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Advanced Electromagnetics:
21st Century Electromagnetics

Lorentz Model for Dielectrics

Lecture Outline

- The Lorentz dielectric function $\tilde{\varepsilon}_{\mathrm{r}}(\omega)$
- Wave parameters
- Notes & observations

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The Lorentz Dielectric Function $\tilde{\varepsilon}_{\mathrm{r}}(\omega)$

Slide 3

Constitutive Relation with Material Polarization $\vec{P}$

Material polarization $\vec{P}$ is incorporated into the electric constitutive relation as follows.

$$\vec{D} = \underbrace{\varepsilon_0 \vec{E}}_{\text{Response of free space}} + \underbrace{\vec{P}}_{\text{Response of material}}$$

Recall that $\vec{P} = \varepsilon_0 \chi_{\mathrm{e}} \vec{E}$. This gives

$$\vec{D} = \varepsilon_0 \vec{E} + \varepsilon_0 \chi_{\mathrm{e}} \vec{E}$$

$$= \varepsilon_0 \left(1 + \chi_{\mathrm{e}}\right) \vec{E}$$

Factor out ε_0 and $\vec{E}$.

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The Complex Relative Permittivity $\tilde{\varepsilon}_{\mathrm{r}}$

The constitutive relation from the previous slide was

$$\vec{D} = \varepsilon_0 \left(1 + \chi_{\mathrm{e}}\right) \vec{E}$$

The constitutive relation can also be written in terms of the complex relative permittivity $\tilde{\varepsilon}_{\mathrm{r}}$.

$$\vec{D} = \varepsilon_0 \tilde{\varepsilon}_{\mathrm{r}} \vec{E}$$

The righthand side of the equations above must be equal.

$$\varepsilon_0 \tilde{\varepsilon}_{\mathrm{r}} \vec{E} = \varepsilon_0 \left(1 + \chi_{\mathrm{e}}\right) \vec{E} \longrightarrow \boxed{\tilde{\varepsilon}_{\mathrm{r}} = 1 + \chi_{\mathrm{e}}}$$

Response of material

Response of free space

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Slide 5

The Lorentz Dielectric Function $\tilde{\varepsilon}_{\mathrm{r}}(\omega)$

Recall the electric susceptibility $\chi_{\mathrm{e}}(\omega)$ derived from the Lorentz oscillator model.

$$\chi_{\mathrm{e}}(\omega) = \frac{\omega_{\mathrm{p}}^2}{\omega_0^2 - \omega^2 - j\omega\Gamma} \qquad \omega_{\mathrm{p}}^2 = \frac{Nq^2}{\varepsilon_0 m_{\mathrm{e}}}$$

It follows that the complex relative permittivity $\tilde{\varepsilon}_{\mathrm{r}}(\omega)$ is

$$\boxed{\tilde{\varepsilon}_{\mathrm{r}}(\omega) = 1 + \chi_{\mathrm{e}} = 1 + \frac{\omega_{\mathrm{p}}^2}{\omega_0^2 - \omega^2 - j\omega\Gamma}}$$

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Slide 6

Real and Imaginary Parts of Permittivity $\tilde{\varepsilon}_{\mathrm{r}}(\omega)$

$$\boxed{\tilde{\varepsilon}_{\mathrm{r}}(\omega)=1+\frac{\omega_{\mathrm{p}}^2}{\omega_0^2-\omega^2-j\omega\Gamma} \qquad \omega_{\mathrm{p}}^2=\frac{Nq^2}{\varepsilon_0 m_{\mathrm{e}}}}$$

