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Advanced Computation:

Computational Electromagnetics

Transfer Matrix Method Using Scattering Matrices

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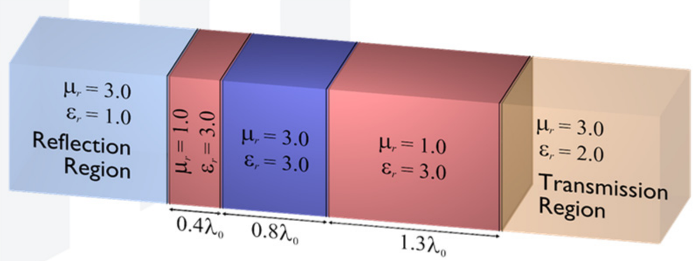
Outline

- Calculating reflected and transmitted power
- Simplifications for 1D transfer matrix method
- Notes on implementation
- Block diagram



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Calculating Transmitted and Reflected Power

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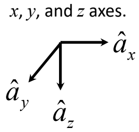
Recall How to Calculate the Source Parameters

Incident Wave Vector

$$\vec{k}_{\text{inc}} = k_0 n_{\text{inc}} \begin{bmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{bmatrix}$$

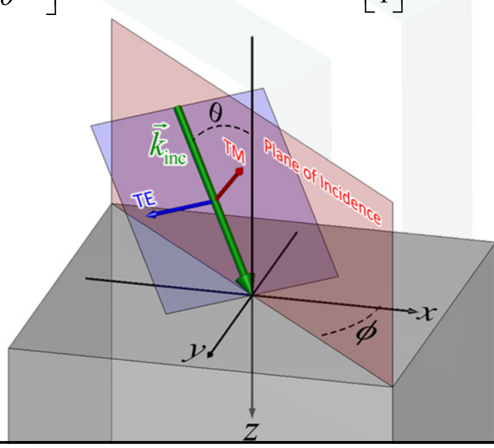
Unit vectors along x, y, and z axes.



Right-handed coordinate system

Surface Normal

$$\hat{n} = \hat{a}_z = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$



Unit Vectors in Directions of TE & TM

$$\hat{a}_{\text{TE}} = \begin{cases} \hat{a}_y & \theta = 0^\circ \\ \dfrac{\hat{n}\times\vec{k}_{\text{inc}}}{\left|\hat{n}\times\vec{k}_{\text{inc}}\right|} & \theta \neq 0^\circ \end{cases}$$

Can be any direction in the x-y plane

$$\hat{a}_{\text{TM}} = \frac{\vec{k}_{\text{inc}}\times\hat{a}_{\text{TE}}}{\left|\vec{k}_{\text{inc}}\times\hat{a}_{\text{TE}}\right|}$$

Composite Polarization Vector

$$\vec{P} = p_{\text{TE}}\hat{a}_{\text{TE}} + p_{\text{TM}}\hat{a}_{\text{TM}}$$

In CEM, the magnitude is usually set to 1.

$$\left|\vec{P}\right| = 1$$

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Solution Using Scattering Matrices

The external fields (i.e. incident wave, reflected wave, transmitted wave) are related through the global transfer matrix.

$$\begin{bmatrix} \mathbf{c}_{\text{ref}} \\ \mathbf{c}_{\text{trn}} \end{bmatrix} = \mathbf{S}^{(\text{global})} \begin{bmatrix} \mathbf{c}_{\text{inc}} \\ \mathbf{0} \end{bmatrix} \qquad \mathbf{c}_{\text{inc}} = \mathbf{W}_{\text{ref}}^{-1} \begin{bmatrix} P_x \\ P_y \end{bmatrix}$$

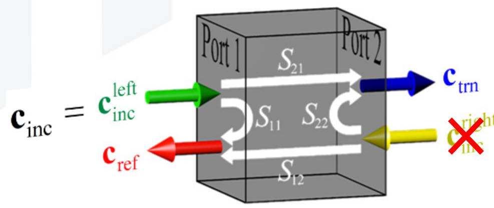
P_x and P_y are obtained from the polarization vector $\vec{P}$.
Note that P_z is not needed.

$$\vec{P} = \begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix}$$

This matrix equation can be solved to calculate the mode coefficients of the reflected and transmitted fields.

$$\begin{bmatrix} \mathbf{c}_{\text{ref}} \\ \mathbf{c}_{\text{trn}} \end{bmatrix} = \begin{bmatrix} \mathbf{S}_{11}^{(\text{global})} & \mathbf{S}_{12}^{(\text{global})} \\ \mathbf{S}_{21}^{(\text{global})} & \mathbf{S}_{22}^{(\text{global})} \end{bmatrix} \begin{bmatrix} \mathbf{c}_{\text{inc}} \\ \mathbf{0} \end{bmatrix} \quad \Rightarrow \quad \begin{aligned} \mathbf{c}_{\text{ref}} &= \mathbf{S}_{11}^{(\text{global})} \mathbf{c}_{\text{inc}} \\ \mathbf{c}_{\text{trn}} &= \mathbf{S}_{21}^{(\text{global})} \mathbf{c}_{\text{inc}} \end{aligned}$$

$\mathbf{c}_{\text{inc}}^{\text{right}}$ not typically used



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Calculation of Transmitted and Reflected Fields

The procedure described thus far calculated the mode coefficients $\mathbf{c}_{\text{ref}}$ and $\mathbf{c}_{\text{trn}}$.

