Advanced Electromagnetics:
21st Century Electromagnetics

Kramers-Kronig Relations

1

Lecture Outline

- Linear Systems
- Derivation
- Kramers-Kronig relations in electromagnetics

2

Linear Systems

Slide 3

3

Impulse Response & Transfer Functions

Linear systems can be characterized by their impulse response,

$$f(t) \rightarrow \boxed{h(t)} \rightarrow g(t) \qquad g(t) = f(t) * h(t) = \int_{-\infty}^{\infty} f(\tau)\, h(t-\tau)\, d\tau$$

…or by their transfer function (Fourier transform of impulse response),

$$F(\omega) \rightarrow \boxed{H(\omega)} \rightarrow G(\omega) \qquad G(\omega) = H(\omega)\, F(\omega)$$

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4

4

Real Systems

If a function $u(t)$ is purely real, then the real and imaginary parts of its Fourier transform $U(\omega) = U'(\omega) + jU''(\omega)$ has symmetry.

$$\text{Re}\left[U(\omega)\right] \text{ even function} \qquad \text{Im}\left[U(\omega)\right] \text{ odd function}$$

$$U'(-\omega) = U'(\omega) \qquad U''(-\omega) = -U''(\omega)$$

If an impulse response $h(t)$ is purely real, then the real and imaginary parts of the transfer function $H(\omega) = H'(\omega) + jH''(\omega)$ have symmetry.

$$F(\omega) \rightarrow \boxed{H(\omega)} \rightarrow G(\omega)$$

$$H(\omega) = H'(\omega) + jH''(\omega)$$

$$H'(-\omega) = H'(\omega)$$

$$H''(-\omega) = -H''(\omega)$$

5

Causal Systems

In physical systems, the output cannot react to an input before the input has happened. These systems are called *causal*. Causality requires

$$f(t) \rightarrow \boxed{h(t)} \rightarrow g(t)$$

$$h(t \leq 0) = g(t \leq 0) = 0$$

The transfer function $H(\omega)$ is the Fourier transform of $h(t)$ so it can be written as

$$F(\omega) \rightarrow \boxed{H(\omega)} \rightarrow G(\omega)$$

$$H(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(t) u(t) e^{-j\omega t} dt$$

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$

Heaviside step function

6

Derivation

7

Apply Convolution Property

The convolution property of Fourier transforms is

$$H(\omega)=F\{h(t)u(t)\}=\frac{H(\omega)}{\sqrt{2\pi}}*F\{u(t)\}$$

$$=\frac{H(\omega)}{\sqrt{2\pi}}*\frac{1}{\sqrt{2\pi}}\left[\pi\delta(\omega)+\frac{1}{j\omega}\right]$$

This is the Fourier transform of the Heaviside step function.

$$=\frac{1}{2\pi}\left[H(\omega)*\pi\delta(\omega)+H(\omega)*\frac{1}{j\omega}\right]$$

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Perform Convolution

$$H(\omega)=F\{h(t)u(t)\}=\frac{1}{2\pi}\left[H(\omega)*\pi\delta(\omega)+H(\omega)*\frac{1}{j\omega}\right]$$

$$=\frac{1}{2\pi}\left[\int_{-\infty}^{\infty}H(\Omega)\pi\delta(\omega-\Omega)d\Omega+\int_{-\infty}^{\infty}H(\Omega)\frac{1}{j(\omega-\Omega)}d\Omega\right]$$

$$=\frac{1}{2\pi}\left[\pi H(\omega)+\int_{-\infty}^{\infty}\frac{H(\Omega)}{j(\omega-\Omega)}d\Omega\right]$$

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9

Solve for $H(\omega)$

$$H(\omega)=\frac{1}{2\pi}\left[\pi H(\omega)+\int_{-\infty}^{\infty}\frac{H(\Omega)}{j(\omega-\Omega)}d\Omega\right]$$

$$=\frac{1}{2}H(\omega)+\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{H(\Omega)}{j(\omega-\Omega)}d\Omega$$

$$H(\omega)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{H(\Omega)}{j(\omega-\Omega)}d\Omega$$

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10

Eliminate Negative Frequencies

$$H(\omega)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{H(\Omega)}{j(\omega-\Omega)}d\Omega \;=\underbrace{\frac{1}{\pi}\int_{-\infty}^{0}\frac{H(\Omega)}{j(\omega-\Omega)}d\Omega}+\frac{1}{\pi}\int_{0}^{\infty}\frac{H(\Omega)}{j(\omega-\Omega)}d\Omega$$

$$=-\frac{1}{\pi}\int_{0}^{-\infty}\frac{H(\Omega)}{j(\omega-\Omega)}d\Omega=-\frac{1}{\pi}\int_{0}^{\infty}\frac{H(-\Omega)}{j(\omega+\Omega)}d(-\Omega)$$

$$=\frac{1}{\pi}\int_{0}^{\infty}\frac{H(-\Omega)}{j(\omega+\Omega)}d\Omega+\frac{1}{\pi}\int_{0}^{\infty}\frac{H(\Omega)}{j(\omega-\Omega)}d\Omega$$