Split this into real and imaginary parts

$$\begin{aligned}
\tilde{\varepsilon}_{\mathrm{r}}(\omega)=\varepsilon_{\mathrm{r}}'(\omega)+j\varepsilon_{\mathrm{r}}''(\omega) &= 1+\frac{\omega_{\mathrm{p}}^2}{\left(\omega_0^2-\omega^2\right)-j\omega\Gamma}\,\frac{\left(\omega_0^2-\omega^2\right)+j\omega\Gamma}{\left(\omega_0^2-\omega^2\right)+j\omega\Gamma} \\
&= 1+\omega_{\mathrm{p}}^2\frac{\left(\omega_0^2-\omega^2\right)+j\omega\Gamma}{\left(\omega_0^2-\omega^2\right)^2+\omega^2\Gamma^2} \\
&= 1+\omega_{\mathrm{p}}^2\frac{\omega_0^2-\omega^2}{\left(\omega_0^2-\omega^2\right)^2+\omega^2\Gamma^2}+j\omega_{\mathrm{p}}^2\frac{\omega\Gamma}{\left(\omega_0^2-\omega^2\right)^2+\omega^2\Gamma^2}
\end{aligned}$$

$$\boxed{\varepsilon_{\mathrm{r}}'(\omega)=1+\omega_{\mathrm{p}}^2\frac{\omega_0^2-\omega^2}{\left(\omega_0^2-\omega^2\right)^2+\omega^2\Gamma^2} \qquad \varepsilon_{\mathrm{r}}''(\omega)=\omega_{\mathrm{p}}^2\frac{\omega\Gamma}{\left(\omega_0^2-\omega^2\right)^2+\omega^2\Gamma^2}}$$

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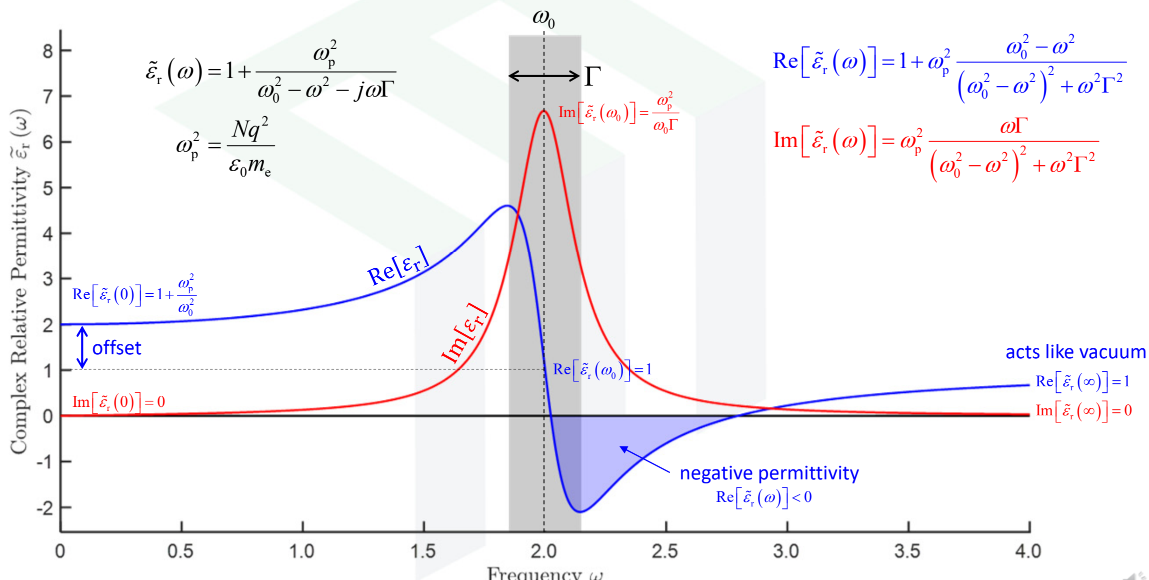
Typical Lorentz Response $\left(\omega_{\mathrm{p}}=2, \omega_0=2, \Gamma=0.3\right)$



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Slide 8

Wave Parameters

Complex Refractive Index $\tilde{n}(\omega)$

The complex refractive index $\tilde{n}$ is related to the complex relative permeability $\tilde{\mu}_r$ and complex relative permittivity $\tilde{\varepsilon}_r$ through

$$\tilde{n} = n_o + j\kappa = \pm\sqrt{\tilde{\mu}_r \tilde{\varepsilon}_r}$$

This sign here is the same as the sign of

$$\frac{\mathrm{Re}[\tilde{\mu}_r]}{|\tilde{\mu}_r|} + \frac{\mathrm{Re}[\tilde{\varepsilon}_r]}{|\tilde{\varepsilon}_r|}$$