The transmitted and reflected fields must be calculated from the mode coefficients.

$$\begin{bmatrix} E_x^{\text{ref}} \\ E_y^{\text{ref}} \end{bmatrix} = \mathbf{W}_{\text{ref}} \mathbf{c}_{\text{ref}}$$

$$\begin{bmatrix} E_x^{\text{trn}} \\ E_y^{\text{trn}} \end{bmatrix} = \mathbf{W}_{\text{trn}} \mathbf{c}_{\text{trn}}$$

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Calculation of the Longitudinal Components

The longitudinal field component E_z must still be calculated on both the reflection and transmission sides.

These are calculated from E_x and E_y using Maxwell's divergence equation.

$$\nabla \bullet \vec{E} = 0$$

$$\frac{\partial}{\partial x}\left(E_{0,x} e^{j\vec{k}\bullet\vec{r}}\right) + \frac{\partial}{\partial y}\left(E_{0,y} e^{j\vec{k}\bullet\vec{r}}\right) + \frac{\partial}{\partial z}\left(E_{0,z} e^{j\vec{k}\bullet\vec{r}}\right) = 0$$

$$jk_x E_{0,x} e^{j\vec{k}\bullet\vec{r}} + jk_y E_{0,y} e^{j\vec{k}\bullet\vec{r}} + jk_z E_{0,z} e^{j\vec{k}\bullet\vec{r}} = 0$$

$$k_x E_{0,x} + k_y E_{0,y} + k_z E_{0,z} = 0$$

$$k_z E_{0,z} = -k_x E_{0,x} - k_y E_{0,y}$$

$$E_{0,z} = -\frac{k_x E_{0,x} + k_y E_{0,y}}{k_z}$$

$$E_z^{\text{ref}} = -\frac{\tilde{k}_x E_x^{\text{ref}} + \tilde{k}_y E_y^{\text{ref}}}{\tilde{k}_z^{\text{ref}}}$$

$$E_z^{\text{trn}} = -\frac{\tilde{k}_x E_x^{\text{trn}} + \tilde{k}_y E_y^{\text{trn}}}{\tilde{k}_z^{\text{trn}}}$$

Note:
$\nabla \bullet \left(\varepsilon \vec{E}\right) = 0$ reduces to
$\nabla \bullet \vec{E} = 0$ when ε is homogeneous.

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Calculation of Power Flow

Reflectance R is defined as the fraction of power reflected from a device.

$$R = \frac{\left|\vec{E}_{\text{ref}}\right|^2}{\left|\vec{E}_{\text{inc}}\right|^2}$$

$$\left|\vec{E}\right|^2 = \left|E_x\right|^2 + \left|E_y\right|^2 + \left|E_z\right|^2$$

Transmittance T is defined as the fraction of power transmitted through a device.

$$T = \frac{\left|\vec{E}_{\text{trn}}\right|^2}{\left|\vec{E}_{\text{inc}}\right|^2} \frac{\text{Re}\left[k_z^{\text{trn}} / \mu_{\text{r,trn}}\right]}{\text{Re}\left[k_z^{\text{inc}} / \mu_{\text{r,inc}}\right]}$$

Note: These formulas will be derived in a later lecture.

$$\vec{E}_{\text{inc}} = \vec{P}$$

It is always good practice to check for conservation of power.

$$R + T \rightarrow \begin{cases} < 1 & \text{materials have loss} \\ = 1 & \text{materials have no loss and no gain} \\ > 1 & \text{materials have gain} \end{cases}$$

Note: Recall $A + R + T = 1$

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Reflectance and Transmittance on a Decibel Scale

Decibel Scale

$$P_{\text{dB}} = 20\log_{10}\left(A\right)$$ How to calculate decibels from an amplitude quantity A.

$$P_{\text{dB}} = 10\log_{10}\left(P\right)$$ How to calculate decibels from a power quantity P.

$P = A^2 \qquad P_{\text{dB}} = 10\log_{10}\left(A^2\right) = 20\log_{10}\left(A\right)$

Reflectance and Transmittance

Reflectance and transmittance are power quantities, so

$$R_{\text{dB}} = 10\log_{10}\left(R\right)$$
$$T_{\text{dB}} = 10\log_{10}\left(T\right)$$

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Simplifications for 1D Transfer Matrix Method

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Analytical Expressions for $\mathbf{W}$ and $\boldsymbol{\lambda}$

The dispersion relation with a normalized wave vector is

$$\mu_r \varepsilon_r = \tilde{k}_x^2 + \tilde{k}_y^2 + \tilde{k}_z^2$$

Using this relation, the matrix equation for $\boldsymbol{\Omega}^2$ can be greatly simplified.

$$\boldsymbol{\Omega}^2 = \mathbf{PQ} = \frac{1}{\mu_r \varepsilon_r}\begin{bmatrix} \tilde{k}_x\tilde{k}_y & \mu_r\varepsilon_r - \tilde{k}_x^2 \\ \tilde{k}_y^2 - \mu_r\varepsilon_r & -\tilde{k}_x\tilde{k}_y \end{bmatrix}\begin{bmatrix} \tilde{k}_x\tilde{k}_y & \mu_r\varepsilon_r - \tilde{k}_x^2 \\ \tilde{k}_y^2 - \mu_r\varepsilon_r & -\tilde{k}_x\tilde{k}_y \end{bmatrix} = \begin{bmatrix} -\tilde{k}_z^2 & 0 \\ 0 & -\tilde{k}_z^2 \end{bmatrix} = -\tilde{k}_z^2 \mathbf{I}$$

