}}\right|^2}{2\tilde{\eta}^*_{\text{inc}}}\cos\tilde{\theta}^*_{\text{inc}}\right\}$$

Transmitted Wave

$$\tilde{\wp}_{z,\text{trn}} = \text{Re}\left\{\frac{\left|\tilde{E}_{0,\text{trn}}\right|^2}{2\tilde{\eta}^*_{\text{trn}}}\cos\tilde{\theta}^*_{\text{trn}}\right\}$$

Cross Term

$$\tilde{\wp}_{z,\text{c}} = \text{Re}\left\{\frac{\cos\tilde{\theta}^*_{\text{inc}}}{2\tilde{\eta}^*_{\text{inc}}}\left(\tilde{E}^*_{0,\text{inc}}\tilde{E}_{0,\text{ref}} - \tilde{E}_{0,\text{inc}}\tilde{E}^*_{0,\text{ref}}\right)\right\}$$

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z Components of Poynting Vectors (TM)

Incident Wave

$$\tilde{\wp}_{z,\text{inc}} = \text{Re}\left\{ \frac{\left|\tilde{E}_{0,\text{inc}}\right|^2}{2\tilde{\eta}^*_{\text{inc}}} \cos\tilde{\theta}_{\text{inc}} \right\}$$

Reflected Wave

$$\tilde{\wp}_{z,\text{ref}} = \text{Re}\left\{ -\frac{\left|\tilde{E}_{0,\text{ref}}\right|^2}{2\tilde{\eta}^*_{\text{inc}}} \cos\tilde{\theta}_{\text{inc}} \right\}$$

Transmitted Wave

$$\tilde{\wp}_{z,\text{trn}} = \text{Re}\left\{ \frac{\left|\tilde{E}_{0,\text{trn}}\right|^2}{2\tilde{\eta}^*_{\text{trn}}} \cos\tilde{\theta}_{\text{trn}} \right\}$$

Cross Term

$$\tilde{\wp}_{z,\text{c}} = \text{Re}\left\{ \frac{\cos\tilde{\theta}_{\text{inc}}}{2\tilde{\eta}^*_{\text{inc}}} \left(\tilde{E}^*_{0,\text{inc}}\tilde{E}_{0,\text{ref}} - \tilde{E}_{0,\text{inc}}\tilde{E}^*_{0,\text{ref}} \right) \right\}$$

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Reflectance & Transmittance

Reflectance (TE)

$$R_{\text{TE}} = \left|\tilde{r}_{\text{TE}}\right|^2$$

Reflectance (TM)

$$R_{\text{TM}} = \left|\tilde{r}_{\text{TM}}\right|^2$$

Transmittance (TE)

$$T_{\text{TE}} = \left|\tilde{t}_{\text{TE}}\right|^2 \text{Re}\left[\frac{\tilde{\eta}^*_{\text{inc}} \cos\tilde{\theta}^*_{\text{trn}}}{\tilde{\eta}^*_{\text{trn}} \cos\tilde{\theta}^*_{\text{inc}}} \right]$$

Transmittance (TM)

$$T_{\text{TM}} = \left|\tilde{t}_{\text{TM}}\right|^2 \text{Re}\left[\frac{\tilde{\eta}^*_{\text{inc}} \cos\tilde{\theta}_{\text{trn}}}{\tilde{\eta}^*_{\text{trn}} \cos\tilde{\theta}_{\text{inc}}} \right]$$

Conservation of Power (TE)

$$R_{\text{TE}} + T_{\text{TE}} = 1$$

Conservation of Power (TM)

$$R_{\text{TM}} + T_{\text{TM}} = 1$$

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Summary for TE Polarization

Fresnel Equations

$$\tilde{r}_{\text{TE}} = \frac{\tilde{E}_{0,\text{r}}}{\tilde{E}_{0,\text{i}}} = \frac{\tilde{\eta}_{\text{t}} \cos\tilde{\theta}_{\text{i}} - \tilde{\eta}_{\text{i}} \cos\tilde{\theta}_{\text{t}}}{\tilde{\eta}_{\text{t}} \cos\tilde{\theta}_{\text{i}} + \tilde{\eta}_{\text{i}} \cos\tilde{\theta}_{\text{t}}} \qquad \tilde{t}_{\text{TE}} = \frac{\tilde{E}_{0,\text{t}}}{\tilde{E}_{0,\text{i}}} = \frac{2\tilde{\eta}_{\text{t}} \cos\tilde{\theta}_{\text{i}}}{\tilde{\eta}_{\text{t}} \cos\tilde{\theta}_{\text{i}} + \tilde{\eta}_{\text{i}} \cos\tilde{\theta}_{\text{t}}} \qquad 1 + \tilde{r}_{\text{TE}} = \tilde{t}_{\text{TE}}$$