No Magnetic Response ($\tilde{\mu}_r = 1$)

The magnetic response $\tilde{\mu}_r$ of most dielectric materials is negligible (i.e. $\tilde{\mu}_r = 1$). When this is the case, the real and imaginary parts of the complex refractive index $\tilde{n}$ can be calculated directly from the real and imaginary parts of complex relative permittivity $\tilde{\varepsilon}_r$.

$$\tilde{n}^2 = (n_o + j\kappa)^2 = \tilde{\varepsilon}_r = \tilde{\varepsilon}'_r + j\tilde{\varepsilon}''_r$$

Calculating $\tilde{\varepsilon}_r$ from $\tilde{n}$

$$\tilde{\varepsilon}'_r = n_o^2 - \kappa^2$$
$$\tilde{\varepsilon}''_r = 2n_o\kappa$$

Calculating $\tilde{n}$ from $\tilde{\varepsilon}_r$

$$n_o = \sqrt{\frac{|\tilde{\varepsilon}_r| + \tilde{\varepsilon}'_r}{2}}$$
$$\kappa = \sqrt{\frac{|\tilde{\varepsilon}_r| - \tilde{\varepsilon}'_r}{2}}$$

Derivation

Expand equation

$$(n_o + j\kappa)^2 = \tilde{\varepsilon}'_r + j\tilde{\varepsilon}''_r$$
$$n_o^2 + j2n_o\kappa - \kappa^2 = \tilde{\varepsilon}'_r + j\tilde{\varepsilon}''_r$$

Collect real and imaginary parts

$$(n_o^2 - \kappa^2) + j(2n_o\kappa) = \tilde{\varepsilon}'_r + j\tilde{\varepsilon}''_r$$

Set real and imaginary parts equal

$$\tilde{\varepsilon}'_r = n_o^2 - \kappa^2 \quad \text{Eq. (1)}$$
$$\tilde{\varepsilon}''_r = 2n_o\kappa \quad \text{Eq. (2)}$$

Solve Eq. (2) for κ and substitute into Eq. (1).

$$\kappa = \frac{\tilde{\varepsilon}''_r}{2n_o} \qquad n_o^2 - \left(\frac{\tilde{\varepsilon}''_r}{2n_o}\right)^2 = \tilde{\varepsilon}'_r$$

Expand into quadratic equation for n_o^2.

$$n_o^4 - \tilde{\varepsilon}'_r n_o^2 - \frac{\tilde{\varepsilon}''^2_r}{4} = 0$$

Solve using quadratic equation.

$$n_o^2 = \frac{\tilde{\varepsilon}'_r + \sqrt{\tilde{\varepsilon}'^2_r + \tilde{\varepsilon}''^2_r}}{2} \rightarrow n_o = \sqrt{\frac{|\tilde{\varepsilon}_r| + \tilde{\varepsilon}'_r}{2}}$$

Repeat last three steps for κ.

$$\kappa = \sqrt{\frac{|\tilde{\varepsilon}_r| - \tilde{\varepsilon}'_r}{2}}$$

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Attenuation Constant α

Waves propagate according to

$$E(z) = E_0 e^{jk_0\tilde{n}z} = E_0 e^{jk_0(n_o + j\kappa)z} = E_0 \boxed{e^{-k_0\kappa z} e^{jk_0 n_o z}}$$

The wave can also be written more intuitively in terms of the attenuation constant α and phase constant β.