A lot of algebra

$$\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{identity matrix}$$

Since $\boldsymbol{\Omega}^2$ is a diagonal matrix, it can be concluded that

$$\mathbf{W} = \mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \Rightarrow \quad \boldsymbol{\lambda} = \begin{bmatrix} j\tilde{k}_z & 0 \\ 0 & j\tilde{k}_z \end{bmatrix} = j\tilde{k}_z \mathbf{I} \quad \Rightarrow \quad e^{\boldsymbol{\lambda} z'} = \begin{bmatrix} e^{j\tilde{k}_z z'} & 0 \\ 0 & e^{j\tilde{k}_z z'} \end{bmatrix}$$

$$\boldsymbol{\lambda}^2 = \boldsymbol{\Omega}^2$$

For isotropic materials and diagonally anisotropic materials, it is not necessary to actually solve the eigen-value problem to obtain the eigen-modes! ☺

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Simplifications for TMM in LHI Media

Given that

$$\mathbf{W}_i = \mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \boldsymbol{\Omega}_i = j\tilde{k}_{z,i}\mathbf{I} \qquad \boldsymbol{\lambda}_i = \boldsymbol{\Omega}_i$$

The expression for the eigen-vectors for the magnetic fields $\mathbf{V}$ reduces to

$$\mathbf{V}_i = \mathbf{Q}_i \mathbf{W}_i \boldsymbol{\lambda}_i^{-1} = \underline{\mathbf{Q}_i \boldsymbol{\Omega}_i^{-1}}$$

When calculating scattering matrices, the intermediate matrices $\mathbf{A}_i$ and $\mathbf{B}_i$ reduce to

$$\mathbf{A}_i = \mathbf{W}_i^{-1}\mathbf{W}_g + \mathbf{V}_i^{-1}\mathbf{V}_g = \underline{\mathbf{I} + \mathbf{V}_i^{-1}\mathbf{V}_g}$$

$$\mathbf{B}_i = \mathbf{W}_i^{-1}\mathbf{W}_g - \mathbf{V}_i^{-1}\mathbf{V}_g = \underline{\mathbf{I} - \mathbf{V}_i^{-1}\mathbf{V}_g}$$

The fields and mode coefficients are now the same thing!

$$\mathbf{c}_{inc} = \mathbf{W}_{ref}^{-1}\begin{bmatrix} P_x \\ P_y \end{bmatrix} = \underline{\begin{bmatrix} P_x \\ P_y \end{bmatrix}} \qquad \begin{bmatrix} E_x^{ref} \\ E_y^{ref} \end{bmatrix} = \mathbf{W}_{ref}\mathbf{S}_{11}\mathbf{c}_{inc} = \underline{\mathbf{S}_{11}\mathbf{c}_{inc}} \qquad \begin{bmatrix} E_x^{trn} \\ E_y^{trn} \end{bmatrix} = \mathbf{W}_{trn}\mathbf{S}_{21}\mathbf{c}_{inc} = \underline{\mathbf{S}_{21}\mathbf{c}_{inc}}$$

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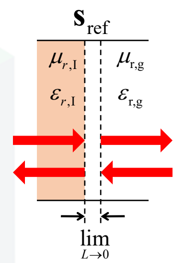
Simplified External S-Matrices in LHI Media

The reflection-side scattering matrix reduces to

$$\mathbf{S}_{11}^{(\mathrm{ref})} = -\mathbf{A}_{\mathrm{ref}}^{-1}\mathbf{B}_{\mathrm{ref}}$$
$$\mathbf{S}_{12}^{(\mathrm{ref})} = 2\mathbf{A}_{\mathrm{ref}}^{-1}$$
$$\mathbf{S}_{21}^{(\mathrm{ref})} = 0.5\left(\mathbf{A}_{\mathrm{ref}} - \mathbf{B}_{\mathrm{ref}}\mathbf{A}_{\mathrm{ref}}^{-1}\mathbf{B}_{\mathrm{ref}}\right)$$
$$\mathbf{S}_{22}^{(\mathrm{ref})} = \mathbf{B}_{\mathrm{ref}}\mathbf{A}_{\mathrm{ref}}^{-1}$$

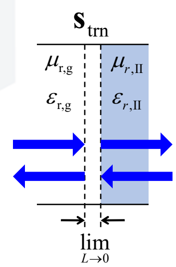
$$\mathbf{A}_{\mathrm{ref}} = \mathbf{I} + \mathbf{V}_{\mathrm{g}}^{-1}\mathbf{V}_{\mathrm{ref}}$$
$$\mathbf{B}_{\mathrm{ref}} = \mathbf{I} - \mathbf{V}_{\mathrm{g}}^{-1}\mathbf{V}_{\mathrm{ref}}$$