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11

11

Relate Real & Imaginary Parts (1 of 2)

$$H(\omega)=\frac{1}{\pi}\int_{0}^{\infty}\frac{H(-\Omega)}{j(\omega+\Omega)}d\Omega+\frac{1}{\pi}\int_{0}^{\infty}\frac{H(\Omega)}{j(\omega-\Omega)}d\Omega$$

Fourier transforms of real functions satisfy $H(-\Omega)=H^*(\Omega)$. Applying this to the above integrals allows them to be combined.

$$H(\omega)=\frac{1}{\pi}\int_{0}^{\infty}\left[\frac{H^*(\Omega)}{j(\omega+\Omega)}+\frac{H(\Omega)}{j(\omega-\Omega)}\right]d\Omega$$

$$=\frac{1}{\pi}\int_{0}^{\infty}\left[\frac{H(\Omega)(\omega+\Omega)+H^*(\Omega)(\omega-\Omega)}{j(\omega^2-\Omega^2)}\right]d\Omega$$

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12

12

Relate Real & Imaginary Parts (1 of 2)

$$H(\omega)=\frac{1}{\pi}\int_0^\infty\left[\frac{H(\Omega)(\omega+\Omega)+H^*(\Omega)(\omega-\Omega)}{j(\omega^2-\Omega^2)}\right]d\Omega$$

Expand the numerator and collect the common ω and Ω terms.

$H(\Omega)+H^*(\Omega)=2H'(\Omega)$

$H(\Omega)-H^*(\Omega)=j2H''(\Omega)$

$$H(\omega)=\frac{1}{\pi}\int_0^\infty\left\{\frac{\left[H(\Omega)+H^*(\Omega)\right]\omega+\left[H(\Omega)-H^*(\Omega)\right]\Omega}{j(\omega^2-\Omega^2)}\right\}d\Omega$$

$$=\frac{2}{\pi}\int_0^\infty\frac{\Omega H''(\Omega)}{\omega^2-\Omega^2}d\Omega-j\frac{2\omega}{\pi}\int_0^\infty\frac{H'(\Omega)}{\omega^2-\Omega^2}d\Omega$$

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13

Set Real and Imaginary Parts Equal

$$H'(\omega)+jH''(\omega)=\frac{2}{\pi}\int_0^\infty\frac{\Omega H''(\Omega)}{\omega^2-\Omega^2}d\Omega-j\frac{2}{\pi}\int_0^\infty\frac{\omega H'(\Omega)}{\omega^2-\Omega^2}d\Omega$$

$$H'(\omega)=\frac{2}{\pi}\int_0^\infty\frac{\Omega H''(\Omega)}{\omega^2-\Omega^2}d\Omega$$

$$H''(\omega)=-\frac{2\omega}{\pi}\int_0^\infty\frac{H'(\Omega)}{\omega^2-\Omega^2}d\Omega$$

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14

Meaning of Kramers-Kronig Relations

The Kramers-Kronig relations shows that the real and imaginary parts of the transfer function of a causal (i.e. physical) system are related. If one is known than the other can be calculated from it.

This limits the freedom of a system. It is not possible to choose or control the real and imaginary parts independently.

$$H(\omega) = H'(\omega) + jH''(\omega)$$

$$H'(\omega) = \frac{2}{\pi}\int_0^\infty \frac{\Omega H''(\Omega)}{\omega^2 - \Omega^2}\,d\Omega \qquad H''(\omega) = -\frac{2\omega}{\pi}\int_0^\infty \frac{H'(\Omega)}{\omega^2 - \Omega^2}\,d\Omega$$

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15

Kramers-Kronig Relations for Electromagnetic Materials

Slide 16

16

Susceptibility is an Impulse Response of an Atom

The electric susceptibility $\chi_e(t)$ is the impulse response of an atomic resonance.

$$\vec{E}(t) \rightarrow \boxed{\varepsilon_0 \chi_e(t)} \rightarrow \vec{P}(t) \qquad \vec{P}(t) = \varepsilon_0 \int_{-\infty}^{\infty} \vec{E}(\tau)\chi_e(t-\tau)\,d\tau$$

…and its Fourier transform $\tilde{\chi}_e(\omega)$ is the transfer function.

$$\vec{E}(\omega) \rightarrow \boxed{\varepsilon_0 \tilde{\chi}_e(\omega)} \rightarrow \vec{P}(\omega) \qquad \vec{P}(\omega) = \varepsilon_0 \tilde{\chi}_e(\omega)\vec{E}(\omega)$$