$$E(z) = \boxed{E_0 e^{-\alpha z} e^{j\beta z}}$$

Comparing these expressions above leads to

$$\alpha = k_0\kappa \qquad \beta = k_0 n_o$$

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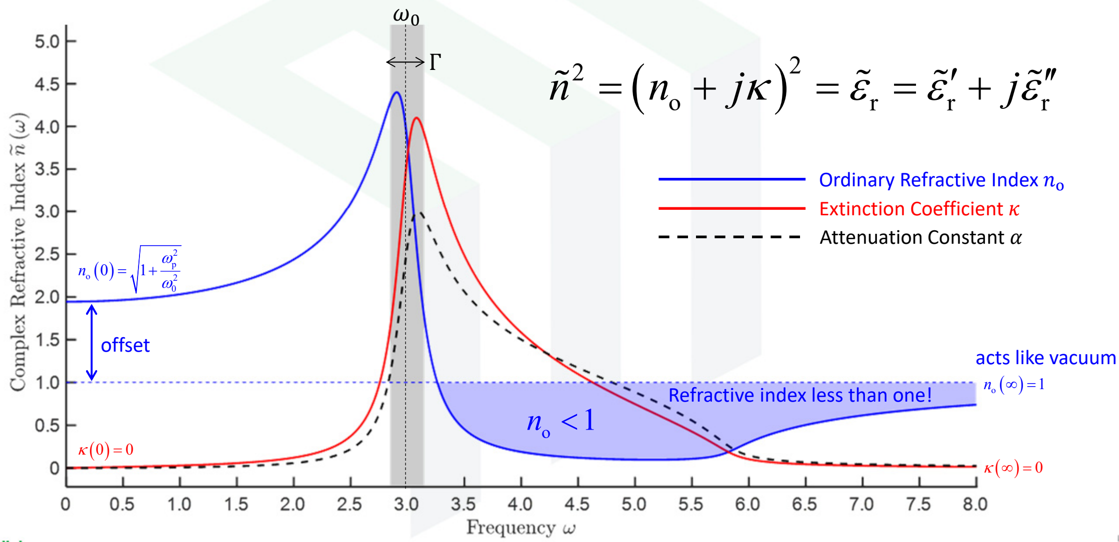
Complex Refractive Index $\tilde{n}(\omega)$ from a Lorentz Oscillator

$(\omega_p = 5, \omega_0 = 3, \Gamma = 0.3)$



Example – Salt Water

M. R. Querry, R. C. Waring, W. E. Holland, M. Hale, W. Nijm, "Optical Constants in the Infrared for Aqueous Solutions of NaCl," J. Opt. Soc. Am. **62**(7), 849–855 (1972)

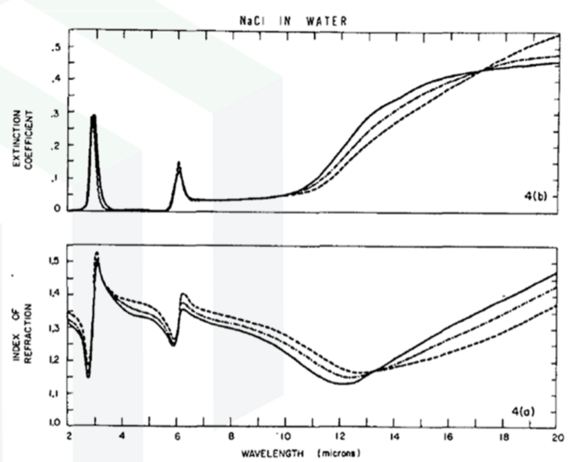


FIG. 4. The optical constants in the infrared for 1*M*——, 3*M*—·—, and 5*M*— — — aqueous solutions of NaCl. The KK analysis and algorithm for computing these quantities are presented in the text along with estimates of the uncertainties of the optical constants.

