



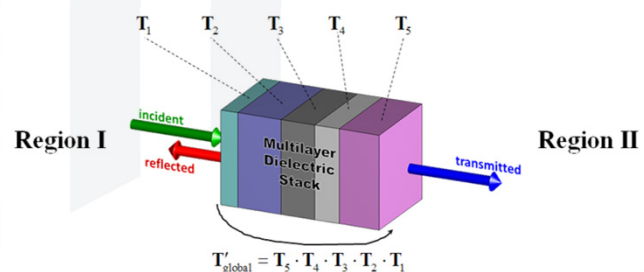
Advanced Computation:
Computational Electromagnetics

Transfer Matrices



Outline

- Concept of transfer matrices
- Wave vector components
- Calculating transfer matrices
- Stability of transfer matrices

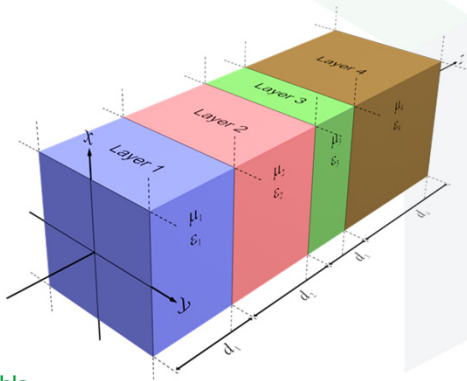


Concept of Transfer Matrices

Slide 3

Concept of Transfer Matrices (1 of 15)

Suppose it is desired to simulate four dielectric slabs.

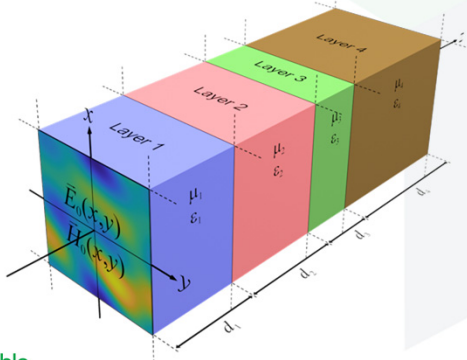


EMPossible

Slide 4

Concept of Transfer Matrices (2 of 15)

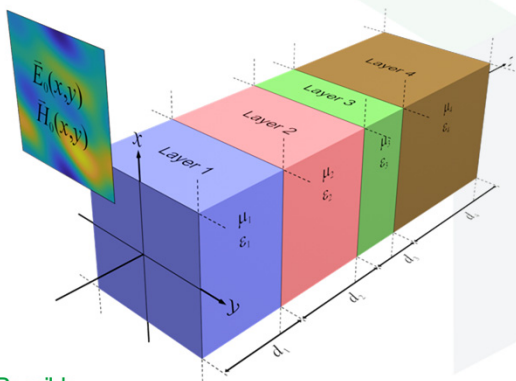
The process starts by defining the field at the input face of the device.



$$\begin{bmatrix} \vec{E}_0(x,y) \\ \vec{H}_0(x,y) \end{bmatrix}$$

Concept of Transfer Matrices (3 of 15)

How can the field at the first interface inside the device be calculated?

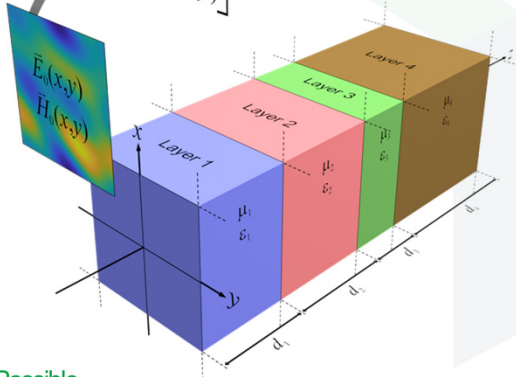


$$\begin{bmatrix} \vec{E}_1(x,y) \\ \vec{H}_1(x,y) \end{bmatrix} = ?$$

$$\begin{bmatrix} \vec{E}_0(x,y) \\ \vec{H}_0(x,y) \end{bmatrix}$$

Concept of Transfer Matrices (4 of 15)

$$\begin{bmatrix} \bar{E}_1(x,y) \\ \bar{H}_1(x,y) \end{bmatrix} = \begin{bmatrix} t_{11}^{(1)} & t_{12}^{(1)} \\ t_{21}^{(1)} & t_{22}^{(1)} \end{bmatrix} \begin{bmatrix} \bar{E}_0(x,y) \\ \bar{H}_0(x,y) \end{bmatrix}$$

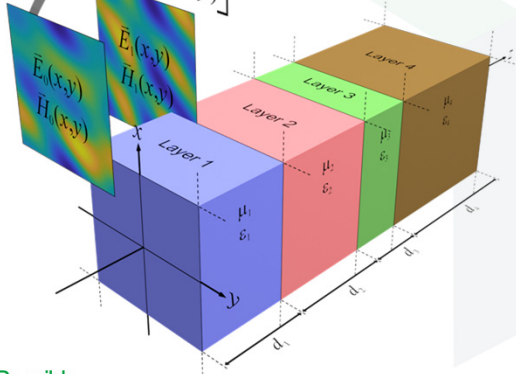


The first transfer matrix $T^{(1)}$ is calculated by analyzing Layer 1.

