Advanced Electromagnetics:
21st Century Electromagnetics

Generalizations & Alternative Materials Models

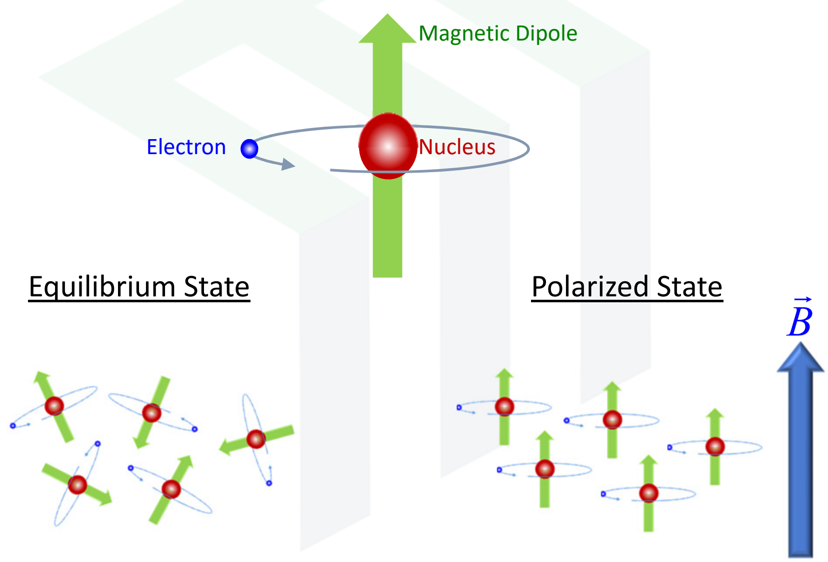
Lecture Outline

- Lorentz model for permeability
- Materials with multiple resonances
- Other materials models

Lorentz Model for Permeability

Slide 3

Magnetic Response of Ordinary Materials



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Slide 4

Lorentz Model for Permeability

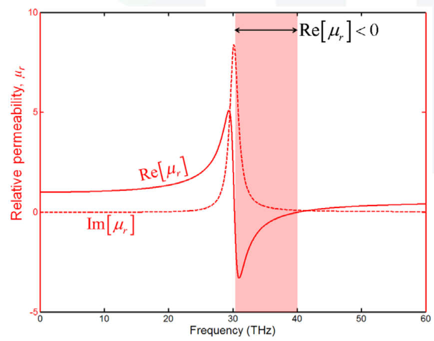
$$\tilde{\mu}_{\mathrm{r}}(\omega) = 1 + \frac{\omega_{\mathrm{mp}}^2}{\omega_{\mathrm{m0}}^2 - \omega^2 - j\omega\Gamma_{\mathrm{m}}}$$

$\omega_{\mathrm{mp}} \equiv$ magnetic plasma frequency

$\omega_{\mathrm{m0}} \equiv$ magnetic resonant frequency

$\Gamma_{\mathrm{m}} \equiv$ magnetic damping rate

Boardman, Allan D., and Kiril Marinov, "Electromagnetic energy in a dispersive metamaterial," Phys. Rev. B, Vol. 73, No.16, pp. 165110, 2006.



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Materials with Multiple Resonances

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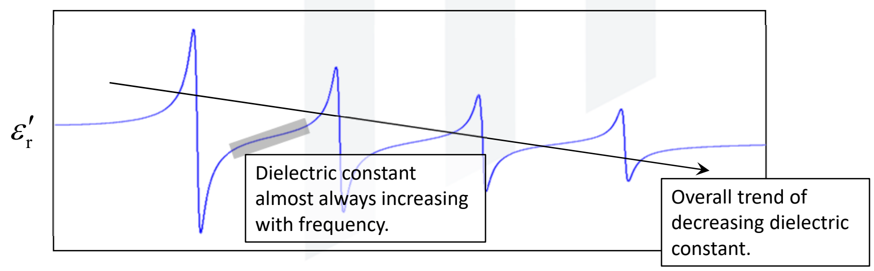
Accounting for Multiple Resonances

At a macroscopic level, all resonance mechanisms can be characterized using the Lorentz model. This allows any number of resonances to be accounted for through a simple summation.

$$\chi_{\mathrm{e}}(\omega)=\omega_{\mathrm{p}}^{2}\sum_{i=1}^{N}\frac{f_i}{\omega_{0,i}^{2}-\omega^{2}-j\omega\Gamma_i} \qquad N_{\mathrm{e}}=\sum_{i=1}^{N}f_i$$