Notes & Observations

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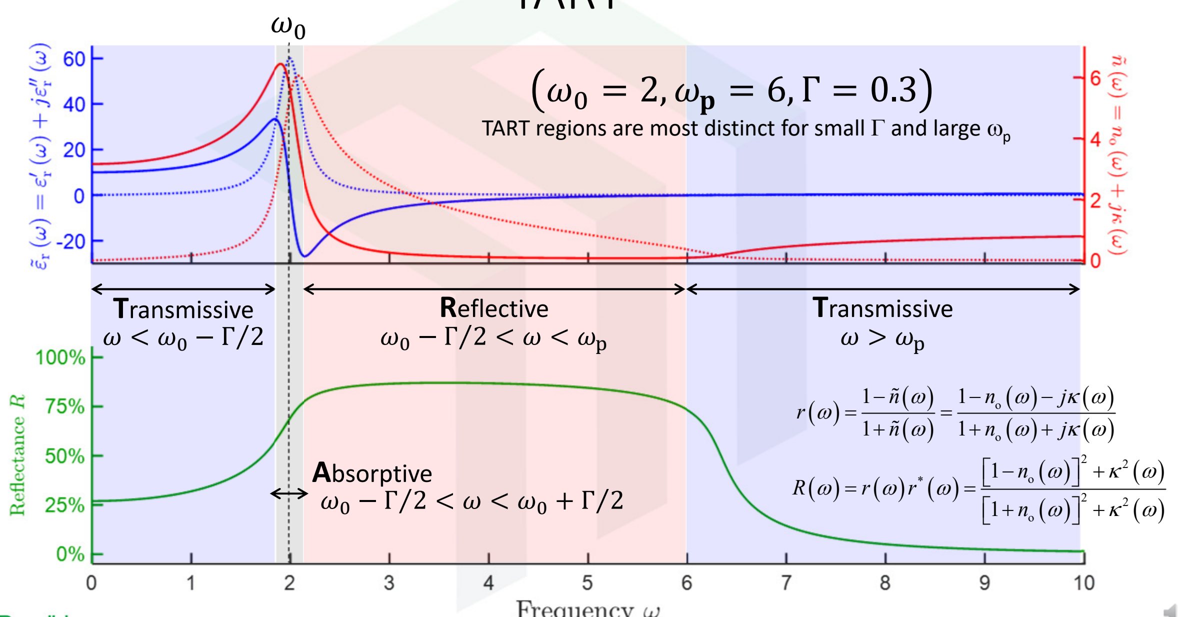

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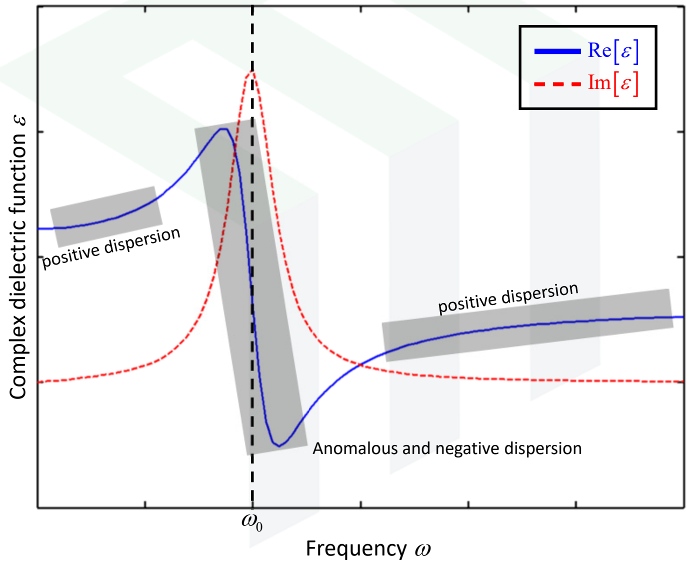
Observation #1 – Dispersion



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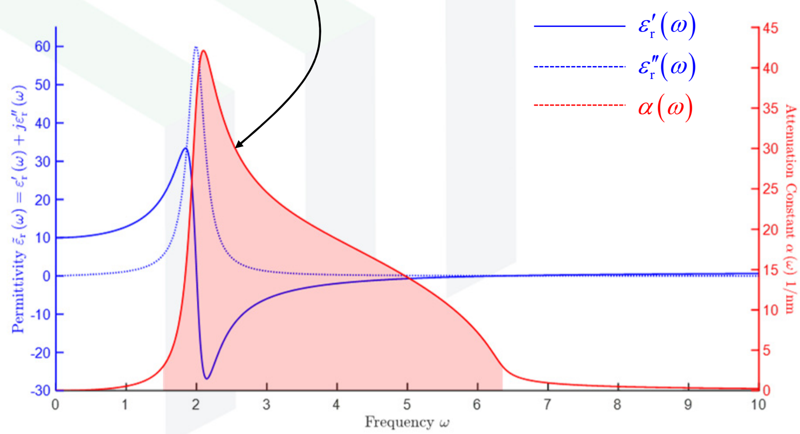
Observation #2 – Loss Near Resonance