The transmission-side scattering matrix reduces to

$$\mathbf{S}_{11}^{(\mathrm{trn})} = \mathbf{B}_{\mathrm{trn}}\mathbf{A}_{\mathrm{trn}}^{-1}$$
$$\mathbf{S}_{12}^{(\mathrm{trn})} = 0.5\left(\mathbf{A}_{\mathrm{trn}} - \mathbf{B}_{\mathrm{trn}}\mathbf{A}_{\mathrm{trn}}^{-1}\mathbf{B}_{\mathrm{trn}}\right)$$
$$\mathbf{S}_{21}^{(\mathrm{trn})} = 2\mathbf{A}_{\mathrm{trn}}^{-1}$$
$$\mathbf{S}_{22}^{(\mathrm{trn})} = -\mathbf{A}_{\mathrm{trn}}^{-1}\mathbf{B}_{\mathrm{trn}}$$

$$\mathbf{A}_{\mathrm{trn}} = \mathbf{I} + \mathbf{V}_{\mathrm{g}}^{-1}\mathbf{V}_{\mathrm{trn}}$$
$$\mathbf{B}_{\mathrm{trn}} = \mathbf{I} - \mathbf{V}_{\mathrm{g}}^{-1}\mathbf{V}_{\mathrm{trn}}$$



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Notes on Implementation

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Outline

- Step 0 – Define problem
- Step 1 – Dashboard
- Step 2 – Describe device layers
- Step 3 – Compute wave vector components
- Step 4 – Compute gap medium parameters
- Step 5 – Initialize global scattering matrix
- Step 6 – Main loop through layers
- Step 7 – Compute reflection side scattering matrix
- Step 8 – Compute transmission side scattering matrix
- Step 9 – Update global scattering matrix
- Step 10 – Compute source
- Step 11 – Compute reflected and transmitted fields
- Step 12 – Compute reflectance and transmittance
- Step 13 – Verify conservation of power

human does this

computer does the rest

Step 6: Iterate through layers

- Compute **P** and **Q**
- Compute eigen-modes
- Compute layer scattering matrix
- Update global scattering matrix

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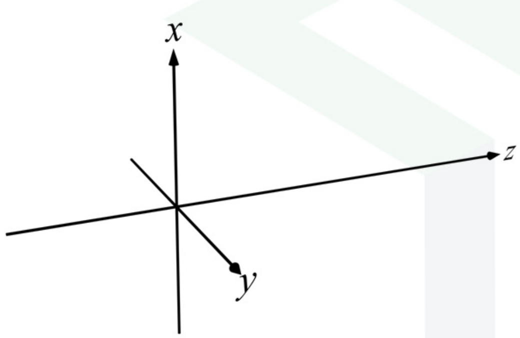
Storing the Problem

How is the problem stored and described in the dashboard of TMM?



Note that this is a right-handed coordinate system.

$$\hat{a}_x \times \hat{a}_y = \hat{a}_z$$

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Storing the Problem

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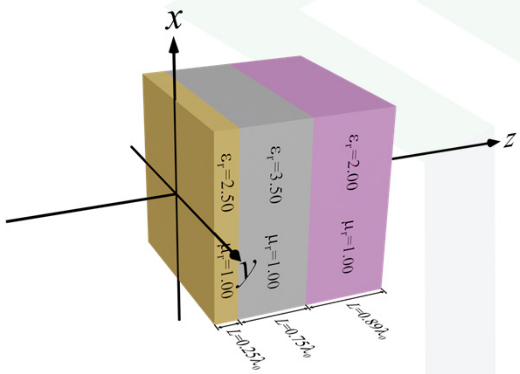

Device is described in three 1D arrays.

```
ER = [ 2.50 , 3.50 , 2.00 ];
UR = [ 1.00 , 1.00 , 1.00 ];
L  = [ 0.25 , 0.75 , 0.89 ];
```

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Storing the Problem

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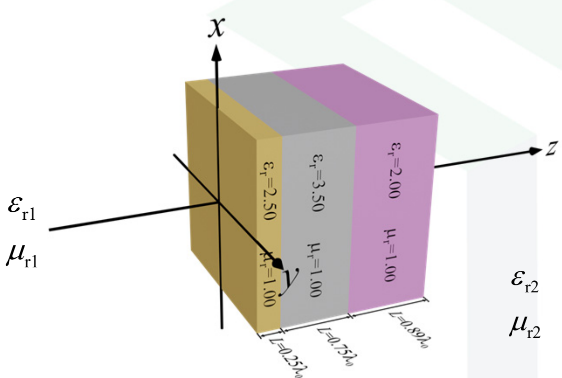

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```

External materials:

`er1`, `er2`, `ur1` and `ur2`

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Storing the Problem

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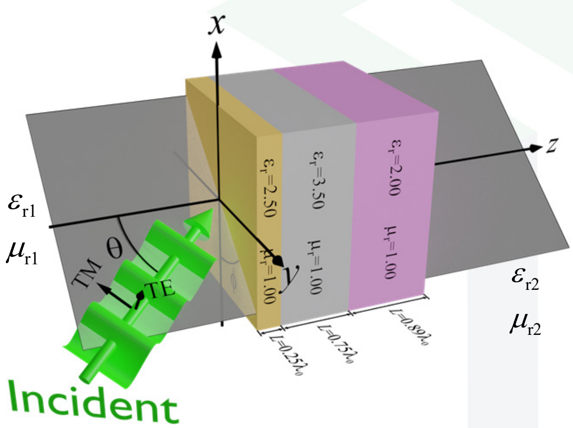

