Advanced Electromagnetics:
21st Century Electromagnetics

Dispersion Relation & Index Ellipsoids

Lecture Outline

- Dispersion relation
- Dispersion surfaces
- Index ellipsoids

Dispersion Relation

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The Wave Vector $\vec{k}$

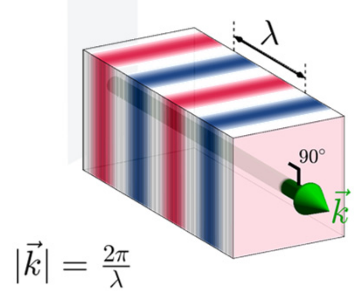
The wave vector (wave momentum) is a vector quantity that conveys two pieces of information:

1. **Wavelength and Refractive Index** – The magnitude of the wave vector conveys the spatial period λ (i.e. wavelength) of the wave inside the material. When the frequency is known, the magnitude $\left|\vec{k}\right|$ conveys the material's refractive index n (more to be said later).

$$\left|\vec{k}\right| = \frac{2\pi}{\lambda} = \frac{2\pi n}{\lambda_0} \qquad \lambda_0 \equiv \text{free space wavelength}$$

2. **Direction** – The direction of the wave is perpendicular to the wave fronts (more to be said later).

$$\vec{k} = k_a\hat{a} + k_b\hat{b} + k_c\hat{c}$$



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The Dispersion Relation

The dispersion relation for a material relates the wave vector $\vec{k}$ to frequency ω. Essentially, it sets a rule for the values of $\vec{k}$ as a function of direction and frequency.

For an ordinary linear, homogeneous and isotropic (LHI) material, the dispersion relation is:

$$k_a^2 + k_b^2 + k_c^2 = \left(\frac{\omega n}{c_0}\right)^2$$

This can also be written as: $\frac{k_a^2 + k_b^2 + k_c^2}{n^2} - k_0^2 = 0 \qquad k_0 = \frac{\omega}{c_0}$

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How to Derive the Dispersion Relation (1 of 2)

The wave equation in a linear homogeneous anisotropic material is:

$$\nabla \times \nabla \times \vec{E} - k_0^2 \mu_0 [\varepsilon_r] \vec{E} = 0$$

Assume no magnetic response (i.e. $\mu_r = 1$).

The solution to this equation is still a plane wave, but the allowed values for $\vec{k}$ (modes) are more complicated.

$$\vec{E} = \vec{E}_0 e^{-j\vec{k}\bullet\vec{r}} \qquad \vec{E}_0 = E_a\hat{a} + E_b\hat{b} + E_c\hat{c}$$

Substituting this solution into the wave equation leads to the following relation:

$$\vec{k}\left(\vec{k}\bullet\vec{E}_0\right) - \left|\vec{k}\right|\vec{E}_0 + k_0^2[\varepsilon_r]\vec{E}_0 = 0$$

This equation has the form: $(\bullet\bullet\bullet)\hat{a} + (\bullet\bullet\bullet)\hat{b} + (\bullet\bullet\bullet)\hat{c} = 0$

Each (•••) term has the form: $(\cdots)E_a + (\cdots)E_b + (\cdots)E_c = 0$

Each vector component must be set to zero independently.

$\hat{a}$ component: $(\cdots)E_a + (\cdots)E_b + (\cdots)E_c = 0$

$\hat{b}$ component: $(\cdots)E_a + (\cdots)E_b + (\cdots)E_c = 0$

$\hat{c}$ component: $(\cdots)E_a + (\cdots)E_b + (\cdots)E_c = 0$

Matrix form...

$$\begin{bmatrix} (\ldots) & (\ldots) & (\ldots) \\ (\ldots) & (\ldots) & (\ldots) \\ (\ldots) & (\ldots) & (\ldots) \end{bmatrix}\begin{bmatrix} E_a \\ E_b \\ E_c \end{bmatrix} = 0$$

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How to Derive the Dispersion Relation (2 of 2)

Solutions for $\vec{k}$ are the eigen-values of the big matrix and derived by setting the determinant to zero.

$$\det\begin{bmatrix}(\ldots) & (\ldots) & (\ldots)\\(\ldots) & (\ldots) & (\ldots)\\(\ldots) & (\ldots) & (\ldots)\end{bmatrix}=0$$

This leads to the following general equation:

$$\frac{k_a^2}{\left|\vec{k}\right|^2-k_0^2n_a^2}+\frac{k_b^2}{\left|\vec{k}\right|^2-k_0^2n_b^2}+\frac{k_c^2}{\left|\vec{k}\right|^2-k_0^2n_c^2}=1$$