$$\begin{bmatrix} \bar{E}_1(x,y) \\ \bar{H}_1(x,y) \end{bmatrix} = ? \quad \begin{bmatrix} \bar{E}_0(x,y) \\ \bar{H}_0(x,y) \end{bmatrix}$$

Concept of Transfer Matrices (5 of 15)

$$\begin{bmatrix} \bar{E}_1(x,y) \\ \bar{H}_1(x,y) \end{bmatrix} = \begin{bmatrix} t_{11}^{(1)} & t_{12}^{(1)} \\ t_{21}^{(1)} & t_{22}^{(1)} \end{bmatrix} \begin{bmatrix} \bar{E}_0(x,y) \\ \bar{H}_0(x,y) \end{bmatrix}$$

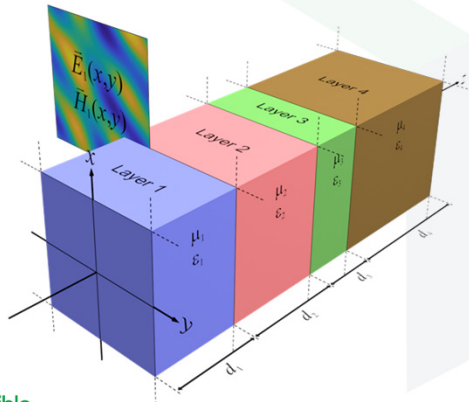


The transfer matrix $T^{(1)}$ is used to calculate the field at the first interface inside the device.

$$\begin{bmatrix} \bar{E}_1(x,y) \\ \bar{H}_1(x,y) \end{bmatrix} = \begin{bmatrix} t_{11}^{(1)} & t_{12}^{(1)} \\ t_{21}^{(1)} & t_{22}^{(1)} \end{bmatrix} \begin{bmatrix} \bar{E}_0(x,y) \\ \bar{H}_0(x,y) \end{bmatrix}$$

Concept of Transfer Matrices (6 of 15)

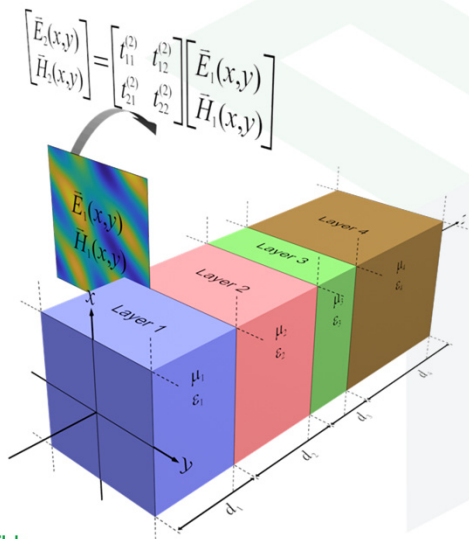
Now the field at the first interface inside the device is known.



$$\begin{bmatrix} \bar{E}_1(x,y) \\ \bar{H}_1(x,y) \end{bmatrix} = \begin{bmatrix} t_{11}^{(1)} & t_{12}^{(1)} \\ t_{21}^{(1)} & t_{22}^{(1)} \end{bmatrix} \begin{bmatrix} \bar{E}_0(x,y) \\ \bar{H}_0(x,y) \end{bmatrix}$$

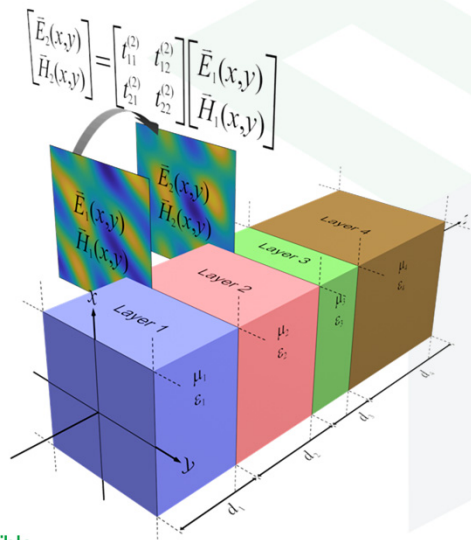
Concept of Transfer Matrices (7 of 15)

The second transfer matrix $T^{(2)}$ is calculated by analyzing Layer 2.



$$\begin{bmatrix} \bar{E}_1(x,y) \\ \bar{H}_1(x,y) \end{bmatrix} = \begin{bmatrix} t_{11}^{(1)} & t_{12}^{(1)} \\ t_{21}^{(1)} & t_{22}^{(1)} \end{bmatrix} \begin{bmatrix} \bar{E}_0(x,y) \\ \bar{H}_0(x,y) \end{bmatrix}$$

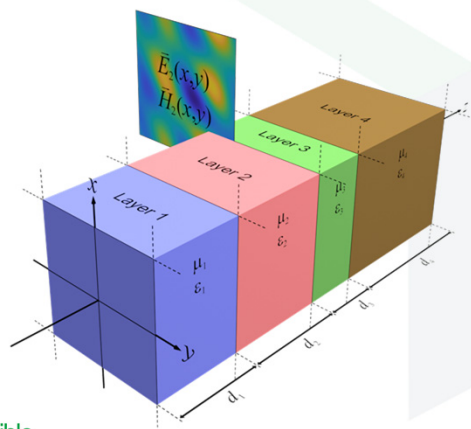
Concept of Transfer Matrices (8 of 15)



