



Advanced Electromagnetics:
21st Century Electromagnetics

Kramers-Kronig Relations



Lecture Outline

- Linear Systems
- Derivation
- Kramers-Kronig relations in electromagnetics



Linear Systems

Slide 3

Impulse Response & Transfer Functions

Linear systems can be characterized by their impulse response,

$$f(t) \rightarrow h(t) \rightarrow g(t) \quad g(t) = f(t) * h(t) = \int_{-\infty}^{\infty} f(\tau) h(t-\tau) d\tau$$

...or by their transfer function (Fourier transform of impulse response),

$$F(\omega) \rightarrow H(\omega) \rightarrow G(\omega) \quad G(\omega) = H(\omega) F(\omega)$$

Real Systems

If a function $u(t)$ is purely real, then the real and imaginary parts of its Fourier transform $U(\omega) = U'(\omega) + jU''(\omega)$ has symmetry.

$\text{Re}[U(\omega)]$ even function

$$U'(-\omega) = U'(\omega)$$

$\text{Im}[U(\omega)]$ odd function

$$U''(-\omega) = -U''(\omega)$$

If an impulse response $h(t)$ is purely real, then the real and imaginary parts of the transfer function $H(\omega) = H'(\omega) + jH''(\omega)$ have symmetry.

$$F(\omega) \rightarrow H(\omega) \rightarrow G(\omega)$$

$$H(\omega) = H'(\omega) + jH''(\omega)$$

$$H'(-\omega) = H'(\omega)$$

$$H''(-\omega) = -H''(\omega)$$

Causal Systems

In physical systems, the output cannot react to an input before the input has happened. These systems are called *causal*. Causality requires

$$f(t) \rightarrow h(t) \rightarrow g(t)$$

$$h(t \leq 0) = g(t \leq 0) = 0$$

The transfer function $H(\omega)$ is the Fourier transform of $h(t)$ so it can be written as

$$H(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(t) u(t) e^{-j\omega t} dt$$

$$F(\omega) \rightarrow H(\omega) \rightarrow G(\omega)$$

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$

Heaviside
step function

Derivation

Slide 7

Apply Convolution Property

The convolution property of Fourier transforms is

$$\begin{aligned}
 H(\omega) = F\{h(t)u(t)\} &= \frac{H(\omega)}{\sqrt{2\pi}} * F\{u(t)\} \\
 &= \frac{H(\omega)}{\sqrt{2\pi}} * \frac{1}{\sqrt{2\pi}} \left[\pi\delta(t) + \frac{1}{j\omega} \right] \\
 &= \frac{1}{2\pi} \left[H(\omega) * \pi\delta(t) + H(\omega) * \frac{1}{j\omega} \right]
 \end{aligned}$$

This is the Fourier transform of the Heaviside step function.

Perform Convolution

$$\begin{aligned}
 H(\omega) &= F\{h(t)u(t)\} = \frac{1}{2\pi} \left[H(\omega) * \pi\delta(t) + H(\omega) * \frac{1}{j\omega} \right] \\
 &= \frac{1}{2\pi} \left[\int_{-\infty}^{\infty} H(\Omega) \pi\delta(\omega - \Omega) d\Omega + \int_{-\infty}^{\infty} H(\Omega) \frac{1}{j(\omega - \Omega)} d\Omega \right] \\
 &= \frac{1}{2\pi} \left[\pi H(\omega) + \int_{-\infty}^{\infty} \frac{H(\Omega)}{j(\omega - \Omega)} d\Omega \right]
 \end{aligned}$$



Solve for $H(\omega)$

$$\begin{aligned}
 H(\omega) &= \frac{1}{2\pi} \left[\pi H(\omega) + \int_{-\infty}^{\infty} \frac{H(\Omega)}{j(\omega - \Omega)} d\Omega \right] \\
 &= \frac{1}{2} H(\omega) + \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{H(\Omega)}{j(\omega - \Omega)} d\Omega \\
 H(\omega) &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{H(\Omega)}{j(\omega - \Omega)} d\Omega
 \end{aligned}$$