It can also be shown that given the wave vector $\vec{k}$, the polarization of the electric field $\vec{E}_0$ is:

$$\vec{E}_0=\left(\frac{k_a}{\left|\vec{k}\right|^2-k_0^2n_a^2}\right)\hat{a}+\left(\frac{k_b}{\left|\vec{k}\right|^2-k_0^2n_b^2}\right)\hat{b}+\left(\frac{k_c}{\left|\vec{k}\right|^2-k_0^2n_c^2}\right)\hat{c}$$

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Dispersion Relation for Anisotropic Media

Given the dielectric tensor…

$$\varepsilon_{\mathrm{r}}=\begin{bmatrix}\varepsilon_a & 0 & 0\\0 & \varepsilon_b & 0\\0 & 0 & \varepsilon_c\end{bmatrix}=\begin{bmatrix}n_a^2 & 0 & 0\\0 & n_b^2 & 0\\0 & 0 & n_c^2\end{bmatrix}$$

The general form of the dispersion relation is:

$$\frac{k_a^2}{\left|\vec{k}\right|^2-k_0^2n_a^2}+\frac{k_b^2}{\left|\vec{k}\right|^2-k_0^2n_b^2}+\frac{k_c^2}{\left|\vec{k}\right|^2-k_0^2n_c^2}=1$$

This can be written in a more useful form as:

$$\left|\vec{k}\right|^2\left(\frac{k_a^2}{n_b^2n_c^2}+\frac{k_b^2}{n_a^2n_c^2}+\frac{k_c^2}{n_a^2n_b^2}\right)-k_0^2\left(\frac{k_b^2+k_c^2}{n_a^2}+\frac{k_a^2+k_c^2}{n_b^2}+\frac{k_a^2+k_b^2}{n_c^2}\right)+k_0^4=1$$

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Dispersion Relation for Uniaxial Crystals

Uniaxial crystals have

$$n_a = n_b = n_o \qquad n_o \equiv \text{ordinary refractive index}$$

$$n_c = n_e \qquad n_e \equiv \text{extraordinary refractive index}$$

The general dispersion relation reduces to:

$$\underbrace{\left(\frac{k_a^2 + k_b^2 + k_c^2}{n_e^2} - k_0^2\right)}_{\substack{\text{Sphere} \\ \text{Ordinary Wave}}}\underbrace{\left(\frac{k_a^2 + k_b^2}{n_o^2} + \frac{k_c^2}{n_e^2} - k_0^2\right)}_{\substack{\text{Ellipse} \\ \text{Extraordinary Wave}}} = 0$$

This has two solutions corresponding to the two polarizations (TE and TM).

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The first solution is the same solution for an isotropic material. The wave behaves like it is propagating through a isotropic material with index n_o so it is called the "ordinary wave."

The second solution is an ellipsoid. Depending on its direction, the effective refractive index will be somewhere between n_o and n_e.

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Dispersion Surfaces

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Dispersion Surface

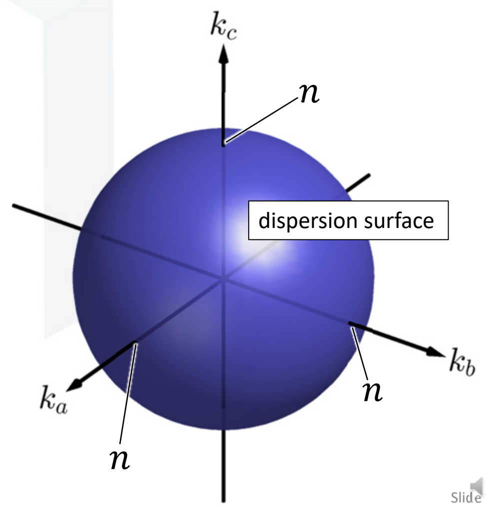
Observe that the dispersion relation for a LHI material is the equation for a sphere:

$$k_a^2 + k_b^2 + k_c^2 = k_0^2 n^2 \qquad \left|\vec{k}\right| = k_0 n$$

This sphere has many names: *dispersion surface*, *k-surface*, and *momentum surface*.

For LHI materials, the index ellipsoid is a sphere indicating that the magnitude of the wave vector is constant in all directions.

This implies the refractive index is constant is all directions for LHI media.