Device is described in three 1D arrays.

```
ER = [ 2.50 , 3.50 , 2.00 ];
UR = [ 1.00 , 1.00 , 1.00 ];
L  = [ 0.25 , 0.75 , 0.89 ];
```

External materials:

`er1`, `er2`, `ur1` and `ur2`

The source:

`theta`, `phi`, `pte`, `ptm`, and `lam0`

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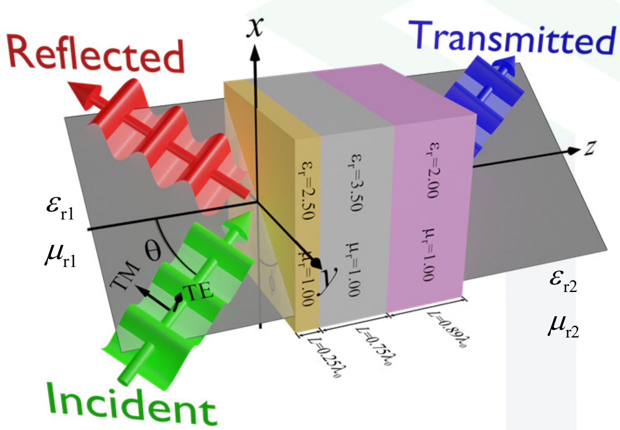
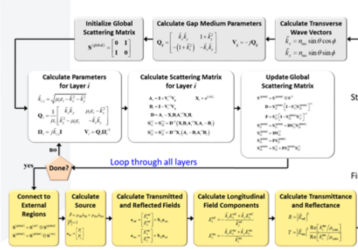
Storing the Problem

How is the problem stored and described in the dashboard of TMM?



After simulation!



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Storing Scattering Matrices

The scattering matrix $\mathbf{S}$ is often talked about as a single matrix.

$$\mathbf{S}=\begin{bmatrix}\mathbf{S}_{11} & \mathbf{S}_{12}\\ \mathbf{S}_{21} & \mathbf{S}_{22}\end{bmatrix}$$

However, the scattering matrix $\mathbf{S}$ is rarely used this way.
Most commonly, the individual terms $\mathbf{S}_{11}$, $\mathbf{S}_{12}$, $\mathbf{S}_{21}$, and $\mathbf{S}_{22}$ are handled separately.

So, scattering matrices are best stored as the four separate components of the scattering matrix.

$$\cancel{\mathbf{S}=\begin{bmatrix}\mathbf{S}_{11} & \mathbf{S}_{12}\\ \mathbf{S}_{21} & \mathbf{S}_{22}\end{bmatrix}} \quad\Rightarrow\quad \mathbf{S}_{11},\ \mathbf{S}_{12},\ \mathbf{S}_{21},\ \text{and } \mathbf{S}_{22}$$

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Initializing the Global Scattering Matrix

Before iterating through all the layers, the global scattering matrix must be initialized as the scattering matrix of "nothing."

What are the ideal properties of nothing?

1. Transmits 100% of power with no phase change.

$$\mathbf{S}_{12}^{(\text{global})}=\mathbf{S}_{21}^{(\text{global})}=\mathbf{I}$$

2. Does not reflect at all.

$$\mathbf{S}_{11}^{(\text{global})}=\mathbf{S}_{22}^{(\text{global})}=\mathbf{0}$$

Therefore, the global scattering matrix is initialized according to

$$\mathbf{S}^{(\text{global})}=\begin{bmatrix}\mathbf{0} & \mathbf{I}\\ \mathbf{I} & \mathbf{0}\end{bmatrix}$$

This is NOT an identity matrix*!*
Look at the position of the $\mathbf{0}$'s and $\mathbf{I}$'s.

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Calculating the Parameters of the Gap Media

The analytical solution for a homogeneous gap medium is

$$\mathbf{Q}_{\mathrm{g}} = \frac{1}{\mu_{\mathrm{r,g}}}\begin{bmatrix} \tilde{k}_x\tilde{k}_y & \mu_{\mathrm{r,g}}\varepsilon_{\mathrm{r,g}} - \tilde{k}_x^2 \\ \tilde{k}_y^2 - \mu_{\mathrm{r,g}}\varepsilon_{\mathrm{r,g}} & -\tilde{k}_x\tilde{k}_y \end{bmatrix} \qquad \tilde{k}_{z,\mathrm{g}}^2 = \mu_{\mathrm{r,g}}\varepsilon_{\mathrm{r,g}} - \tilde{k}_x^2 - \tilde{k}_y^2 \qquad \boldsymbol{\lambda}_{\mathrm{g}} = j\tilde{k}_{z,\mathrm{g}}\mathbf{I}$$

$$\mathbf{W}_{\mathrm{g}} = \mathbf{I} \qquad \mathbf{V}_{\mathrm{g}} = \mathbf{Q}_{\mathrm{g}}\boldsymbol{\lambda}_{\mathrm{g}}^{-1}$$

Any choice of $\mu_{\mathrm{r,g}}$ and $\varepsilon_{\mathrm{r,g}}$ is possible. However, It is best to avoid the case of $k_{z,\mathrm{g}} = 0$.