Eliminate Negative Frequencies

$$\begin{aligned}
 H(\omega) &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{H(\Omega)}{j(\omega - \Omega)} d\Omega = \frac{1}{\pi} \int_{-\infty}^0 \frac{H(\Omega)}{j(\omega - \Omega)} d\Omega + \frac{1}{\pi} \int_0^{\infty} \frac{H(\Omega)}{j(\omega - \Omega)} d\Omega \\
 &= -\frac{1}{\pi} \int_0^{\infty} \frac{H(\Omega)}{j(\omega - \Omega)} d\Omega = -\frac{1}{\pi} \int_0^{\infty} \frac{H(-\Omega)}{j(\omega + \Omega)} d(-\Omega) \\
 &= \frac{1}{\pi} \int_0^{\infty} \frac{H(-\Omega)}{j(\omega + \Omega)} d\Omega + \frac{1}{\pi} \int_0^{\infty} \frac{H(\Omega)}{j(\omega - \Omega)} d\Omega
 \end{aligned}$$

Relate Real & Imaginary Parts (1 of 2)

$$H(\omega) = \frac{1}{\pi} \int_0^{\infty} \frac{H(-\Omega)}{j(\omega + \Omega)} d\Omega + \frac{1}{\pi} \int_0^{\infty} \frac{H(\Omega)}{j(\omega - \Omega)} d\Omega$$

Fourier transforms of real functions satisfy $H(-\Omega) = H^*(\Omega)$. Applying this to the above integrals allows them to be combined.

$$\begin{aligned}
 H(\omega) &= \frac{1}{\pi} \int_0^{\infty} \left[\frac{H^*(\Omega)}{j(\omega + \Omega)} + \frac{H(\Omega)}{j(\omega - \Omega)} \right] d\Omega \\
 &= \frac{1}{\pi} \int_0^{\infty} \left[\frac{H(\Omega)(\omega + \Omega) + H^*(\Omega)(\omega - \Omega)}{j(\omega^2 - \Omega^2)} \right] d\Omega
 \end{aligned}$$

Relate Real & Imaginary Parts (1 of 2)

$$H(\omega) = \frac{1}{\pi} \int_0^{\infty} \left[\frac{H(\Omega)(\omega + \Omega) + H^*(\Omega)(\omega - \Omega)}{j(\omega^2 - \Omega^2)} \right] d\Omega$$

Expand the numerator and collect the common ω and Ω terms.

$$\begin{aligned}
 H(\omega) &= \frac{1}{\pi} \int_0^{\infty} \left\{ \frac{\left[\overset{H(\Omega) + H^*(\Omega) = 2H'(\Omega)}{H(\Omega) + H^*(\Omega)} \right] \omega + \left[\overset{H(\Omega) - H^*(\Omega) = j2H''(\Omega)}{H(\Omega) - H^*(\Omega)} \right] \Omega}{j(\omega^2 - \Omega^2)} \right\} d\Omega \\
 &= \frac{2}{\pi} \int_0^{\infty} \frac{\Omega H''(\Omega)}{\omega^2 - \Omega^2} d\Omega - j \frac{2\omega}{\pi} \int_0^{\infty} \frac{H'(\Omega)}{\omega^2 - \Omega^2} d\Omega
 \end{aligned}$$

Set Real and Imaginary Parts Equal

$$H'(\omega) + jH''(\omega) = \frac{2}{\pi} \int_0^{\infty} \frac{\Omega H''(\Omega)}{\omega^2 - \Omega^2} d\Omega - j \frac{2\omega}{\pi} \int_0^{\infty} \frac{H'(\Omega)}{\omega^2 - \Omega^2} d\Omega$$

$$H'(\omega) = \frac{2}{\pi} \int_0^{\infty} \frac{\Omega H''(\Omega)}{\omega^2 - \Omega^2} d\Omega$$

$$H''(\omega) = -\frac{2\omega}{\pi} \int_0^{\infty} \frac{H'(\Omega)}{\omega^2 - \Omega^2} d\Omega$$

Meaning of Kramers-Kronig Relations

The Kramers-Kronig relations shows that the real and imaginary parts of the transfer function of a causal (i.e. physical) system are related. If one is known than the other can be calculated from it.