The second transfer matrix $T^{(2)}$ is used to calculate the field at the second interface inside the device.

$$\begin{bmatrix} \vec{E}_2(x,y) \\ \vec{H}_2(x,y) \end{bmatrix} = \begin{bmatrix} t_{11}^{(2)} & t_{12}^{(2)} \\ t_{21}^{(2)} & t_{22}^{(2)} \end{bmatrix} \begin{bmatrix} t_{11}^{(1)} & t_{12}^{(1)} \\ t_{21}^{(1)} & t_{22}^{(1)} \end{bmatrix} \begin{bmatrix} \vec{E}_0(x,y) \\ \vec{H}_0(x,y) \end{bmatrix}$$

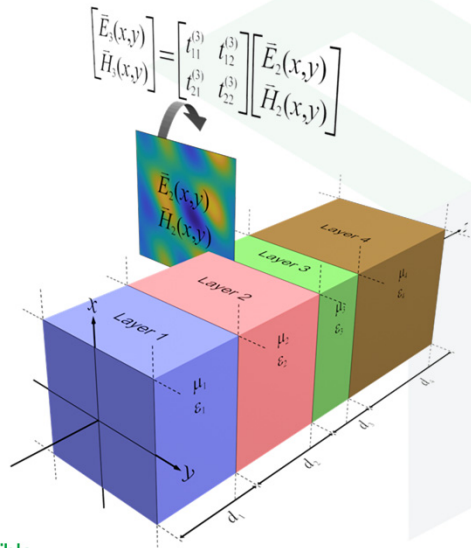
Concept of Transfer Matrices (9 of 15)



Now the field at the second interface inside the device is known.

$$\begin{bmatrix} \vec{E}_3(x,y) \\ \vec{H}_3(x,y) \end{bmatrix} = \begin{bmatrix} t_{11}^{(2)} & t_{12}^{(2)} \\ t_{21}^{(2)} & t_{22}^{(2)} \end{bmatrix} \begin{bmatrix} t_{11}^{(1)} & t_{12}^{(1)} \\ t_{21}^{(1)} & t_{22}^{(1)} \end{bmatrix} \begin{bmatrix} \vec{E}_0(x,y) \\ \vec{H}_0(x,y) \end{bmatrix}$$

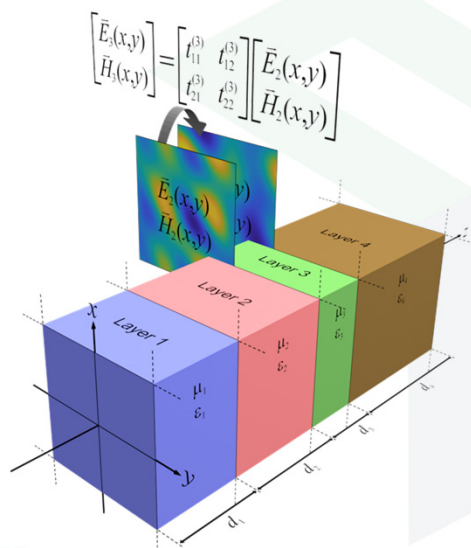
Concept of Transfer Matrices (10 of 15)



The third transfer matrix $T^{(3)}$ is calculated by analyzing Layer 3.

$$\begin{bmatrix} \bar{E}_2(x,y) \\ \bar{H}_2(x,y) \end{bmatrix} = \begin{bmatrix} t_{11}^{(2)} & t_{12}^{(2)} \\ t_{21}^{(2)} & t_{22}^{(2)} \end{bmatrix} \begin{bmatrix} t_{11}^{(1)} & t_{12}^{(1)} \\ t_{21}^{(1)} & t_{22}^{(1)} \end{bmatrix} \begin{bmatrix} \bar{E}_0(x,y) \\ \bar{H}_0(x,y) \end{bmatrix}$$

Concept of Transfer Matrices (11 of 15)

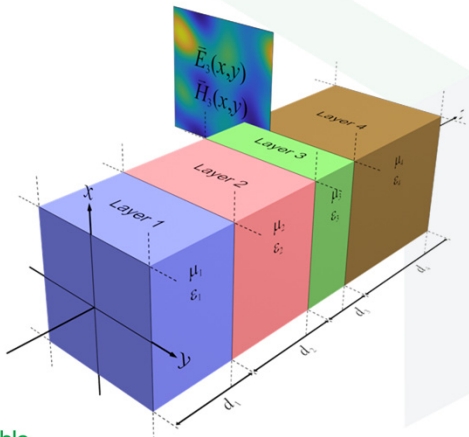


The third transfer matrix $T^{(3)}$ is used to calculate the field at the third interface inside the device.

$$\begin{bmatrix} \bar{E}_3(x,y) \\ \bar{H}_3(x,y) \end{bmatrix} = \begin{bmatrix} t_{11}^{(3)} & t_{12}^{(3)} \\ t_{21}^{(3)} & t_{22}^{(3)} \end{bmatrix} \begin{bmatrix} t_{11}^{(2)} & t_{12}^{(2)} \\ t_{21}^{(2)} & t_{22}^{(2)} \end{bmatrix} \begin{bmatrix} t_{11}^{(1)} & t_{12}^{(1)} \\ t_{21}^{(1)} & t_{22}^{(1)} \end{bmatrix} \begin{bmatrix} \bar{E}_0(x,y) \\ \bar{H}_0(x,y) \end{bmatrix}$$