To do this in a mathematically convenient way, choose

$$\mu_{\mathrm{r,g}} = 1.0 \quad \text{and} \quad \varepsilon_{\mathrm{r,g}} = 1 + \tilde{k}_x^2 + \tilde{k}_y^2$$

Given this choice, the parameters reduce to

$$\mathbf{Q}_{\mathrm{g}} = \begin{bmatrix} \tilde{k}_x\tilde{k}_y & 1 + \tilde{k}_y^2 \\ -\left(1 + \tilde{k}_x^2\right) & -\tilde{k}_x\tilde{k}_y \end{bmatrix} \qquad \mathbf{W}_{\mathrm{g}} = \mathbf{I} \qquad \mathbf{V}_{\mathrm{g}} = -j\mathbf{Q}_{\mathrm{g}}$$

$\mathbf{W}$ is not even used in isotropic TMM!

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Mode Coefficients = Fields (Sometimes)

Recall that the fields are calculated from the mode coefficients through

$$\begin{bmatrix} E_x^{\mathrm{inc}} \\ E_y^{\mathrm{inc}} \end{bmatrix} = \mathbf{W}_{\mathrm{inc}}\mathbf{c}_{\mathrm{inc}} \qquad \begin{bmatrix} E_x^{\mathrm{ref}} \\ E_y^{\mathrm{ref}} \end{bmatrix} = \mathbf{W}_{\mathrm{ref}}\mathbf{c}_{\mathrm{ref}} \qquad \begin{bmatrix} E_x^{\mathrm{trn}} \\ E_y^{\mathrm{trn}} \end{bmatrix} = \mathbf{W}_{\mathrm{trn}}\mathbf{c}_{\mathrm{trn}}$$

However, for TMM using LHI materials, $\mathbf{W} = \mathbf{I}$ always.

$$\begin{bmatrix} E_x^{\mathrm{inc}} \\ E_y^{\mathrm{inc}} \end{bmatrix} = \mathbf{c}_{\mathrm{inc}} \qquad \begin{bmatrix} E_x^{\mathrm{ref}} \\ E_y^{\mathrm{ref}} \end{bmatrix} = \mathbf{c}_{\mathrm{ref}} \qquad \begin{bmatrix} E_x^{\mathrm{trn}} \\ E_y^{\mathrm{trn}} \end{bmatrix} = \mathbf{c}_{\mathrm{trn}}$$

This means that in TMM, the mode coefficients are the fields.

This is a special thing for TMM with LHI materials. It does not hold for other methods or when anisotropic materials are being simulated.

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Calculating $\mathbf{X}_i = \exp(\mathbf{\Omega}_i k_0 L_i)$

Recall the correct answer:

$$\mathbf{X}_i = e^{\mathbf{\Omega}_i k_0 L_i} = \begin{bmatrix} e^{j\tilde{k}_z k_0 L_i} & 0 \\ 0 & e^{j\tilde{k}_z k_0 L_i} \end{bmatrix}$$

It is incorrect to use the function `exp()` directly on the matrix $\mathbf{\Omega}$ because in MATLAB `exp()` calculates a point-by-point exponential, not a matrix exponential.

WRONG

```
X = exp(OMEGA*k0*L);
```

→

```
X =
   0.0135 + 0.9999i    1.0000
   1.0000              0.0135 + 0.9999i
```

Approach #1: `expm()`

```
X = expm(OMEGA*k0*L);

X =
   0.0135 + 0.9999i         0
        0           0.0135 + 0.9999i
```

Approach #2: `diag()`

```
X = diag(exp(diag(OMEGA)*k0*L));

X =
   0.0135 + 0.9999i         0
        0           0.0135 + 0.9999i
```

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Efficient Calculation of Layer S-Matrices

There are redundant calculations in the equations for the scattering matrix elements.

$$\mathbf{S}_{11}^{(i)} = \mathbf{S}_{22}^{(i)} = \left(\mathbf{A}_i - \mathbf{X}_i\mathbf{B}_i\mathbf{A}_i^{-1}\mathbf{X}_i\mathbf{B}_i\right)^{-1}\left(\mathbf{X}_i\mathbf{B}_i\mathbf{A}_i^{-1}\mathbf{X}_i\mathbf{A}_i - \mathbf{B}_i\right)$$

$$\mathbf{S}_{12}^{(i)} = \mathbf{S}_{21}^{(i)} = \left(\mathbf{A}_i - \mathbf{X}_i\mathbf{B}_i\mathbf{A}_i^{-1}\mathbf{X}_i\mathbf{B}_i\right)^{-1}\mathbf{X}_i\left(\mathbf{A}_i - \mathbf{B}_i\mathbf{A}_i^{-1}\mathbf{B}_i\right)$$