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Dispersion Surfaces for Uniaxial Crystals (1 of 2)

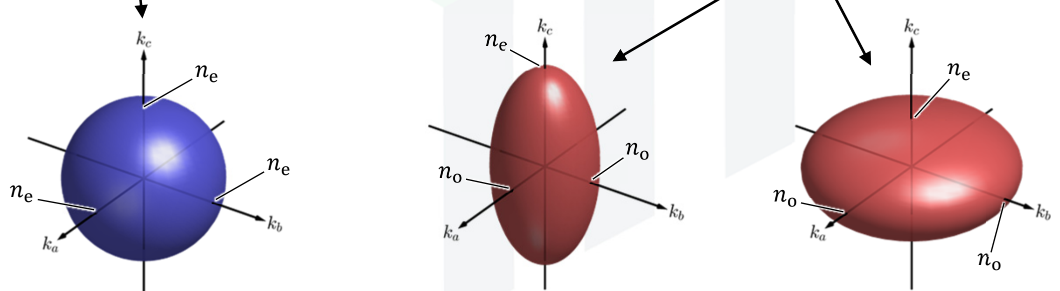
$$\left(\frac{k_a^2 + k_b^2 + k_c^2}{n_e^2} - k_0^2\right)\left(\frac{k_a^2 + k_b^2}{n_o^2} + \frac{k_c^2}{n_e^2} - k_0^2\right) = 0$$

Ordinary Wave

$$\left(\frac{k_a^2 + k_b^2 + k_c^2}{n_e^2} - k_0^2\right) = 0$$

Extraordinary Wave

$$\left(\frac{k_a^2 + k_b^2}{n_o^2} + \frac{k_c^2}{n_e^2} - k_0^2\right) = 0$$



$n_e > n_o$ $\qquad$ $n_e < n_o$

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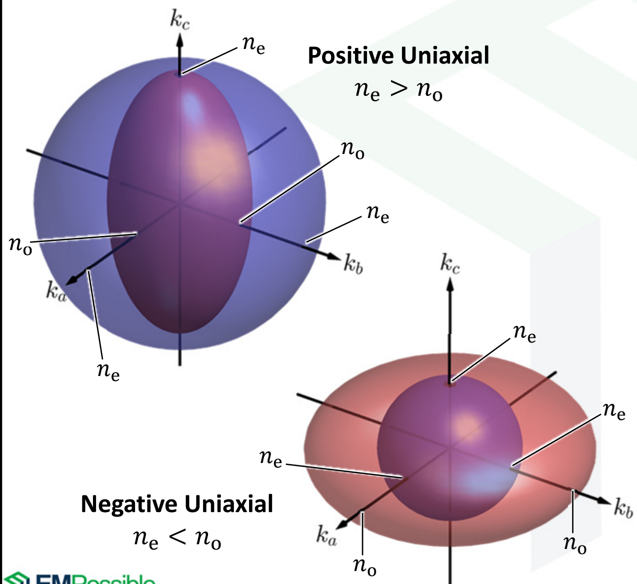
Dispersion Surfaces for Uniaxial Crystals (2 of 2)



<u>Observations</u>

- Both solutions share a common axis.
- This common axis looks isotropic with refractive index n_e regardless of direction.
- Since both solutions share only a single common axis, these crystals are called *uniaxial*.
- The common axis is called:
  - *Optic axis*
  - *Ordinary axis*
  - *C axis*
  - *Uniaxial axis*
- Deviation from the optic axis will result in two separate possible modes.

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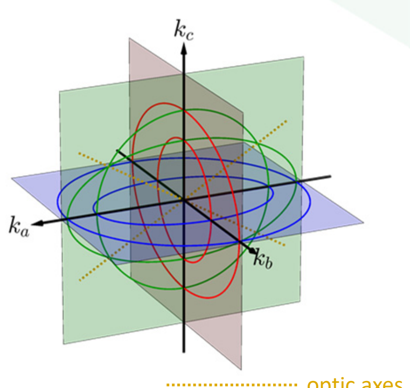
Dispersion Surface for Biaxial Crystals (1 of 2)

Biaxial crystals have three unique refractive indices. Most texts adopt the convention where

$$n_a < n_b < n_c$$

The general dispersion relation cannot be reduced. ☹



<u>Notes and Observations</u>

- The convention $n_a < n_b < n_c$ causes the optic axes to lie in the $\hat{a}$-$\hat{c}$ plane.
- The two solutions can be envisioned as one balloon inside another, pinched together so they touch at only four points.
- Propagation along either of the two optic axes looks isotropic, thus the name *biaxial*.