This limits the freedom of a system. It is possible to independently choose or control the real and imaginary parts.

$$H(\omega) = H'(\omega) + jH''(\omega)$$

$$H'(\omega) = \frac{2}{\pi} \int_0^{\infty} \frac{\Omega H''(\Omega)}{\omega^2 - \Omega^2} d\Omega \quad H''(\omega) = -\frac{2\omega}{\pi} \int_0^{\infty} \frac{H'(\Omega)}{\omega^2 - \Omega^2} d\Omega$$

Kramers-Kronig Relations for Electromagnetic Materials

Susceptibility is an Impulse Response of an Atom

The electric susceptibility $\chi_e(t)$ is the impulse response of an atomic resonance.

$$\vec{E}(t) \rightarrow \varepsilon_0 \chi_e(t) \rightarrow \vec{P}(t) \quad \vec{P}(t) = \varepsilon_0 \int_{-\infty}^{\infty} \vec{E}(\tau) \chi_e(t - \tau) d\tau$$

...and its Fourier transform $\tilde{\chi}_e(\omega)$ is the transfer function.

$$\vec{E}(\omega) \rightarrow \varepsilon_0 \tilde{\chi}_e(\omega) \rightarrow \vec{P}(\omega) \quad \vec{P}(\omega) = \varepsilon_0 \tilde{\chi}_e(\omega) \vec{E}(\omega)$$

Kramers-Kronig Relations for Susceptibility

If susceptibility is an impulse response, the Kramers-Kronig relations can be written for it.

$$\begin{aligned} H(\omega) &= H'(\omega) + jH''(\omega) \\ H'(\omega) &= \frac{2}{\pi} \int_0^{\infty} \frac{\Omega H''(\Omega)}{\omega^2 - \Omega^2} d\Omega \\ H''(\omega) &= -\frac{2\omega}{\pi} \int_0^{\infty} \frac{H'(\Omega)}{\omega^2 - \Omega^2} d\Omega \end{aligned} \quad \rightarrow \quad \begin{aligned} \tilde{\chi}_e(\omega) &= \chi'_e(\omega) + j\chi''_e(\omega) \\ \chi'_e(\omega) &= \frac{2}{\pi} \int_0^{\infty} \frac{\Omega \chi''_e(\Omega)}{\omega^2 - \Omega^2} d\Omega \\ \chi''_e(\omega) &= -\frac{2\omega}{\pi} \int_0^{\infty} \frac{\chi'_e(\Omega)}{\omega^2 - \Omega^2} d\Omega \end{aligned}$$

Kramers-Kronig Relations for Permittivity

From the Kramers-Kronig relations for susceptibility, it follows that for permittivity,

$$\begin{aligned} \tilde{\chi}_e(\omega) &= \chi'_e(\omega) + j\chi''_e(\omega) \\ \tilde{\chi}_e(\omega) &= \chi'_e(\omega) + j\chi''_e(\omega) \\ \chi'_e(\omega) &= \frac{2}{\pi} \int_0^\infty \frac{\Omega \chi''_e(\Omega)}{\omega^2 - \Omega^2} d\Omega \\ \chi''_e(\omega) &= -\frac{2\omega}{\pi} \int_0^\infty \frac{\chi'_e(\Omega)}{\omega^2 - \Omega^2} d\Omega \end{aligned} \quad \rightarrow \quad \begin{aligned} \tilde{\epsilon}_r(\omega) &= \epsilon'_r(\omega) + j\epsilon''_r(\omega) \\ \epsilon'_r(\omega) &= 1 + \frac{2}{\pi} \int_0^\infty \frac{\Omega \epsilon''_r(\Omega)}{\omega^2 - \Omega^2} d\Omega \\ \epsilon''_r(\omega) &= -\frac{2\omega}{\pi} \int_0^\infty \frac{[\epsilon'_r(\Omega) - 1]}{\omega^2 - \Omega^2} d\Omega \end{aligned}$$

Relation Between \tilde{n} and $\tilde{\chi}_e$ for Dilute Media

Recall how refractive index and susceptibility are related.

$$n_o(\omega) + j\kappa(\omega) = \sqrt{1 + \tilde{\chi}_e(\omega)}$$

For small susceptibility (i.e. $|\chi_e| < 1$), the above equation can be simplified using the binomial approximation.

$$n_o(\omega) + j\kappa(\omega) = 1 + \frac{\tilde{\chi}_e(\omega)}{2} = 1 + \frac{\chi'_e(\omega)}{2} + j\frac{\chi''_e(\omega)}{2} \quad |\tilde{\chi}_e(\omega)| < 1$$

Setting the real and imaginary parts on both sides of the equation equal gives

$$n_o(\omega) = 1 + \frac{\chi'_e(\omega)}{2} \quad \kappa(\omega) = \frac{\chi''_e(\omega)}{2}$$

