Advanced Electromagnetics:
21st Century Electromagnetics

Drude Model for Metals

Lecture Outline

- Derivation of Drude model
- Plasmons & the plasma frequency ω_{p}
- Notes & observations

Derivation of Drude Model

Drude Model for Metals

In metals, most electrons are free because they are not bound to a nucleus. For this reason, the restoring force is negligible and there is no natural frequency (i.e. $\omega_0 = 0$).

The Drude model for metals is derived from the Lorentz model by setting $\omega_0 = 0$.

$$\tilde{\varepsilon}_{\mathrm{r}}(\omega) = 1 + \frac{\omega_{\mathrm{p}}^2}{\cancel{\omega_0^2} - \omega^2 - j\omega\Gamma}$$

$$\omega_{\mathrm{p}}^2 = \frac{N_{\mathrm{e}} q^2}{\varepsilon_0 m_{\mathrm{e}}}$$

$$\boxed{\tilde{\varepsilon}_{\mathrm{r}}(\omega) = 1 - \frac{\omega_{\mathrm{p}}^2}{\omega^2 + j\omega\Gamma}}$$

N is now interpreted as free electron density N_{e}.
Typical metals have $N_{\mathrm{e}} \sim 10^{22}\ \mathrm{cm}^{-3}$.

m_{e} is the effective mass of the electron.

Typical metals have $\omega_p \sim 10^{16}\ \mathrm{rad/s}$

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Mean Collision Rate τ

When describing metals, it is often more meaningful to put the equation in terms of the *mean collision rate* τ. This is also called the *momentum scattering time*.

$$\tilde{\varepsilon}_{\mathrm{r}}(\omega) = 1 - \frac{\omega_{\mathrm{p}}^2}{\omega^2 + j\omega\tau^{-1}} \qquad \tau = \frac{1}{\Gamma}$$

τ is the average time an electron travels between consecutive collisions.

The collision rate in metals is ~10^{-14} seconds.

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Real & Imaginary Parts of Permittivity

The complex permittivity can be written in terms of the real and imaginary components.

$$\tilde{\varepsilon}_{\mathrm{r}}(\omega) = \tilde{\varepsilon}'_{\mathrm{r}}(\omega) + j\tilde{\varepsilon}''_{\mathrm{r}}(\omega) = 1 - \frac{\omega_{\mathrm{p}}^2}{\omega^2 + j\omega\tau^{-1}}$$

$$\tilde{\varepsilon}'_{\mathrm{r}}(\omega) = 1 - \frac{\omega_{\mathrm{p}}^2\tau^2}{1+\omega^2\tau^2}$$

$$\tilde{\varepsilon}''_{\mathrm{r}}(\omega) = \frac{\omega_{\mathrm{p}}^2\tau/\omega}{1+\omega^2\tau^2}$$

Derivation

$$\begin{aligned}
\tilde{\varepsilon}'_{\mathrm{r}}(\omega) + j\tilde{\varepsilon}''_{\mathrm{r}}(\omega) &= 1 - \frac{\omega_{\mathrm{p}}^2}{\omega^2 + j\omega\tau^{-1}}\frac{\omega^2 - j\omega\tau^{-1}}{\omega^2 - j\omega\tau^{-1}} \\
&= 1 - \frac{\omega_{\mathrm{p}}^2\omega^2 - j\omega_{\mathrm{p}}^2\omega\tau^{-1}}{\omega^4 + \omega^2\tau^{-2}} \\
&= 1 - \frac{\omega\omega_{\mathrm{p}}^2\tau^2 - j\omega_{\mathrm{p}}^2\tau}{\omega^3\tau^2 + \omega} \\
&= 1 - \frac{\omega\omega_{\mathrm{p}}^2\tau^2}{\omega^3\tau^2 + \omega} + j\frac{\omega_{\mathrm{p}}^2\tau}{\omega^3\tau^2 + \omega} \\
&= 1 - \frac{\omega_{\mathrm{p}}^2\tau^2}{\omega^2\tau^2 + 1} + j\frac{\omega_{\mathrm{p}}^2\tau/\omega}{\omega^2\tau^2 + 1}
\end{aligned}$$