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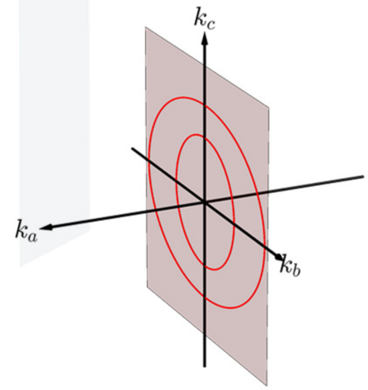
Dispersion Surface for Biaxial Crystals (2 of 2)

There are three special cases when the biaxial case can be simplified. These conditions can be produced in practice by launching electromagnetic waves at the proper orientation. Each special case has two separate solutions corresponding to the two possible polarizations.

$$k_a = 0: \quad \left(k_b^2 + k_c^2 - k_0^2 n_a^2\right)\left(\frac{k_b^2}{n_c^2} + \frac{k_c^2}{n_b^2} - k_0^2\right) = 0$$

$$k_b = 0: \quad \left(k_a^2 + k_c^2 - k_0^2 n_b^2\right)\left(\frac{k_a^2}{n_c^2} + \frac{k_c^2}{n_a^2} - k_0^2\right) = 0$$

$$k_c = 0: \quad \left(k_a^2 + k_b^2 - k_0^2 n_c^2\right)\left(\frac{k_a^2}{n_b^2} + \frac{k_b^2}{n_a^2} - k_0^2\right) = 0$$



Dispersion Surface for Biaxial Crystals (2 of 2)

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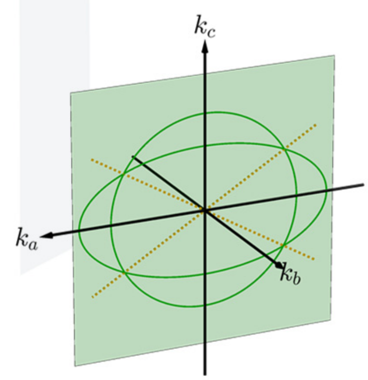
$$k_b = 0: \quad \left(k_a^2 + k_c^2 - k_0^2 n_b^2\right)\left(\frac{k_a^2}{n_c^2} + \frac{k_c^2}{n_a^2} - k_0^2\right) = 0$$

$$k_c = 0: \quad \left(k_a^2 + k_b^2 - k_0^2 n_c^2\right)\left(\frac{k_a^2}{n_b^2} + \frac{k_b^2}{n_a^2} - k_0^2\right) = 0$$



Dispersion Surface for Biaxial Crystals (2 of 2)

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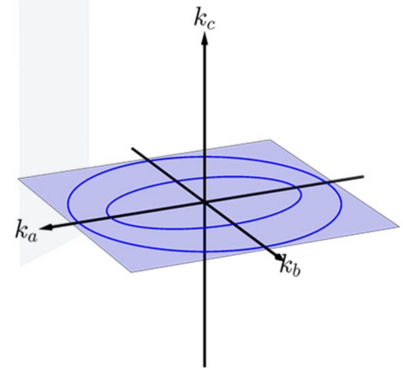
$$k_c = 0: \quad \left(k_a^2 + k_b^2 - k_0^2 n_c^2\right)\left(\frac{k_a^2}{n_b^2} + \frac{k_b^2}{n_a^2} - k_0^2\right) = 0$$



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Dispersion Surface for Biaxial Crystals (2 of 2)

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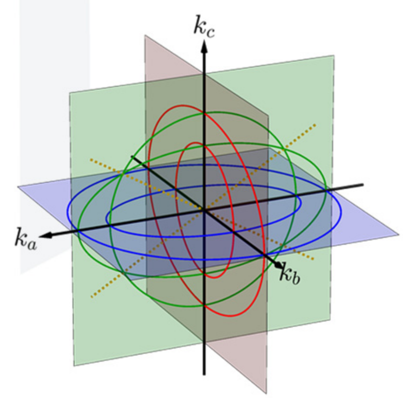
$$k_b = 0: \quad \left(k_a^2 + k_c^2 - k_0^2 n_b^2\right)\left(\frac{k_a^2}{n_c^2} + \frac{k_c^2}{n_a^2} - k_0^2\right) = 0$$

$$k_c = 0: \quad \left(k_a^2 + k_b^2 - k_0^2 n_c^2\right)\left(\frac{k_a^2}{n_b^2} + \frac{k_b^2}{n_a^2} - k_0^2\right) = 0$$