Concept of Transfer Matrices (12 of 15)

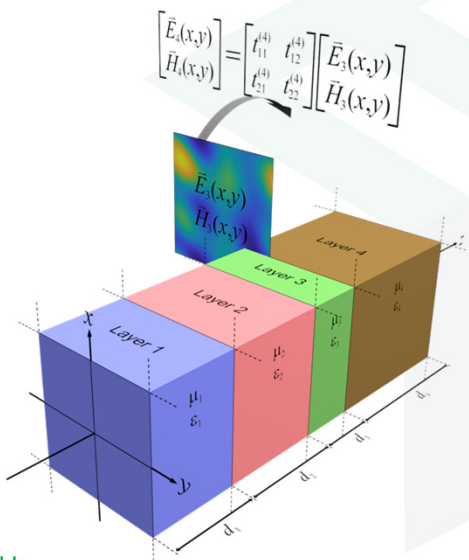
Now the field at the third interface inside the device is known.



$$\begin{bmatrix} \bar{E}_3(x,y) \\ \bar{H}_3(x,y) \end{bmatrix} = \begin{bmatrix} t_{11}^{(3)} & t_{12}^{(3)} \\ t_{21}^{(3)} & t_{22}^{(3)} \end{bmatrix} \begin{bmatrix} t_{11}^{(2)} & t_{12}^{(2)} \\ t_{21}^{(2)} & t_{22}^{(2)} \end{bmatrix} \begin{bmatrix} t_{11}^{(1)} & t_{12}^{(1)} \\ t_{21}^{(1)} & t_{22}^{(1)} \end{bmatrix} \begin{bmatrix} \bar{E}_0(x,y) \\ \bar{H}_0(x,y) \end{bmatrix}$$

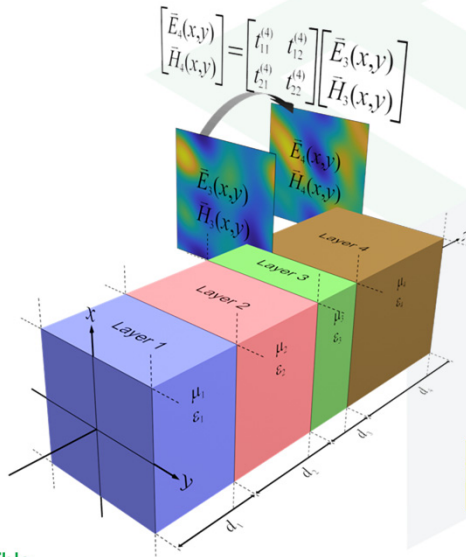
Concept of Transfer Matrices (13 of 15)

The fourth transfer matrix $T^{(4)}$ is calculated by analyzing Layer 4.



$$\begin{bmatrix} \bar{E}_3(x,y) \\ \bar{H}_3(x,y) \end{bmatrix} = \begin{bmatrix} t_{11}^{(3)} & t_{12}^{(3)} \\ t_{21}^{(3)} & t_{22}^{(3)} \end{bmatrix} \begin{bmatrix} t_{11}^{(2)} & t_{12}^{(2)} \\ t_{21}^{(2)} & t_{22}^{(2)} \end{bmatrix} \begin{bmatrix} t_{11}^{(1)} & t_{12}^{(1)} \\ t_{21}^{(1)} & t_{22}^{(1)} \end{bmatrix} \begin{bmatrix} \bar{E}_0(x,y) \\ \bar{H}_0(x,y) \end{bmatrix}$$

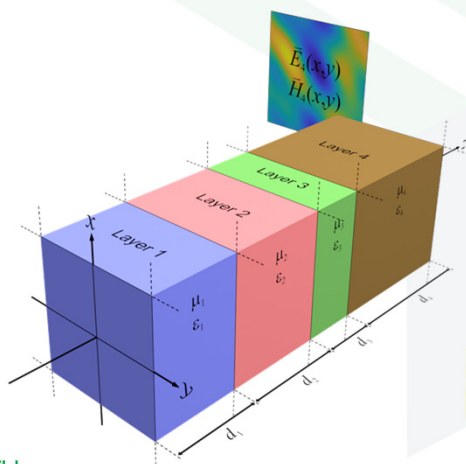
Concept of Transfer Matrices (14 of 15)



The fourth transfer matrix $T^{(4)}$ is used to calculate the field at the output face of the device.

$$\begin{bmatrix} \bar{E}_4(x, y) \\ \bar{H}_4(x, y) \end{bmatrix} = \begin{bmatrix} t_{11}^{(4)} & t_{12}^{(4)} \\ t_{21}^{(4)} & t_{22}^{(4)} \end{bmatrix} \begin{bmatrix} t_{11}^{(3)} & t_{12}^{(3)} \\ t_{21}^{(3)} & t_{22}^{(3)} \end{bmatrix} \begin{bmatrix} t_{11}^{(2)} & t_{12}^{(2)} \\ t_{21}^{(2)} & t_{22}^{(2)} \end{bmatrix} \begin{bmatrix} t_{11}^{(1)} & t_{12}^{(1)} \\ t_{21}^{(1)} & t_{22}^{(1)} \end{bmatrix} \begin{bmatrix} \bar{E}_0(x, y) \\ \bar{H}_0(x, y) \end{bmatrix}$$