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17

Kramers-Kronig Relations for Susceptibility

If susceptibility is an impulse response, the Kramers-Kronig relations can be written for it.

$$H(\omega) = H'(\omega) + jH''(\omega)$$
$$H'(\omega) = \frac{2}{\pi}\int_0^{\infty} \frac{\Omega H''(\Omega)}{\omega^2 - \Omega^2}\,d\Omega$$
$$H''(\omega) = -\frac{2\omega}{\pi}\int_0^{\infty} \frac{H'(\Omega)}{\omega^2 - \Omega^2}\,d\Omega$$

$\Rightarrow$

$$\tilde{\chi}_e(\omega) = \chi_e'(\omega) + j\chi_e''(\omega)$$
$$\chi_e'(\omega) = \frac{2}{\pi}\int_0^{\infty} \frac{\Omega \chi_e''(\Omega)}{\omega^2 - \Omega^2}\,d\Omega$$
$$\chi_e''(\omega) = -\frac{2\omega}{\pi}\int_0^{\infty} \frac{\chi_e'(\Omega)}{\omega^2 - \Omega^2}\,d\Omega$$

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18

Kramers-Kronig Relations for Permittivity

From the Kramers-Kronig relations for susceptibility, it follows that for permittivity,

$$\tilde{\chi}_e(\omega) = \chi'_e(\omega) + j\chi''_e(\omega)$$

$$\chi'_e(\omega) = \frac{2}{\pi}\int_0^\infty \frac{\Omega\chi''_e(\Omega)}{\omega^2 - \Omega^2} d\Omega$$

$$\chi''_e(\omega) = -\frac{2\omega}{\pi}\int_0^\infty \frac{\chi'_e(\Omega)}{\omega^2 - \Omega^2} d\Omega$$

$\Rightarrow$

$$\tilde{\varepsilon}_r(\omega) = \varepsilon'_r(\omega) + j\varepsilon''_r(\omega)$$

$$\varepsilon'_r(\omega) = 1 + \frac{2}{\pi}\int_0^\infty \frac{\Omega\varepsilon''_r(\Omega)}{\omega^2 - \Omega^2} d\Omega$$

$$\varepsilon''_r(\omega) = -\frac{2\omega}{\pi}\int_0^\infty \frac{\left[\varepsilon'_r(\Omega) - 1\right]}{\omega^2 - \Omega^2} d\Omega$$

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19

Relation Between $\tilde{n}$ and $\tilde{\chi}_e$ for Dilute Media

Recall how refractive index and susceptibility are related.

$$n_o(\omega) + j\kappa(\omega) = \sqrt{1 + \tilde{\chi}_e(\omega)}$$

For small susceptibility (i.e. $|\chi_e| < 1$), the above equation can be simplified using the binomial approximation.

$$n_o(\omega) + j\kappa(\omega) = 1 + \frac{\tilde{\chi}_e(\omega)}{2} = 1 + \frac{\chi'_e(\omega)}{2} + j\frac{\chi''_e(\omega)}{2} \qquad \left|\tilde{\chi}_e(\omega)\right| < 1$$

Setting the real and imaginary parts on both sides of the equation equal gives

$$n_o(\omega) = 1 + \frac{\chi'_e(\omega)}{2} \qquad \kappa(\omega) = \frac{\chi''_e(\omega)}{2}$$

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20

Kramers-Kronig Relations for Refractive Index

It follows that

$$\tilde{\chi}_e(\omega) = \chi'_e(\omega) + j\chi''_e(\omega)$$
$$\chi'_e(\omega) = \frac{2}{\pi}\int_0^\infty \frac{\Omega\chi''_e(\Omega)}{\omega^2 - \Omega^2}d\Omega$$
$$\chi''_e(\omega) = -\frac{2\omega}{\pi}\int_0^\infty \frac{\chi'_e(\Omega)}{\omega^2 - \Omega^2}d\Omega$$

$\Rightarrow$

$$\tilde{n}(\omega) = n_o(\omega) + j\kappa(\omega)$$
$$n_o(\omega) = 1 + \frac{2}{\pi}\int_0^\infty \frac{\Omega\kappa(\Omega)}{\omega^2 - \Omega^2}d\Omega$$
$$\kappa(\omega) = -\frac{2\omega}{\pi}\int_0^\infty \frac{n_o(\Omega) - 1}{\omega^2 - \Omega^2}d\Omega$$