These are more efficiently calculated as

$$\mathbf{A}_i = \mathbf{I} + \mathbf{V}_i^{-1}\mathbf{V}_g$$

$$\mathbf{B}_i = \mathbf{I} - \mathbf{V}_i^{-1}\mathbf{V}_g$$

$$\mathbf{X}_i = e^{\mathbf{\lambda}_i k_0 L_i}$$

$$\mathbf{D} = \mathbf{A}_i - \mathbf{X}_i\mathbf{B}_i\mathbf{A}_i^{-1}\mathbf{X}_i\mathbf{B}_i$$

$$\mathbf{S}_{11}^{(i)} = \mathbf{S}_{22}^{(i)} = \mathbf{D}^{-1}\left(\mathbf{X}_i\mathbf{B}_i\mathbf{A}_i^{-1}\mathbf{X}_i\mathbf{A}_i - \mathbf{B}_i\right)$$

$$\mathbf{S}_{12}^{(i)} = \mathbf{S}_{21}^{(i)} = \mathbf{D}^{-1}\mathbf{X}_i\left(\mathbf{A}_i - \mathbf{B}_i\mathbf{A}_i^{-1}\mathbf{B}_i\right)$$

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Efficient Star Product

After observing the equations to implement the Redheffer star product, some common terms can be identified. Calculating these multiple times is inefficient so they should be calculated only once.

$$\mathbf{S}^{(\mathrm{AB})} = \mathbf{S}^{(\mathrm{A})} \otimes \mathbf{S}^{(\mathrm{B})}$$

$$\mathbf{S}_{11}^{(\mathrm{AB})} = \mathbf{S}_{11}^{(\mathrm{A})} + \mathbf{S}_{12}^{(\mathrm{A})}\left[\mathbf{I} - \mathbf{S}_{11}^{(\mathrm{B})}\mathbf{S}_{22}^{(\mathrm{A})}\right]^{-1}\mathbf{S}_{11}^{(\mathrm{B})}\mathbf{S}_{21}^{(\mathrm{A})}$$

$$\mathbf{S}_{12}^{(\mathrm{AB})} = \mathbf{S}_{12}^{(\mathrm{A})}\left[\mathbf{I} - \mathbf{S}_{11}^{(\mathrm{B})}\mathbf{S}_{22}^{(\mathrm{A})}\right]^{-1}\mathbf{S}_{12}^{(\mathrm{B})}$$

$$\mathbf{S}_{21}^{(\mathrm{AB})} = \mathbf{S}_{21}^{(\mathrm{B})}\left[\mathbf{I} - \mathbf{S}_{22}^{(\mathrm{A})}\mathbf{S}_{11}^{(\mathrm{B})}\right]^{-1}\mathbf{S}_{21}^{(\mathrm{A})}$$

$$\mathbf{S}_{22}^{(\mathrm{AB})} = \mathbf{S}_{22}^{(\mathrm{B})} + \mathbf{S}_{21}^{(\mathrm{B})}\left[\mathbf{I} - \mathbf{S}_{22}^{(\mathrm{A})}\mathbf{S}_{11}^{(\mathrm{B})}\right]^{-1}\mathbf{S}_{22}^{(\mathrm{A})}\mathbf{S}_{12}^{(\mathrm{B})}$$

→

$$\mathbf{D} = \mathbf{S}_{12}^{(\mathrm{A})}\left[\mathbf{I} - \mathbf{S}_{11}^{(\mathrm{B})}\mathbf{S}_{22}^{(\mathrm{A})}\right]^{-1}$$

$$\mathbf{F} = \mathbf{S}_{21}^{(\mathrm{B})}\left[\mathbf{I} - \mathbf{S}_{22}^{(\mathrm{A})}\mathbf{S}_{11}^{(\mathrm{B})}\right]^{-1}$$

$$\mathbf{S}_{11}^{(AB)} = \mathbf{S}_{11}^{(\mathrm{A})} + \mathbf{D}\mathbf{S}_{11}^{(\mathrm{B})}\mathbf{S}_{21}^{(\mathrm{A})}$$

$$\mathbf{S}_{12}^{(AB)} = \mathbf{D}\mathbf{S}_{12}^{(\mathrm{B})}$$

$$\mathbf{S}_{21}^{(AB)} = \mathbf{F}\mathbf{S}_{21}^{(\mathrm{A})}$$

$$\mathbf{S}_{22}^{(AB)} = \mathbf{S}_{22}^{(\mathrm{B})} + \mathbf{F}\mathbf{S}_{22}^{(\mathrm{A})}\mathbf{S}_{12}^{(\mathrm{B})}$$

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Using the Redheffer Star Product as an Update

The global scattering matrix is updated using a Redheffer star product.

Close attention MUST be paid to the order that the equations are implemented so that values are not overwritten.

$$\mathbf{S}^{(\mathrm{global})} = \mathbf{S}^{(\mathrm{global})} \otimes \mathbf{S}^{(i)}$$

$$\mathbf{D} = \mathbf{S}_{12}^{(\mathrm{global})}\left[\mathbf{I} - \mathbf{S}_{11}^{(i)}\mathbf{S}_{22}^{(\mathrm{global})}\right]^{-1}$$

$$\mathbf{F} = \mathbf{S}_{21}^{(i)}\left[\mathbf{I} - \mathbf{S}_{22}^{(\mathrm{global})}\mathbf{S}_{11}^{(i)}\right]^{-1}$$

standard order ↓

$$\mathbf{S}_{11}^{(\mathrm{global})} = \mathbf{S}_{11}^{(\mathrm{global})} + \mathbf{D}\mathbf{S}_{11}^{(i)}\mathbf{S}_{21}^{(\mathrm{global})}$$