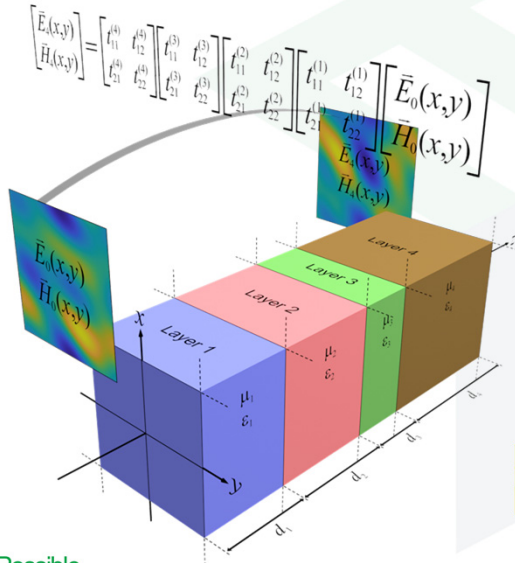
Concept of Transfer Matrices (15 of 15)



Finally the field at the output face the device is known.

$$\begin{bmatrix} \bar{E}_4(x, y) \\ \bar{H}_4(x, y) \end{bmatrix} = \begin{bmatrix} t_{11}^{(4)} & t_{12}^{(4)} \\ t_{21}^{(4)} & t_{22}^{(4)} \end{bmatrix} \begin{bmatrix} t_{11}^{(3)} & t_{12}^{(3)} \\ t_{21}^{(3)} & t_{22}^{(3)} \end{bmatrix} \begin{bmatrix} t_{11}^{(2)} & t_{12}^{(2)} \\ t_{21}^{(2)} & t_{22}^{(2)} \end{bmatrix} \begin{bmatrix} t_{11}^{(1)} & t_{12}^{(1)} \\ t_{21}^{(1)} & t_{22}^{(1)} \end{bmatrix} \begin{bmatrix} \bar{E}_0(x, y) \\ \bar{H}_0(x, y) \end{bmatrix}$$

The Global Transfer Matrix



Alternatively, all of the intermediate transfer matrices can be multiplied together to calculate the global transfer matrix that directly relates the field at the input and output faces.

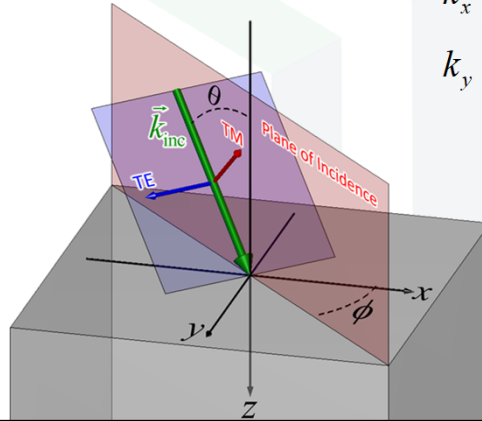
Global Transfer Matrix
 $T = T^{(4)} T^{(3)} T^{(2)} T^{(1)}$

$$\begin{bmatrix} \bar{E}_4(x,y) \\ \bar{H}_4(x,y) \end{bmatrix} = \begin{bmatrix} t_{11}^{(4)} & t_{12}^{(4)} \\ t_{21}^{(4)} & t_{22}^{(4)} \end{bmatrix} \begin{bmatrix} t_{11}^{(3)} & t_{12}^{(3)} \\ t_{21}^{(3)} & t_{22}^{(3)} \end{bmatrix} \begin{bmatrix} t_{11}^{(2)} & t_{12}^{(2)} \\ t_{21}^{(2)} & t_{22}^{(2)} \end{bmatrix} \begin{bmatrix} t_{11}^{(1)} & t_{12}^{(1)} \\ t_{21}^{(1)} & t_{22}^{(1)} \end{bmatrix} \begin{bmatrix} \bar{E}_0(x,y) \\ \bar{H}_0(x,y) \end{bmatrix}$$

Wave Vector Components

Calculation of the Wave Vector Components

The components k_x and k_y are determined by the incident wave and are equal throughout the entire device. The k_z component is different in each layer and is calculated from the dispersion relation in that layer.



$$k_x = k_{x,inc} = k_0 \sqrt{\mu_{r,inc} \epsilon_{r,inc}} \sin \theta \cos \phi$$

$$k_y = k_{y,inc} = k_0 \sqrt{\mu_{r,inc} \epsilon_{r,inc}} \sin \theta \sin \phi$$

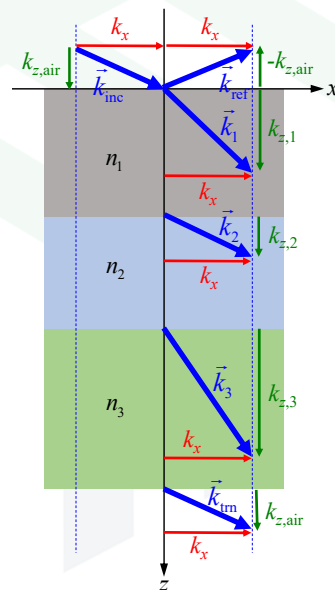
$$k_{z,i} = \sqrt{k_0^2 \mu_{r,i} \epsilon_{r,i} - k_x^2 - k_y^2}$$