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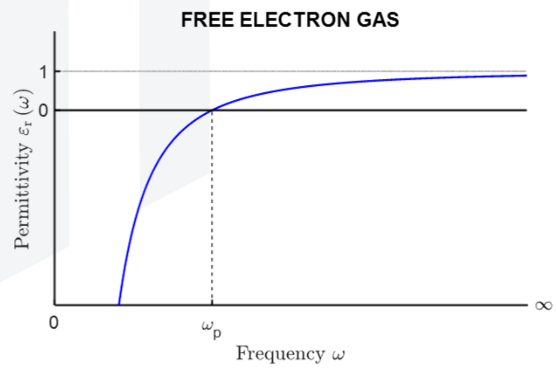
Ideal Metals – Free Electron Gas

The ideal metal has $\tau \rightarrow \infty$ and behaves like a free electron gas.

$$\tilde{\varepsilon}_{\mathrm{r}}(\omega) = \tilde{\varepsilon}'_{\mathrm{r}}(\omega) + j\tilde{\varepsilon}''_{\mathrm{r}}(\omega) = 1 - \frac{\omega_{\mathrm{p}}^2}{\omega^2}$$

$$\tilde{\varepsilon}'_{\mathrm{r}}(\omega) = 1 - \frac{\omega_{\mathrm{p}}^2}{\omega^2}$$

$$\tilde{\varepsilon}''_{\mathrm{r}}(\omega) = 0$$



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Electrical Conductivity Model

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Two Ways to Specify Materials with Loss

Complex Permittivity

$$\nabla \times \vec{H} = j\omega\varepsilon_0\tilde{\varepsilon}_\mathrm{r}\vec{E}$$

Real Permittivity & Real Conductivity

$$\nabla \times \vec{H} = \vec{J} + j\omega\vec{D}$$
$$= \sigma\vec{E} + j\omega\varepsilon_0\varepsilon_\mathrm{r}\vec{E}$$

Both systems have two numbers

$\tilde{\varepsilon}'_\mathrm{r}$ $\quad$ ε_r

$\tilde{\varepsilon}''_\mathrm{r}$ $\quad$ σ

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Relation Between the Systems

Complex Permittivity

$$\nabla \times \vec{H} = j\omega\varepsilon_0\tilde{\varepsilon}_\mathrm{r}\vec{E}$$

Real Permittivity & Real Conductivity

$$\nabla \times \vec{H} = \sigma\vec{E} + j\omega\varepsilon_0\varepsilon_\mathrm{r}\vec{E}$$

$$j\omega\varepsilon_0\tilde{\varepsilon}_\mathrm{r}\vec{E} = \sigma\vec{E} + j\omega\varepsilon_0\varepsilon_\mathrm{r}\vec{E}$$

Set righthand side of these equations equal.

$$j\omega\varepsilon_0\tilde{\varepsilon}_\mathrm{r}\vec{E} = j\omega\varepsilon_0\left(\frac{\sigma}{j\omega\varepsilon_0} + \varepsilon_\mathrm{r}\right)\vec{E}$$

Factor out $j\omega\varepsilon_0$ and $\vec{E}$.

$$\tilde{\varepsilon}_\mathrm{r} = \varepsilon_\mathrm{r} + \frac{\sigma}{j\omega\varepsilon_0}$$

Compare expressions and relate material systems.

$$\tilde{\varepsilon}'_\mathrm{r} = \varepsilon_\mathrm{r} \qquad \tilde{\varepsilon}''_\mathrm{r} = \frac{\sigma}{\omega\varepsilon_0}$$

Set real and imaginary parts equal.

Note $\tilde{\varepsilon}_\mathrm{r} = \tilde{\varepsilon}'_\mathrm{r} - j\tilde{\varepsilon}''_\mathrm{r}$ in the negative sign convention.

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Drude Model for Conductivity $\sigma(\omega)$

From the previous slide, conductivity σ is related to $\tilde{\varepsilon}_r''$ through

$$\tilde{\varepsilon}_r'' = \frac{\sigma}{\omega\varepsilon_0} \longrightarrow \sigma = \omega\varepsilon_0\tilde{\varepsilon}_r''$$

Replace $\tilde{\varepsilon}_r''$ with the expression from the Drude model.

$$\sigma(\omega) = \omega\varepsilon_0\left(\frac{\omega_p^2\tau/\omega}{1+\omega^2\tau^2}\right) \longrightarrow \sigma(\omega) = \frac{\varepsilon_0\omega_p^2\tau}{1+\omega^2\tau^2}$$

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DC Conductivity σ_0

The electrical conductivity at DC is found by setting $\omega = 0$ in the equation for $\sigma(\omega)$.