Kramers-Kronig Relations for Refractive Index

It follows that

$$\begin{aligned} \tilde{\chi}_e(\omega) &= \chi'_e(\omega) + j\chi''_e(\omega) \\ \chi'_e(\omega) &= \frac{2}{\pi} \int_0^\infty \frac{\Omega \chi''_e(\Omega)}{\omega^2 - \Omega^2} d\Omega \\ \chi''_e(\omega) &= -\frac{2\omega}{\pi} \int_0^\infty \frac{\chi'_e(\Omega)}{\omega^2 - \Omega^2} d\Omega \end{aligned} \quad \longrightarrow \quad \begin{aligned} \tilde{n}(\omega) &= n_o(\omega) + j\kappa(\omega) \\ n_o(\omega) &= 1 + \frac{2}{\pi} \int_0^\infty \frac{\Omega \kappa(\Omega)}{\omega^2 - \Omega^2} d\Omega \\ \kappa(\omega) &= -\frac{2\omega}{\pi} \int_0^\infty \frac{n_o(\Omega) - 1}{\omega^2 - \Omega^2} d\Omega \end{aligned}$$

Kramers-Kronig Relations for $n_o(\omega)$ and $\alpha(\omega)$

Given the absorption coefficient is related to the extinction coefficient through $\alpha(\omega) = 2k_0\kappa(\omega)$, it follows that

$$\begin{aligned} \tilde{n}(\omega) &= n_o(\omega) + j\kappa(\omega) \\ n_o(\omega) &= 1 + \frac{2}{\pi} \int_0^\infty \frac{\Omega \kappa(\Omega)}{\omega^2 - \Omega^2} d\Omega \\ \kappa(\omega) &= -\frac{2\omega}{\pi} \int_0^\infty \frac{n_o(\Omega) - 1}{\omega^2 - \Omega^2} d\Omega \end{aligned} \quad \longrightarrow \quad \begin{aligned} n_o(\omega) &= 1 + \frac{c_0}{\pi} \int_0^\infty \frac{\alpha(\Omega)}{\omega^2 - \Omega^2} d\Omega \\ \alpha(\omega) &= -\frac{4\omega^2}{\pi c_0} \int_0^\infty \frac{n_o(\Omega) - 1}{\omega^2 - \Omega^2} d\Omega \end{aligned}$$

Kramers-Kronig Relations for $R(\omega)$ and $\phi(\omega)$

The Kramers-Kronig relation can also be written for the reflectivity $\rho(\omega)$ and phase from reflection $\phi(\omega)$.

$$\begin{aligned} \tilde{r}(\omega) &= \rho(\omega) e^{i\phi(\omega)} \\ \ln[\tilde{r}(\omega)] &= \ln[\rho(\omega)] + i\phi(\omega) \\ \ln[\rho(\omega)] &= -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\phi(\Omega)}{\Omega - \omega} d\Omega \\ \phi(\omega) &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\ln[\rho(\omega)]}{\Omega - \omega} d\Omega \end{aligned} \quad \begin{aligned} \phi(\Omega) &= \frac{2\omega}{\pi} \int_0^{\infty} \frac{\ln[\rho(\omega)]}{\omega^2 - \Omega^2} d\omega \\ &= \frac{\omega}{\pi} \int_0^{\infty} \frac{\ln[R(\Omega)/R(\omega)]}{\omega^2 - \Omega^2} d\omega \end{aligned}$$

This provides a great way to calculate phase of reflection $\phi(\omega)$ just from intensity measurements.

The Meaning & Utility of Kramers-Kronig Relations in Electromagnetics

$$n_o(\omega) = 1 + \frac{c_0}{\pi} \int_0^{\infty} \frac{\alpha(\Omega)}{\omega^2 - \Omega^2} d\Omega$$

Measurement of loss through a sample is easy. The Kramers-Kronig relations provide a way of calculating refractive index from the absorption measurements.

Intensity measurements are easy. The Kramers-Kronig relations provide a way of calculating phase of reflection from just intensity measurements.

$$\phi(\Omega) = \frac{\omega}{\pi} \int_0^{\infty} \frac{\ln[R(\Omega)/R(\omega)]}{\omega^2 - \Omega^2} d\omega$$

Kramers-Kronig relations also illustrate how degrees of freedom are limited in material properties. Real and imaginary parts of many parameters are not independent.