$i \equiv \text{Layer \#}$

k_x and k_y are Continuous Throughout Device

$$k_{x,inc} = k_0 n_{air} \cos \phi \sin \theta$$

$$k_{z,inc} = k_0 n_{air} \cos \theta$$



$$k_{z,air}^2 = -[(k_0 n_{air})^2 - k_x^2]$$

$$k_{z,1}^2 = (k_0 n_1)^2 - k_x^2$$

$$k_{z,2}^2 = (k_0 n_2)^2 - k_x^2$$

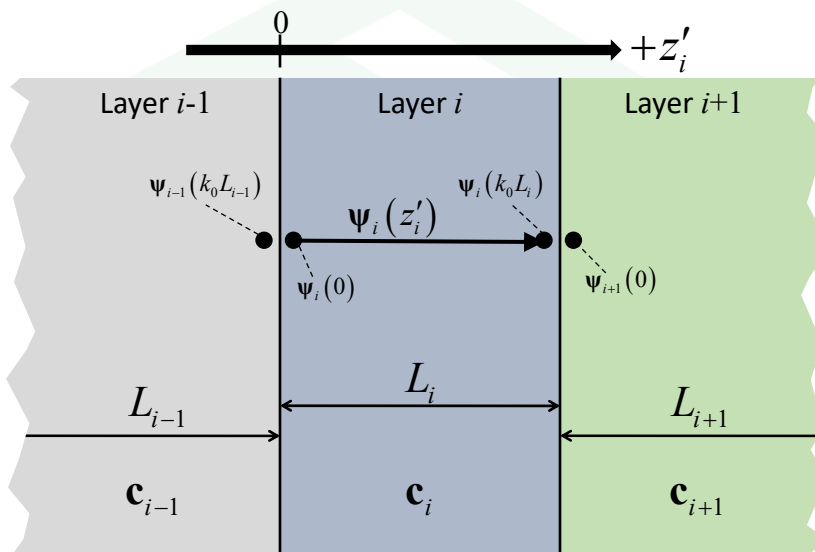
$$k_{z,3}^2 = (k_0 n_3)^2 - k_x^2$$

$$k_{z,tm}^2 = (k_0 n_{air})^2 - k_x^2 = k_{z,air}^2$$

Calculating Transfer Matrices

Slide 23

Geometry of an Intermediate Layer



Slide 24

Enforce the Boundary Conditions

Field inside the i^{th} layer:

$$\Psi_i(z'_i) = \begin{bmatrix} E_{x,i}(z'_i) \\ E_{y,i}(z'_i) \\ \tilde{H}_{x,i}(z'_i) \\ \tilde{H}_{y,i}(z'_i) \end{bmatrix} = \mathbf{W}_i e^{\lambda_i z'_i} \mathbf{c}_i$$

Boundary conditions at the first interface:

$$\Psi_{i-1}(k_0 L_{i-1}) = \Psi_i(0)$$

$$\mathbf{W}_{i-1} e^{\lambda_{i-1} k_0 L_{i-1}} \mathbf{c}_{i-1} = \mathbf{W}_i \mathbf{c}_i$$

Must include k_0 in the exponential to normalize L_{i-1} because the parameter λ_{i-1} expects to multiply a normalized coordinate.

Boundary conditions at the second interface:

$$\Psi_i(k_0 L_i) = \Psi_{i+1}(0)$$

$$\mathbf{W}_i e^{\lambda_i k_0 L_i} \mathbf{c}_i = \mathbf{W}_{i+1} \mathbf{c}_{i+1}$$

Note: Must equate the field Ψ on either side of the interfaces and not the mode coefficients \mathbf{c} .

The Transfer Matrix

The transfer matrix \mathbf{T}_i of the i^{th} layer is defined as:

$$\mathbf{c}_{i+1} = \mathbf{T}_i \cdot \mathbf{c}_i$$

Start with the boundary condition equation from the second interface and rearrange terms...

$$\mathbf{W}_i e^{\lambda_i k_0 L_i} \mathbf{c}_i = \mathbf{W}_{i+1} \mathbf{c}_{i+1} \quad \rightarrow \quad \mathbf{c}_{i+1} = \underbrace{\mathbf{W}_{i+1}^{-1} \mathbf{W}_i e^{\lambda_i k_0 L_i}}_{\mathbf{T}_i} \mathbf{c}_i$$

...then read off the transfer matrix. $\mathbf{T}_i = \mathbf{W}_{i+1}^{-1} \mathbf{W}_i e^{\lambda_i k_0 L_i}$

Stability of Transfer Matrices

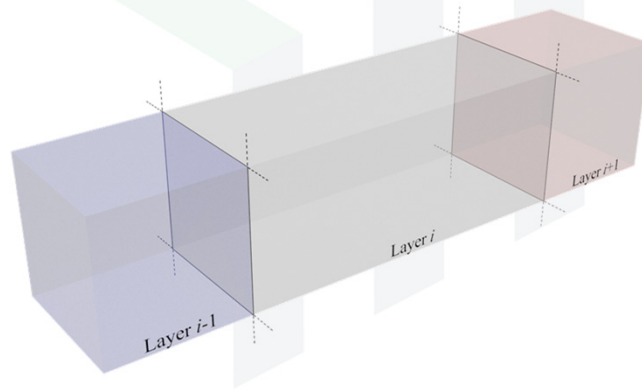


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Slide 27

The Multi-Layer Problem

The diagram below is focused on the i th layer somewhere in the middle of a stack of multiple layers.