$$\sigma(\omega) = \frac{\varepsilon_0\omega_p^2\tau}{1+\omega^2\tau^2}$$

$$\sigma(0) = \frac{\varepsilon_0\omega_p^2\tau}{1+(0)^2\tau^2} = \varepsilon_0\omega_p^2\tau \longrightarrow \sigma_0 = \varepsilon_0\omega_p^2\tau$$

DC conductivity

The Drude conductivity $\sigma(\omega)$ can now be written in terms of the DC conductivity σ_0.

$$\sigma(\omega) = \frac{\sigma_0}{1+\omega^2\tau^2}$$

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Notes & Observations

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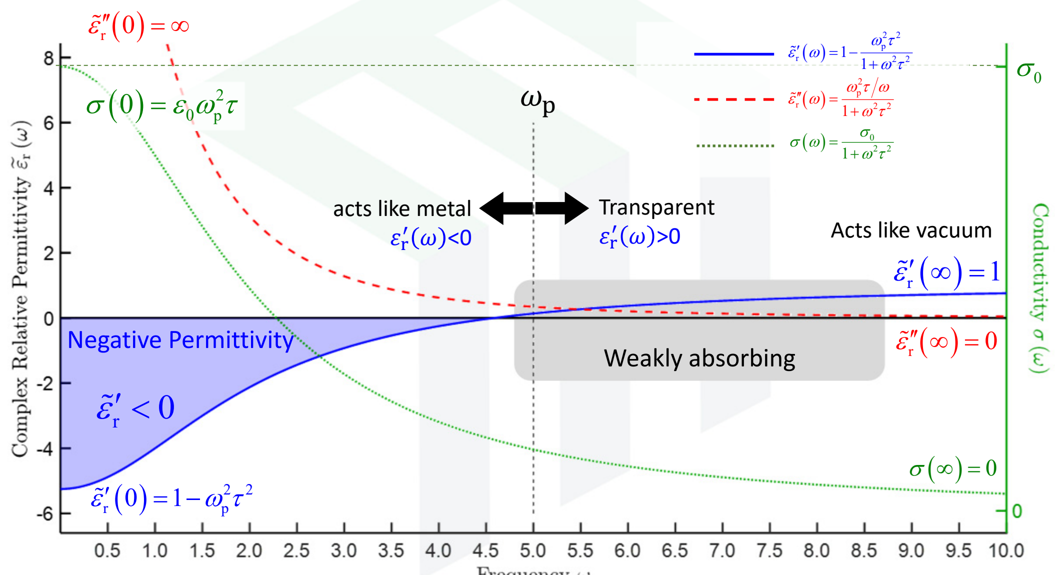
Typical Drude Response for $\tilde{\varepsilon}_{\mathrm{r}}(\omega)$ and $\sigma(\omega)$



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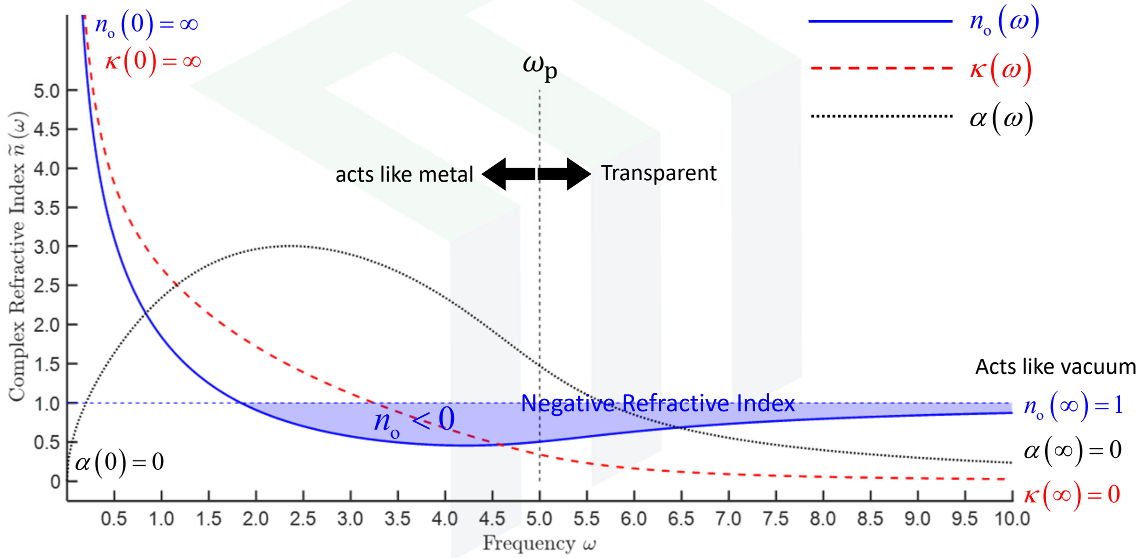
Typical Drude Response for $\tilde{n}(\omega)$ and $\alpha(\omega)$