$$\mathbf{S}_{12}^{(\mathrm{global})} = \mathbf{D}\mathbf{S}_{12}^{(i)}$$

$$\mathbf{S}_{21}^{(\mathrm{global})} = \mathbf{F}\mathbf{S}_{21}^{(\mathrm{global})}$$

$$\mathbf{S}_{22}^{(\mathrm{global})} = \mathbf{S}_{22}^{(i)} + \mathbf{F}\mathbf{S}_{22}^{(\mathrm{global})}\mathbf{S}_{12}^{(i)}$$

$$\mathbf{S}^{(\mathrm{global})} = \mathbf{S}^{(i)} \otimes \mathbf{S}^{(\mathrm{global})}$$

$$\mathbf{D} = \mathbf{S}_{12}^{(i)}\left[\mathbf{I} - \mathbf{S}_{11}^{(\mathrm{global})}\mathbf{S}_{22}^{(i)}\right]^{-1}$$

$$\mathbf{F} = \mathbf{S}_{21}^{(\mathrm{global})}\left[\mathbf{I} - \mathbf{S}_{22}^{(i)}\mathbf{S}_{11}^{(\mathrm{global})}\right]^{-1}$$

reverse order ↑

$$\mathbf{S}_{22}^{(\mathrm{global})} = \mathbf{S}_{22}^{(\mathrm{global})} + \mathbf{F}\mathbf{S}_{22}^{(i)}\mathbf{S}_{12}^{(\mathrm{global})}$$

$$\mathbf{S}_{21}^{(\mathrm{global})} = \mathbf{F}\mathbf{S}_{21}^{(i)}$$

$$\mathbf{S}_{12}^{(\mathrm{global})} = \mathbf{D}\mathbf{S}_{12}^{(\mathrm{global})}$$

$$\mathbf{S}_{11}^{(\mathrm{global})} = \mathbf{S}_{11}^{(i)} + \mathbf{D}\mathbf{S}_{11}^{(\mathrm{global})}\mathbf{S}_{21}^{(i)}$$

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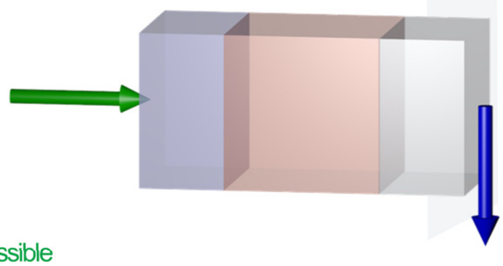
Can TMM Fail?

Yes!

The TMM can fail to give an answer and behave numerically strange any time $k_z = 0$. This happens at a critical angle when the transmitted wave is at or very near its cutoff.

This problem was prevented in the gap medium, but this can also happen in any of the physical layers or in the transmission region under the right conditions.



This happens in any layer where

$$\mu_r \varepsilon_r = \tilde{k}_x^2 + \tilde{k}_y^2$$

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Block Diagram of TMM

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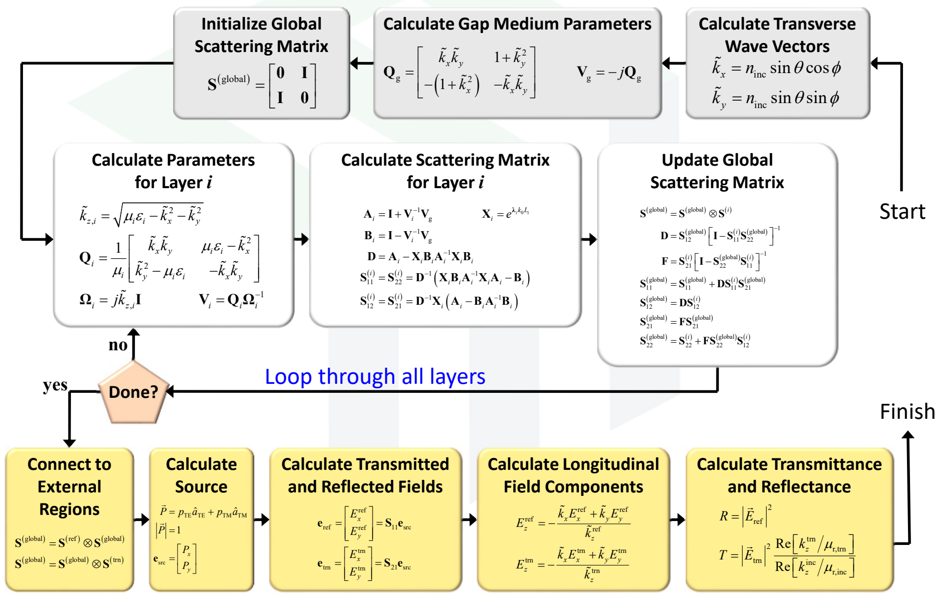
Block Diagram of TMM Using S-Matrices



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How to Handle Zero Number of Layers

Follow the block diagram!!

Setup your loop this way…

```
NLAY = length(L);
for nlay = 1 : NLAY
    ...
end
```

For zero layers:

```
ER = [];
UR = [];
L  = [];
```

If $\text{NLAY} = 0$, then the loop will not execute and the global scattering matrix will remain as it was initialized.

$$\mathbf{S}^{(\text{global})} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{bmatrix}$$

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