EMPossible

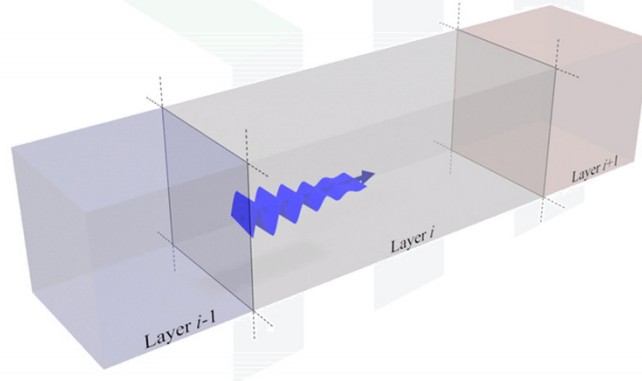
Slide 28

Forward Waves in the i th Layer

Recall that the wave number k is in general complex. Waves can oscillate, decay or both.

$$k^+ = k' + jk''$$

oscillation
decay

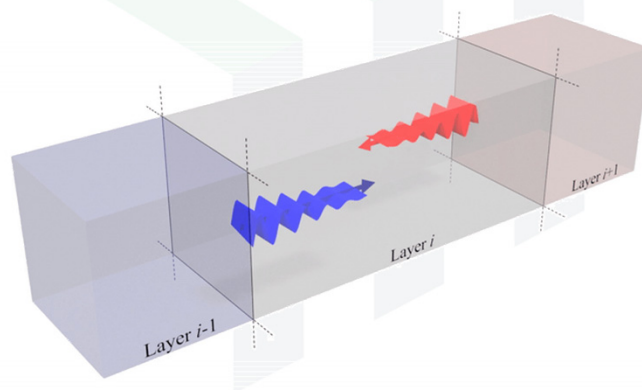


Backward Waves in the i th Layer

Due to reflections at the interfaces, there will be backward waves. These can also oscillate, decay or both.

$$k^+ = k' + jk''$$

oscillation
decay



$$k^- = k' + jk''$$

oscillation
decay

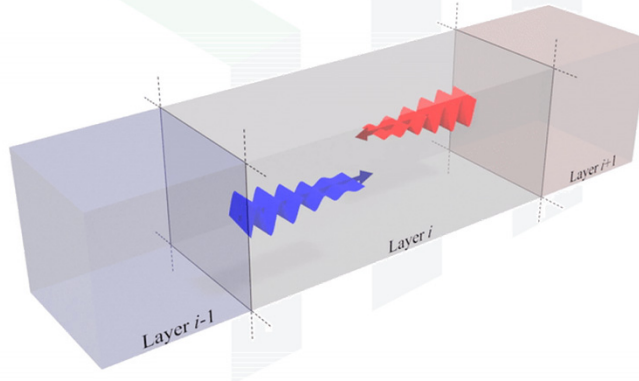
Waves According to Transfer Matrices

In terms of signs, transfer matrices treat all waves as forward propagating, even the backward waves.

$$\Psi_i(z'_i) = \begin{bmatrix} E_{x,i}(z'_i) \\ E_{y,i}(z'_i) \\ \tilde{H}_{x,i}(z'_i) \\ \tilde{H}_{y,i}(z'_i) \end{bmatrix} = \mathbf{W}_i e^{\lambda_i z'_i} \mathbf{c}_i$$

$$k^+ = k' + jk''$$

oscillation decay



$$k^- = k' + jk''$$

oscillation decay

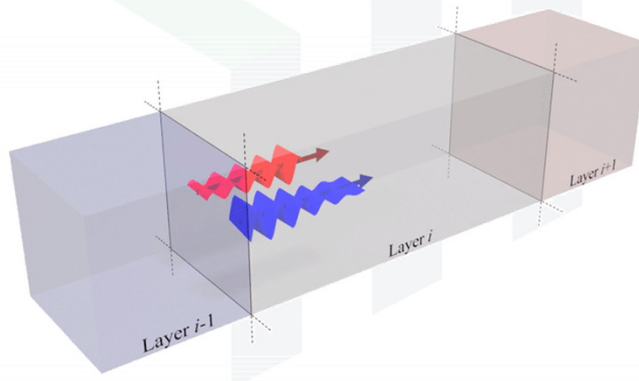
?

Pure Transfer Matrices are Unstable

Transfer matrices treat all waves as if they are forward propagating. Backward waves that also decay increase exponentially and become numerically unstable.

$$k^+ = k' + jk''$$

oscillation decay



$$k^- = -k' - jk''$$

backward oscillation explode

The Fix – Identify Forward and Backward Waves

Clearly, the first part of the fix is to identify forward and backward propagating waves.