Power Flow

Conservation: $R_{\text{TE}} + T_{\text{TE}} = 1$

Reflectance (Lossy + Lossless): $R_{\text{TE}} = \left|\tilde{r}_{\text{TE}}\right|^2$

Transmittance (Lossy): $T_{\text{TE}} = \left|\tilde{t}_{\text{TE}}\right|^2 \text{Re}\left[\frac{\tilde{\eta}_{\text{inc}}^* \cos\tilde{\theta}_{\text{trn}}^*}{\tilde{\eta}_{\text{trn}}^* \cos\tilde{\theta}_{\text{inc}}^*}\right]$

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Summary for TM Polarization

Fresnel Equations

$$\tilde{r}_{\text{TM}} = \frac{\tilde{E}_{0,\text{r}}}{\tilde{E}_{0,\text{i}}} = \frac{\tilde{\eta}_{\text{t}} \cos\tilde{\theta}_{\text{t}} - \tilde{\eta}_{\text{i}} \cos\tilde{\theta}_{\text{i}}}{\tilde{\eta}_{\text{t}} \cos\tilde{\theta}_{\text{t}} + \tilde{\eta}_{\text{i}} \cos\tilde{\theta}_{\text{i}}} \qquad \tilde{t}_{\text{TM}} = \frac{\tilde{E}_{0,\text{t}}}{\tilde{E}_{0,\text{i}}} = \frac{2\tilde{\eta}_{\text{t}} \cos\tilde{\theta}_{\text{i}}}{\tilde{\eta}_{\text{t}} \cos\tilde{\theta}_{\text{t}} + \tilde{\eta}_{\text{i}} \cos\tilde{\theta}_{\text{i}}} \qquad 1 + \tilde{r}_{\text{TM}} = \frac{\cos\tilde{\theta}_{\text{t}}}{\cos\tilde{\theta}_{\text{i}}}\tilde{t}_{\text{TM}}$$

Power Flow

Conservation: $R_{\text{TM}} + T_{\text{TM}} = 1$

Reflectance (Lossy + Lossless): $R_{\text{TM}} = \left|\tilde{r}_{\text{TM}}\right|^2$

Transmittance (Lossy): $T_{\text{TM}} = \left|\tilde{t}_{\text{TM}}\right|^2 \text{Re}\left[\frac{\tilde{\eta}_{\text{inc}}^* \cos\tilde{\theta}_{\text{trn}}}{\tilde{\eta}_{\text{trn}}^* \cos\tilde{\theta}_{\text{inc}}}\right]$

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Relation Between the Parameters

For lossless materials, the transmittance reduces to

$$T = \left|t\right|^2 \frac{\eta_{\text{i}} \cos\theta_{\text{t}}}{\eta_{\text{t}} \cos\theta_{\text{t}}}$$

It becomes possible to derive a relation between T_{TE} and T_{TM} for lossless materials.

$$\frac{T_{\text{TE}}}{T_{\text{TM}}} = \frac{\left|t_{\text{TE}}\right|^2 \dfrac{\eta_1}{\eta_2} \dfrac{\cos\theta_{\text{t}}}{\cos\theta_{\text{i}}}}{\left|t_{\text{TM}}\right|^2 \dfrac{\eta_1}{\eta_2} \dfrac{\cos\theta_{\text{t}}}{\cos\theta_{\text{i}}}} = \frac{\left|t_{\text{TE}}\right|^2}{\left|t_{\text{TM}}\right|^2}$$

$$\boxed{\frac{T_{\text{TE}}}{T_{\text{TM}}} = \frac{\left|t_{\text{TE}}\right|^2}{\left|t_{\text{TM}}\right|^2}}$$

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Total Reflectance and Transmittance

A wave incident onto a surface may have both TE and TM components. Power in the source wave is therefore

$$P_{\text{inc}} = P_{\text{TE,inc}} + P_{\text{TM,inc}}$$

It follows that the total power reflected and transmitted is

$$P_{\text{ref}} = R_{\text{TE}} P_{\text{TE,inc}} + R_{\text{TM}} P_{\text{TM,inc}} \qquad P_{\text{trn}} = T_{\text{TE}} P_{\text{TE,inc}} + T_{\text{TM}} P_{\text{TM,inc}}$$