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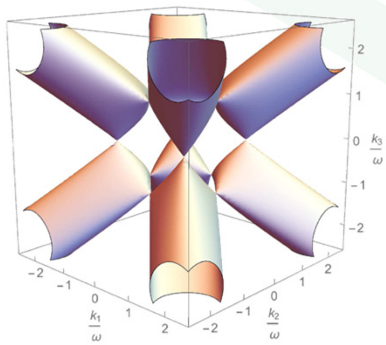
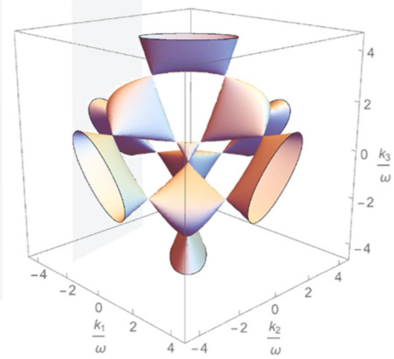
Dispersion Surfaces of Magnetoelectric Materials

Magnetoelectric materials can exhibit up to 16 singularities.

$$\vec{D}=[\varepsilon]\vec{E}+[\zeta]\vec{H} \quad \text{and} \quad \vec{B}=[\xi]\vec{H}+[\varepsilon]\vec{E}$$





Alberto Favaro and Friedrich W. Hehl, "Light propagation in local and linear media: Fresnel-Kummer wave surfaces with 16 singular points," arXiv preprint arXiv:1510.05566 (2015).

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Index Ellipsoid

Index Ellipsoid for LHI Media

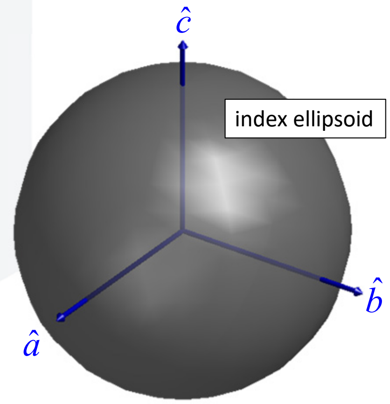
Dispersion surfaces and index ellipsoids are essentially the same thing. They are just scaled by a constant.

$$k_a^2 + k_b^2 + k_c^2 = k_0^2 n^2 \qquad \left|\vec{k}\right| = k_0 n$$

$$\Downarrow$$

$$x^2 + y^2 + z^2 = n^2$$

The surface becomes a map of refractive index as a function of direction of the waves.



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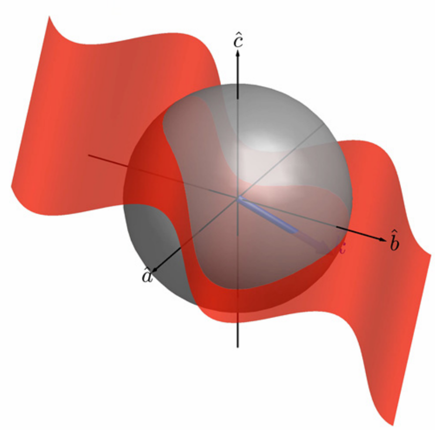
Physical Interpretation of Index Ellipsoid in Isotropic Media

Refractive index is the same in all directions.

$$n_a = n_b = n_c$$

Wavelength is the same in all directions.



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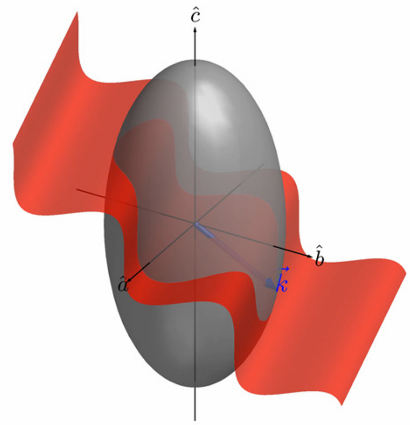
Physical Interpretation of Index Ellipsoid in Positive Uniaxial Media

Refractive index is higher in one direction than the other two directions.

$$n_a = n_b < n_c$$

Wavelength is smaller for waves propagating in the high-index direction.

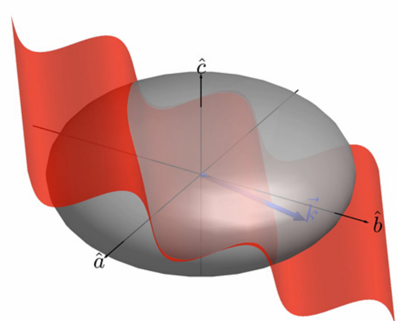


Physical Interpretation of Index Ellipsoid in Negative Uniaxial Media

Refractive index is lower in one direction than the other two directions.