This can be accomplished by calculating the Poynting vector $\vec{\rho}$ associated with the eigen-modes and looking at the sign of the z component. Be careful! A normalized magnetic field is being used.

$$\vec{\rho} = \vec{E} \times \vec{H}^*$$

$$\rho_z = E_x H_y^* - E_y H_x^*$$

$$\rho_z = E_x \left(\frac{\tilde{H}_y}{j\eta_0} \right)^* - E_y \left(\frac{\tilde{H}_x}{j\eta_0} \right)^*$$

$$\rho_z = \frac{1}{-j\eta_0} (E_x \tilde{H}_y^* - E_y \tilde{H}_x^*)$$

$$\mathbf{W} = \begin{bmatrix} -i0.32 & i0.32 & 0 & 0 \\ 0 & 0 & i0.32 & -i0.32 \\ 0 & 0 & 0.95 & 0.95 \\ 0.95 & 0.95 & 0 & 0 \end{bmatrix}$$

+ - + -

Rearrange the Eigen Modes

Now that it is known which eigen-modes are forward and backward propagating, they can be rearranged to group them together.

Original Matrices		Rearranged Matrices
$\mathbf{W} = \begin{bmatrix} -i0.32 & i0.32 & 0 & 0 \\ 0 & 0 & i0.32 & -i0.32 \\ 0 & 0 & 0.95 & 0.95 \\ 0.95 & 0.95 & 0 & 0 \end{bmatrix}$	$\xrightarrow{\text{rearrange eigen-modes}}$	$\mathbf{W}' = \begin{bmatrix} -i0.32 & 0 & i0.32 & 0 \\ 0 & i0.32 & 0 & -i0.32 \\ 0 & 0.95 & 0 & 0.95 \\ 0.95 & 0 & 0.95 & 0 \end{bmatrix}$
$\lambda = \begin{bmatrix} i3.0 & 0 & 0 & 0 \\ 0 & -i3.0 & 0 & 0 \\ 0 & 0 & i3.0 & 0 \\ 0 & 0 & 0 & -i3.0 \end{bmatrix}$		$\lambda' = \begin{bmatrix} i3.0 & 0 & 0 & 0 \\ 0 & i3.0 & 0 & 0 \\ 0 & 0 & -i3.0 & 0 \\ 0 & 0 & 0 & -i3.0 \end{bmatrix}$
+ - + -		+ + - -

Also need to adjust the vertical positions of the eigen-values so that λ' remains a diagonal matrix.

New Interpretation of the Matrices

$$\mathbf{W}' = \begin{bmatrix} -i0.32 & 0 & i0.32 & 0 \\ 0 & i0.32 & 0 & -i0.32 \\ 0 & 0.95 & 0 & 0.95 \\ 0.95 & 0 & 0.95 & 0 \end{bmatrix} \begin{matrix} E_x \\ E_y \\ \tilde{H}_x \\ \tilde{H}_y \end{matrix}$$

$$\boldsymbol{\lambda}' = \begin{bmatrix} i3.0 & 0 & 0 & 0 \\ 0 & i3.0 & 0 & 0 \\ 0 & 0 & -i3.0 & 0 \\ 0 & 0 & 0 & -i3.0 \end{bmatrix}$$

+ + - -

The matrices are now partitioned into forward and backward propagating elements.

$$\mathbf{W}' = \begin{bmatrix} \mathbf{W}_E^+ & \mathbf{W}_E^- \\ \mathbf{W}_H^+ & \mathbf{W}_H^- \end{bmatrix} \begin{matrix} E_x \\ E_y \\ \tilde{H}_x \\ \tilde{H}_y \end{matrix}$$

$$e^{\boldsymbol{\lambda}'z'} = \begin{bmatrix} e^{\lambda^+ z'} & \mathbf{0} \\ \mathbf{0} & e^{-\lambda^- z'} \end{bmatrix}$$

Note: For anisotropic materials, all the eigenvectors and eigen-values are in general unique.

$$\lambda^+ = \begin{bmatrix} i3.0 & 0 \\ 0 & i3.0 \end{bmatrix} \quad \lambda^- = \begin{bmatrix} -i3.0 & 0 \\ 0 & -i3.0 \end{bmatrix}$$

Revised Solution to Differential Equation

The matrix differential equation and its original solution was

$$\frac{d\boldsymbol{\Psi}}{dz'} - \boldsymbol{\Omega}\boldsymbol{\Psi} = \mathbf{0} \quad \rightarrow \quad \boldsymbol{\Psi}(z') = \mathbf{W}e^{\boldsymbol{\lambda}'z'} \mathbf{c}$$

After distinguishing between forward and backward propagating waves and grouping them in the matrices, the solution can be written as

$$\boldsymbol{\Psi}(z') = \begin{bmatrix} \mathbf{W}_E^+ & \mathbf{W}_E^- \\ \mathbf{W}_H^+ & \mathbf{W}_H^- \end{bmatrix} \begin{bmatrix} e^{\lambda^+ z'} & \mathbf{0} \\ \mathbf{0} & e^{-\lambda^- z'} \end{bmatrix} \begin{bmatrix} \mathbf{c}^+ \\ \mathbf{c}^- \end{bmatrix}$$

There are now separate mode coefficients \mathbf{c}^+ and \mathbf{c}^- for forward and backward propagating modes, respectively.