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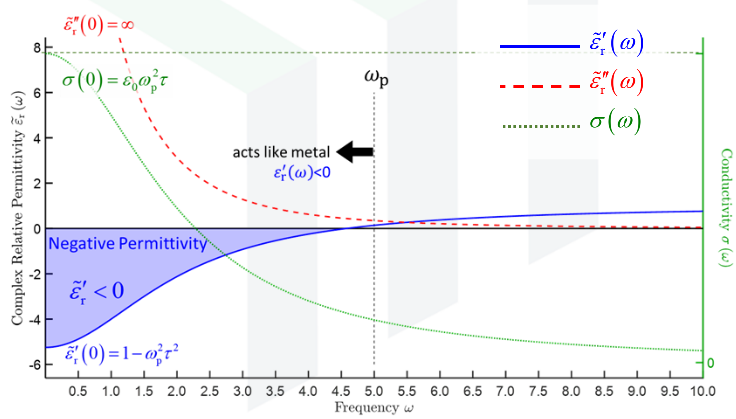
Observation #1 – Below the Plasma Frequency

Below the plasma frequency, the dielectric constant is mostly imaginary and metals behave like good conductors.



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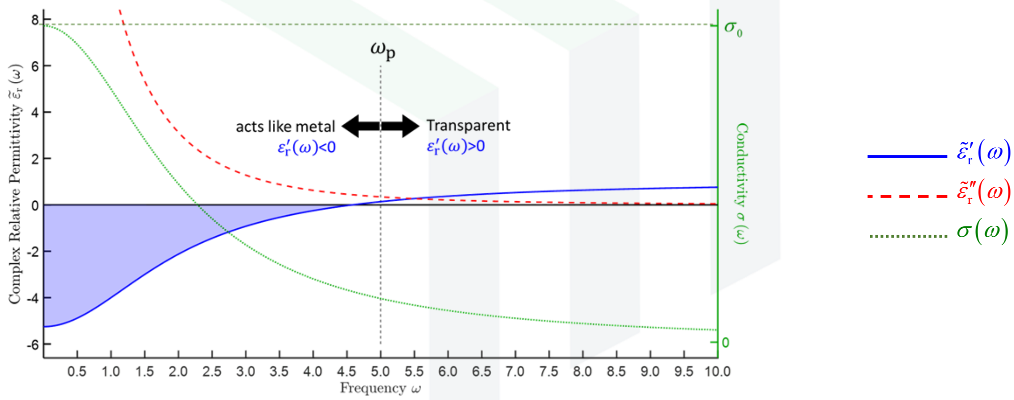
Observation #2 – Near the Plasma Frequency

Near the plasma frequency, both the real and imaginary parts of permittivity are significant and metals are **very lossy**.



This is a big problem for optics and currently the #1 limitation for optical metamaterials.

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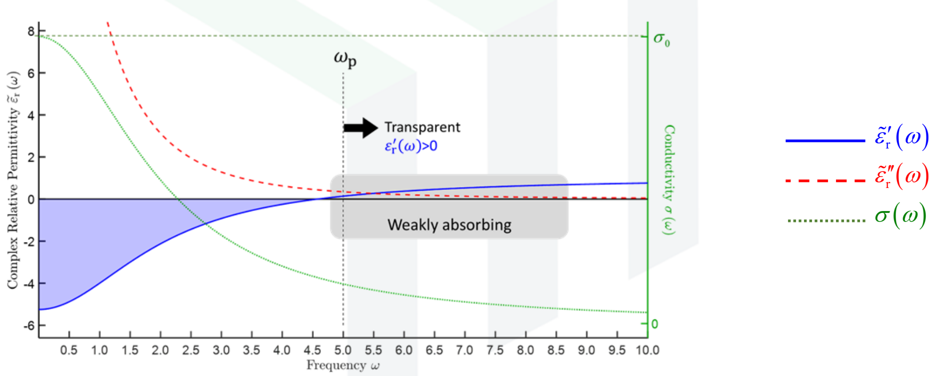
Observation #3 – Above the Plasma Frequency

At very high frequencies above the plasma frequency, loss vanishes and metals become transparent!

Note: more accurately stated as weakly absorbing



This is why x-rays are used to image through things.

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