Loss is very high near resonance.

$$\omega_0 = 2.0$$
$$\omega_p = 6.0$$
$$\Gamma = 0.3$$



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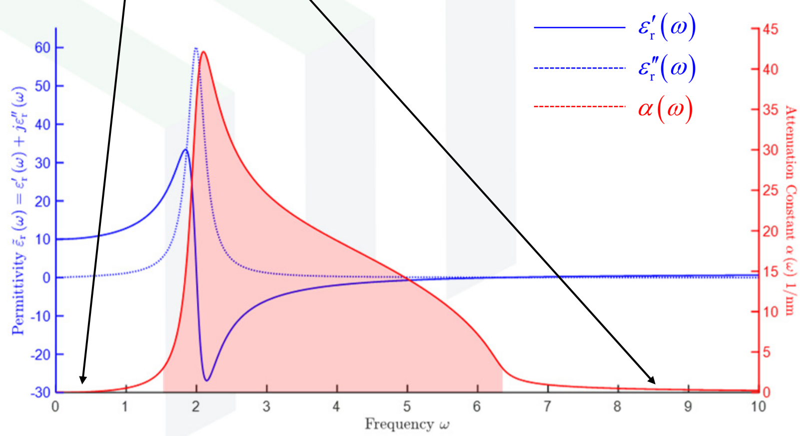
Observation #3 – Loss Far From Resonance

Far from resonance, loss is very low.

$$\omega_0 = 2.0$$
$$\omega_p = 6.0$$
$$\Gamma = 0.3$$



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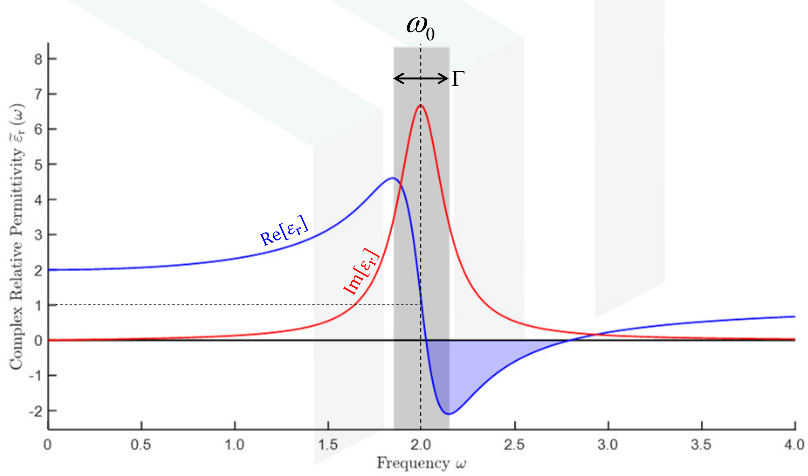
Observation #4 – Bandwidth

Damping rate determines width of resonance.