$N \equiv$ Number of resonators
$N_{\mathrm{e}} \equiv$ Number of electrons
$f_i \equiv$ Oscillator strength of the i^{th} resonator
$\omega_{0,i} \equiv$ Natural frequency of the i^{th} resonator
$\Gamma_i \equiv$ Damping rate of the i^{th} resonator



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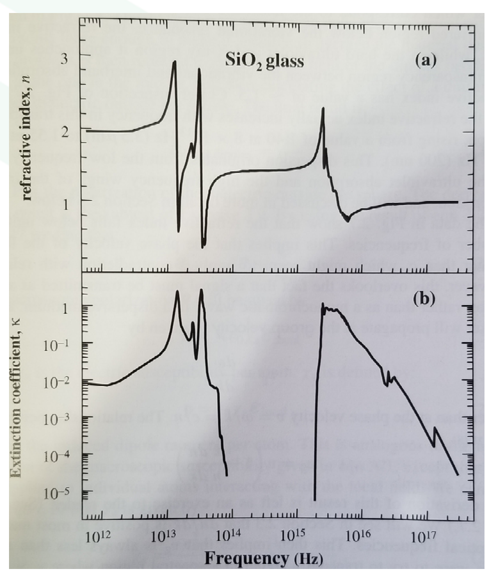
Example – Fused Silica SiO_2

Fox, Mark. "*Optical properties of solids*." Oxford (2002): 1269-1270.



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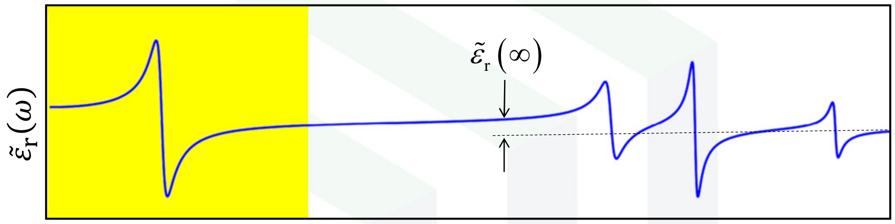
Concept of $\varepsilon_r(\infty)$



Suppose obtaining $\varepsilon_r(\omega)$ only within this span of frequencies is of interest.

In this frequency range, the high frequency resonances only contribute a combined DC offset that is written as $\varepsilon_r(\infty)$.

It is not necessary to resolve the full shape of the high frequency resonances so they are accounted for solely through $\varepsilon_r(\infty)$.

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Generalized Lorentz-Drude Model of Arbitrary Order

A very general equation for modeling complicated dielectrics and metals is the following:

$$\tilde{\varepsilon}_r(\omega) = \tilde{\varepsilon}_r(\infty) + \omega_p^2 \sum_{m=1}^{M} \frac{f_m}{\omega_{0,m}^2 - \omega^2 - j\omega\Gamma_m}$$

This is used to account for the offset produced by resonances at frequencies higher than where you care about.

LORENTZ-DRUDE PARAMETERS (eV)											
Parameter	Ag	Au	Cu	Al	Be	Cr	Ni	Pd	Pt	Ti	W
wp	9.010	9.030	10.830	14.980	18.510	10.750	15.920	9.720	9.590	7.290	13.220
f0	0.845	0.760	0.575	0.523	0.084	0.168	0.096	0.330	0.333	0.148	0.206
G0	0.048	0.053	0.030	0.047	0.035	0.047	0.048	0.008	0.080	0.082	0.064
w0	0	0	0	0	0	0	0	0	0	0	0
f1	0.065	0.024	0.061	0.227	0.031	0.151	0.100	0.649	0.191	0.899	0.054
G1	3.886	0.241	0.378	0.333	1.664	3.175	4.511	2.950	0.517	2.276	0.530
w1	0.816	0.415	0.291	0.162	0.100	0.121	0.174	0.336	0.780	0.777	1.004
f2	0.124	0.010	0.104	0.050	0.140	0.150	0.135	0.121	0.659	0.393	0.166
G2	0.452	0.345	1.056	0.312	3.395	1.305	1.334	0.555	1.838	2.518	1.281
w2	4.481	0.830	2.957	1.544	1.032	0.543	0.582	0.501	1.314	1.545	1.917
f3	0.011	0.071	0.723	0.166	0.530	1.149	0.106	0.638	0.547	0.187	0.706
G3	0.065	0.870	3.213	1.351	4.454	2.676	2.178	4.621	3.668	1.663	3.332
w3	8.185	2.969	5.300	1.808	3.183	1.970	1.597	1.659	3.141	2.509	3.580
f4	0.840	0.601	0.638	0.030	0.130	0.825	0.729	0.453	3.576	0.001	2.590
G4	0.916	2.494	4.305	3.382	1.802	1.335	6.292	3.236	8.517	1.762	5.836
w4	9.083	4.304	11.180	3.473	4.604	8.775	6.089	5.715	9.249	19.430	7.498
f5	5.646	4.384									
G5	2.419	2.214									
w5	20.290	13.320									
Min n:	0.130	0.080	0.100	0.033	1.350	0.310	0.860	0.720	0.730	0.760	0.490

The first resonance is not actually a resonance. Setting $\omega_0 = 0$ converts the Lorentz model into the Drude model.

