



Electromagnetics:
Microwave Engineering

Numerical Analysis of Transmission Lines



Outline

- Governing equations
- Numerical representation
- Finite-difference approximations
- Matrix solution of electric potential $V(x, y)$
- Calculating the transmission line parameters

Governing Equations

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Maxwell's Equations

Start with Maxwell's equations.

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \cdot \vec{D} = 0$$

$$\nabla \times \vec{H} = \vec{J} + \partial \vec{D} / \partial t$$

$$\nabla \times \vec{E} = -\partial \vec{B} / \partial t$$

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Electrostatic Approximation

The dimensions of a transmission line are typically much smaller than the operating wavelength so the wave nature of electromagnetics is less important to consider. Therefore, Maxwell's equations are essential be solved in the limit as $d/dt \rightarrow 0$.

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \cdot \vec{D} = 0$$

$$\nabla \times \vec{H} = \vec{J} + \cancel{\partial \vec{D} / \partial t}$$

$$\nabla \times \vec{E} = -\cancel{\partial \vec{B} / \partial t}$$

Electrostatics & Magnetostatics

Maxwell's equations have decoupled into two sets of equations. One describes electrostatics while the other describes magnetostatics.

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \cdot \vec{D} = 0$$

$$\nabla \times \vec{H} = \vec{J}$$

$$\nabla \times \vec{E} = 0$$

Electrostatics

$$\nabla \cdot \vec{D} = 0$$

$$\nabla \times \vec{E} = 0$$

Magnetostatics

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{H} = \vec{J}$$

Governing Equations

Maxwell's equations:

$$\nabla \cdot \vec{D} = 0 \quad \text{Eq. (1)}$$

$$\nabla \times \vec{E} = 0 \quad \text{Eq. (2)}$$

In addition, there is the constitutive relation

$$\vec{D} = \epsilon_r \vec{E} \quad \text{Eq. (3)}$$

It is not preferred to solve vector equations if there is a way to avoid it. Electrostatic fields are completely characterized by the electric potential $V(x, y)$.

$$\vec{E} = -\nabla V \quad \text{Eq. (4)}$$

Differential Equation to Solve

$$\nabla \cdot \vec{D} = 0 \quad \text{Eq. (1)}$$

$$\nabla \times \vec{E} = 0 \quad \text{Eq. (2)}$$

$$\vec{D} = \epsilon_r \vec{E} \quad \text{Eq. (3)}$$

$$\vec{E} = -\nabla V \quad \text{Eq. (4)}$$

1. Substitute Eq. (3) into Eq. (1) to eliminate \vec{D} .

$$\nabla \cdot (\epsilon_r \vec{E}) = 0 \quad \text{Eq. (5)}$$

2. Substitute Eq. (4) into Eq. (5) to eliminate \vec{E} .

$$\boxed{\nabla \cdot [\epsilon_r (\nabla V)] = 0} \quad \text{Eq. (6)}$$

Inhomogeneous Laplace's Equation

Differential Equation in a Homogeneous Medium

When the dielectric is homogeneous, our differential equation simplifies to

$$\nabla \cdot [\epsilon_r (\nabla V)] = 0 \rightarrow \nabla \cdot (\nabla V) = 0 \rightarrow \nabla^2 V = 0$$

In Cartesian coordinates, this expands to

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

Transmission lines are uniform in the z direction so $\partial^2/\partial z^2 = 0$ and Laplace's equation reduces to

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$



Numerical Representations

Discrete Function for Electric Potential $V(x, y)$

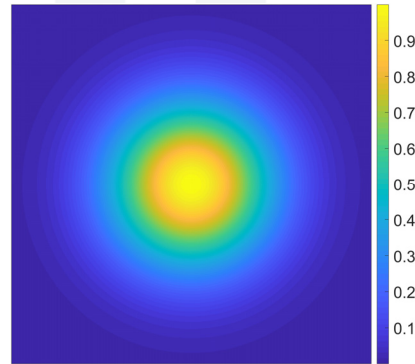
Analytical functions contain an infinite amount of information.