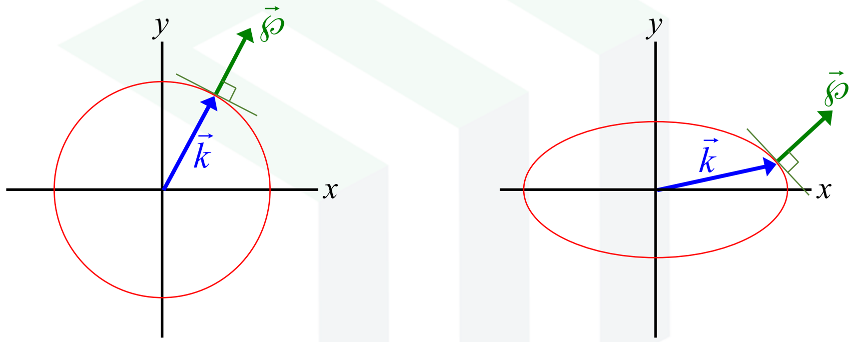
$$n_a = n_b > n_c$$

Wavelength is larger for waves propagating in the low-index direction.



Direction of Power Flow

Isotropic Materials | Anisotropic Materials



Phase propagates in the direction of $\vec{k}$. Therefore, the refractive index derived from $|\vec{k}|$ is best described as the phase refractive index n_{p}. Velocity here is the phase velocity $\vec{v}_{\text{p}}$.

Power propagates in the direction of the Poynting vector $\vec{\wp}$ which is always normal to the surface of the index ellipsoid. From this, group velocity $\vec{v}_{\text{g}}$ and group refractive index n_{g} can be defined.

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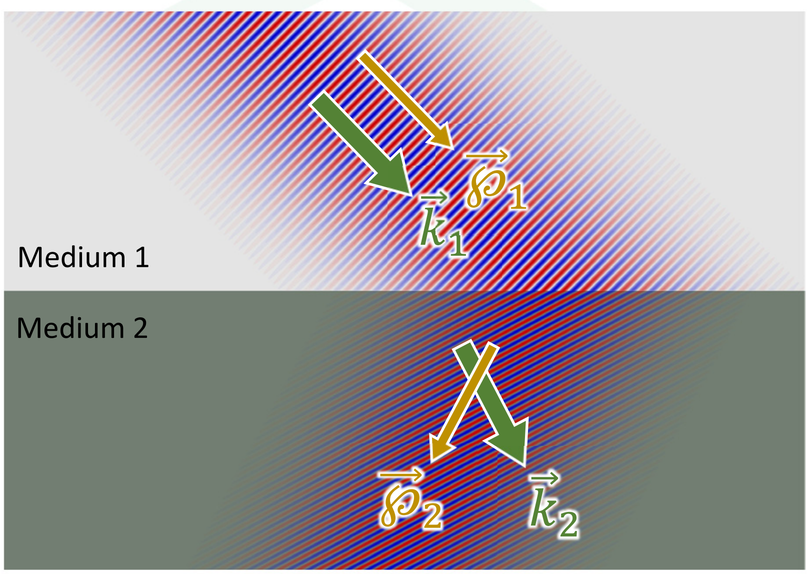
Animation of $\vec{k}$ versus $\vec{\wp}$



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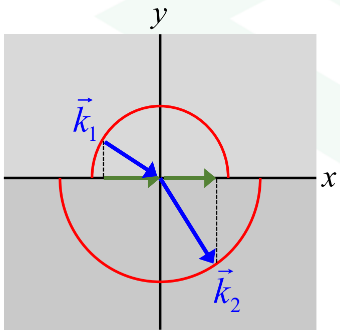
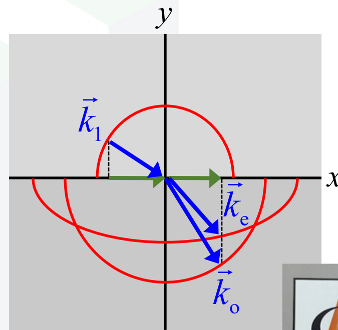
Double Refraction in Anisotropic Materials

Isotropic Materials



Anisotropic Materials



Anisotropic materials have two index ellipsoids – one for each polarization.
Wave power can split between the two to produce both an ordinary and an extraordinary wave.

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