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Isolated Absorbers in a Transparent Host

The overall material polarization is a superposition of the host and the absorber.

$$\vec{P}_{\text{total}} = \vec{P}_{\text{host}} + \vec{P}_{\text{absorber}}$$

The overall dielectric function is then

$$\tilde{\varepsilon}_{\text{r}} = 1 + \chi_{\text{host}} + \frac{\omega_{\text{p}}^2}{\omega_0^2 - \omega^2 - j\omega\Gamma}$$

At very high frequencies relative to the absorber, this becomes

$$\tilde{\varepsilon}_{\text{r}}(\infty) = 1 + \chi_{\text{host}}$$

At very low frequencies relative to the absorber, this becomes

$$\tilde{\varepsilon}_{\text{r}}(0) = \tilde{\varepsilon}_{\text{r}}(\infty) + \frac{\omega_{\text{p}}^2}{\omega_0^2}$$

$$\tilde{\varepsilon}_{\text{r}}(0) - \tilde{\varepsilon}_{\text{r}}(\infty) = \frac{\omega_{\text{p}}^2}{\omega_0^2}$$

This provides a neat way to measure the plasma frequency.

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Other Material Models

Cole-Cole Models

Cole-Cole models are physics-based compact representations of wideband frequency-dependent dielectric properties or polymers and organic materials.

$$\tilde{\varepsilon}(\omega) = \underbrace{\varepsilon_\infty + \frac{\sigma}{j\omega\varepsilon_0}}_{\text{DC or average response}} + \underbrace{\frac{\Delta\varepsilon}{1+(j\omega\tau)^{(1-\alpha)}}}_{\text{Dispersive response}}$$

$\varepsilon_\infty \geq 1$

$\sigma \geq 0$

$\Delta\varepsilon \geq 0$

$\tau \geq 0$

$\alpha \sim 0.1$

α is an empirical parameter that accounts for the observed broad distribution of relaxation time constants.

K. S. Cole, R. H. Cole, "Dispersion and Absorption in Dielectrics I. Alternating Current Characteristics," J. of Chem. Phys. **9**, 341 (1941).

Cauchy Equation

This is an empirical relationship between refractive index and wavelength for transparent media at optical frequencies.

$$n(\lambda_0) = B + \frac{C}{\lambda_0^2} + \frac{D}{\lambda_0^4} + \cdots$$

$\lambda_0 \equiv$ free space wavelength in micrometers (μm)

B, C, D, etc. are called Cauchy coefficients.

For most materials, only B and C are needed.

$$n(\lambda_0) = B + \frac{C}{\lambda_0^2}$$

Material	B	C (μm²)
Fused silica	1.4580	0.00354
Borosilicate glass BK7	1.5046	0.00420
Hard crown glass K5	1.5220	0.00459
Barium crown glass BaK4	1.5690	0.00531
Barium flint glass BaF10	1.6700	0.00743
Dense flint glass SF10	1.7280	0.01342

Sellmeier Equation

This is an empirical relationship between refractive index and wavelength for transparent media at optical frequencies.

$$n^2(\lambda_0) = 1 + \frac{B_1\lambda_0^2}{\lambda_0^2 - C_1} + \frac{B_2\lambda_0^2}{\lambda_0^2 - C_2} + \frac{B_3\lambda_0^2}{\lambda_0^2 - C_3}$$

$\lambda_0 \equiv$ free space wavelength in micrometers (μm)

B_1, B_2, B_3, C_1, C_2, and C_3 are called Sellmeier coefficients.

Each term represents the contribution of a different resonance to refractive index. B_i is the strength of the resonance while $\sqrt{C_i}$ is the wavelength of the resonance in micrometers.

Crown Glass (BK7)

Coefficient	Value
B_1	1.03961212
B_2	0.231792344
B_3	1.01046945
C_1	6.00069867×10^{-3}
C_2	2.00179144×10^{-2}
C_3	1.03560653×10^{2}

The Sellmeier exists in other forms to account for additional physics.

$$n^2(\lambda_0) = A + \sum_i \frac{B_i\lambda_0^2}{\lambda_0^2 - C_i} \qquad A \equiv n_\infty^2$$

There are other forms that account for temperature, pressure, and other parameters.