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21

21

Kramers-Kronig Relations for $n_o(\omega)$ and $\alpha(\omega)$

Given the absorption coefficient is related to the extinction coefficient through $\alpha(\omega) = 2k_0\kappa(\omega)$, it follows that

$$\tilde{n}(\omega) = n_o(\omega) + j\kappa(\omega)$$
$$n_o(\omega) = 1 + \frac{2}{\pi}\int_0^\infty \frac{\Omega\kappa(\Omega)}{\omega^2 - \Omega^2}d\Omega$$
$$\kappa(\omega) = -\frac{2\omega}{\pi}\int_0^\infty \frac{n_o(\Omega) - 1}{\omega^2 - \Omega^2}d\Omega$$

$\Rightarrow$

$$n_o(\omega) = 1 + \frac{c_0}{\pi}\int_0^\infty \frac{\alpha(\Omega)}{\omega^2 - \Omega^2}d\Omega$$
$$\alpha(\omega) = -\frac{4\omega^2}{\pi c_0}\int_0^\infty \frac{n_o(\Omega) - 1}{\omega^2 - \Omega^2}d\Omega$$

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22

22

Kramers-Kronig Relations for $R(\omega)$ and $\phi(\omega)$

The Kramers-Kronig relation can also be written for the reflectivity $\rho(\omega)$ and phase from reflection $\phi(\omega)$.

$$\tilde{r}(\omega) = \rho(\omega) e^{j\phi(\omega)}$$

$$\ln\left[\tilde{r}(\omega)\right] = \ln\left[\rho(\omega)\right] + j\phi(\omega)$$

$$\ln\left[\rho(\omega)\right] = -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\phi(\Omega)}{\Omega - \omega} d\Omega$$

$$\phi(\Omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\ln\left[\rho(\Omega)\right]}{\Omega - \omega} d\Omega$$

$$\phi(\Omega) = \frac{2\omega}{\pi}\int_{0}^{\infty} \frac{\ln\left[\rho(\Omega)\right]}{\omega^2 - \Omega^2} d\Omega = \frac{\omega}{\pi}\int_{0}^{\infty} \frac{\ln\left[R(\Omega)/R(\omega)\right]}{\omega^2 - \Omega^2} d\Omega$$

This provides a great way to calculate phase of reflection $\phi(\omega)$ just from intensity measurements.

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23

The Meaning & Utility of Kramers-Kronig Relations in Electromagnetics

$$n_o(\omega) = 1 + \frac{c_0}{\pi}\int_{0}^{\infty} \frac{\alpha(\Omega)}{\omega^2 - \Omega^2} d\Omega$$

Measurement of loss through a sample is easy. The Kramers-Kronig relations provide a way of calculating refractive index from the absorption measurements.

Intensity measurements are easy. The Kramers-Kronig relations provide a way of calculating phase of reflection from just intensity measurements.

$$\phi(\Omega) = \frac{\omega}{\pi}\int_{0}^{\infty} \frac{\ln\left[R(\Omega)/R(\omega)\right]}{\omega^2 - \Omega^2} d\Omega$$

Kramers-Kronig relations also illustrate how degrees of freedom are limited in material properties. Real and imaginary parts of many parameters are not independent.

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24

Appendix

Cauchy Principal Value

$$n_{\mathrm{o}}(\omega)=1+\frac{c_0}{\pi}\int_0^{\infty}\frac{\alpha(\Omega)}{\omega^2-\Omega^2}d\Omega$$

What about when $\omega=\Omega$?

$$\int_0^{\infty}\frac{\alpha(\Omega)}{\omega^2-\Omega^2}d\Omega=\lim_{\Delta\omega\to 0}\left[\int_0^{\omega-\Delta\omega}\frac{\alpha(\Omega)}{\omega^2-\Omega^2}d\Omega+\int_{\omega+\Delta\omega}^{\infty}\frac{\alpha(\Omega)}{\omega^2-\Omega^2}d\Omega\right]$$

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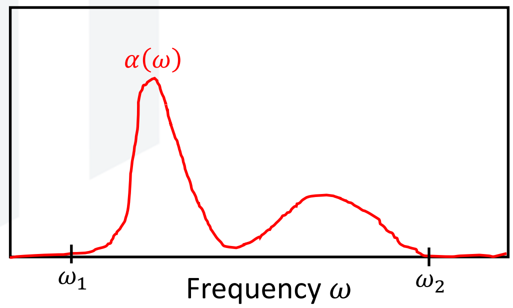
Integrating to Infinity?

How can this be integrated to infinity?

$$n_o(\omega) = 1 + \frac{c_0}{\pi} \int_0^{\infty} \frac{\alpha(\Omega)}{\omega^2 - \Omega^2} d\Omega$$

$$\int_0^{\infty} \frac{\alpha(\Omega)}{\omega^2 - \Omega^2} d\Omega \approx \int_{\omega_1}^{\omega_2} \frac{\alpha(\Omega)}{\omega^2 - \Omega^2} d\Omega$$



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Slide 27

27