Overall reflectance R and transmittance T are derived by dividing these equations by the first equation.

$$\boxed{R = \frac{P_{\text{ref}}}{P_{\text{inc}}} = \frac{R_{\text{TE}} P_{\text{TE,inc}} + R_{\text{TM}} P_{\text{TM,inc}}}{P_{\text{TE,inc}} + P_{\text{TM,inc}}}} \qquad \boxed{T = \frac{P_{\text{trn}}}{P_{\text{inc}}} = \frac{T_{\text{TE}} P_{\text{TE,inc}} + T_{\text{TM}} P_{\text{TM,inc}}}{P_{\text{TE,inc}} + P_{\text{TM,inc}}}}$$

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Example #5 – Fresnel Equations

In spherical coordinates, a wave is incident at $\theta = 30°$ and $\phi = 120°$ from air into water. Calculate the reflectance and transmittance for both TE and TM polarizations.

Solution

The reflectance for the TE polarization is

$$R_{\text{TE}} = |r_{\text{TE}}|^2 = |-0.1749|^2 = 0.0306 = \boxed{3.1\%}$$

The reflectance for the TM polarization is

$$R_{\text{TM}} = |r_{\text{TM}}|^2 = |-0.1080|^2 = 0.0117 = \boxed{1.2\%}$$

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Example #5 – Fresnel Equations

In spherical coordinates, a wave is incident at $\theta = 30°$ and $\phi = 120°$ from air into water. Calculate the reflectance and transmittance for both TE and TM polarizations.

Solution, cont'd

The transmittance for the TE polarization is

$$T_{\text{TE}} = |t_{\text{TE}}|^2 \frac{\eta_1}{\eta_2} \frac{\cos\theta_{\text{t}}}{\cos\theta_{\text{i}}} = |0.8251|^2 \frac{376.73\ \Omega}{283.26\ \Omega} \frac{\cos 22°}{\cos 30°} = 0.9694 = \boxed{96.9\%}$$

The transmittance for the TM polarization is

$$T_{\text{TM}} = |t_{\text{TM}}|^2 \frac{\eta_1}{\eta_2} \frac{\cos\theta_{\text{t}}}{\cos\theta_{\text{i}}} = |0.8332|^2 \frac{376.73\ \Omega}{283.26\ \Omega} \frac{\cos 22°}{\cos 30°} = 0.9885 = \boxed{98.8\%}$$

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Example #6 – Power Conservation

In spherical coordinates, a wave is incident at θ = 30° and ϕ = 120° from air into water. Confirm power conservation for reflectance and transmittance of the TE and TM polarizations.

Solution

Power conservation for the TE polarization is

$$R_{\text{TE}} + T_{\text{TE}} = 3.1\% + 96.9\% = \boxed{100\%}$$

Power conservation for the TM polarization is

$$R_{\text{TM}} + T_{\text{TM}} = 1.2\% + 98.8\% = \boxed{100\%}$$

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Example – Plot of Fresnel Equations

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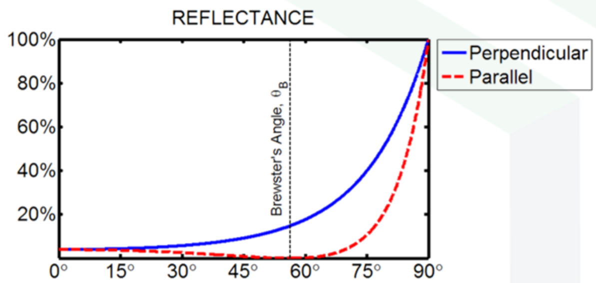
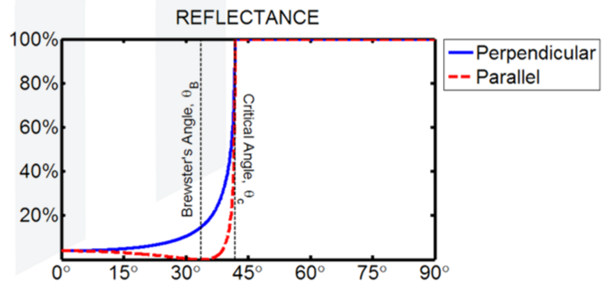
Plots of the Fresnel Equations

Low to High Index
($n_1 = 1.0$ and $n_2 = 1.5$)



High to Low Index
($n_1 = 1.5$ and $n_2 = 1.0$)



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