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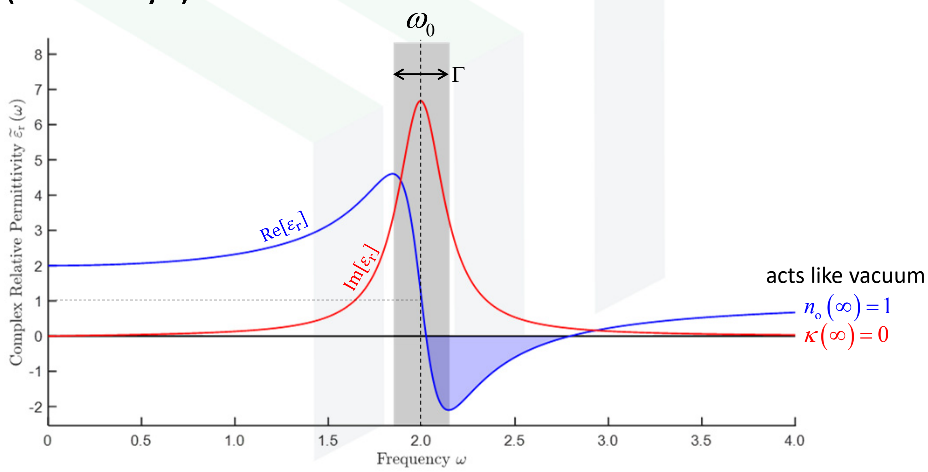
Observation #5 – Far Above Resonance

Materials act like vacuum at frequencies far above resonance (i.e. x-rays).



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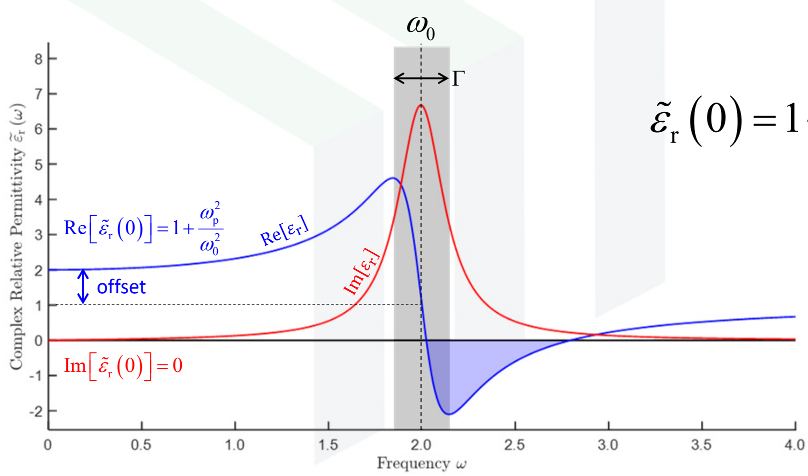
Observation #6 – Below Resonance

Dielectric constant contributes a DC offset below resonance.



$$\tilde{\varepsilon}_r(0)=1+\frac{\omega_p^2}{\omega_0^2}$$

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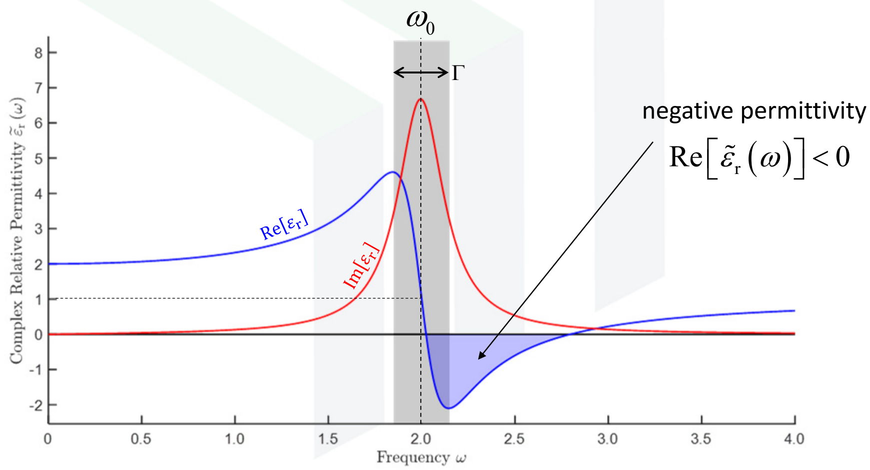
Observation #7 – Anomalous Permittivity

Dielectric constant can be negative and/or less than one.



Observation #8 – Anomalous Refractive Index

Refractive index can be less than one.