In order to store $V(x, y)$ on a computer, it is stored in an array where function values are only known at discrete points.

Start with the analytical function $V(x, y)$.

$$V(x, y)$$

y



x

Discrete Function for Electric Potential $V(x, y)$

Analytical functions contain an infinite amount of information.

In order to store $V(x, y)$ on a computer, it is stored in an array where function values are only known at discrete points.

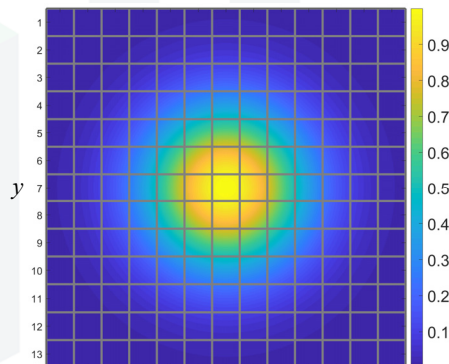
Divide space into a grid.

$$\Delta y \updownarrow \square$$

$$\Delta x \leftarrow \rightarrow$$

$$V(x, y)$$

y



x

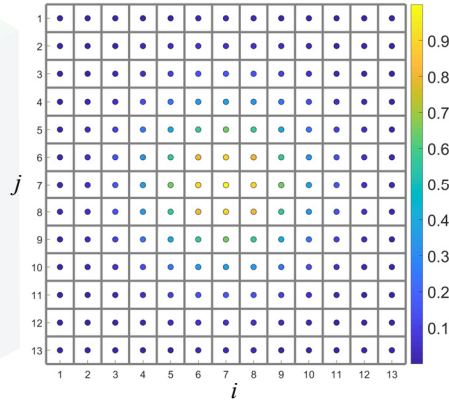
Discrete Function for Electric Potential $V(x, y)$

Analytical functions contain an infinite amount of information.

In order to store $V(x, y)$ on a computer, it is stored in an array where function values are only known at discrete points.

Store the analytical function only at a discrete points inside of each cell on the grid.

$$V(i, j)$$



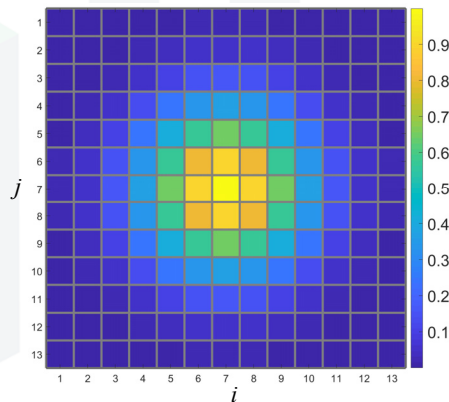
Discrete Function for Electric Potential $V(x, y)$

Analytical functions contain an infinite amount of information.

In order to store $V(x, y)$ on a computer, it is stored in an array where function values are only known at discrete points.

The discrete function tends to be thought of as being uniform throughout the cell, but this is technically incorrect.

$$V(i, j)$$



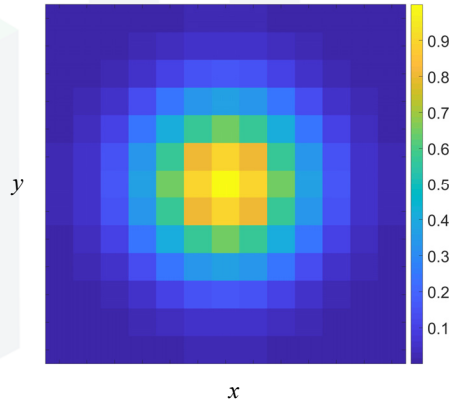
Discrete Function for Electric Potential $V(x, y)$

Analytical functions contain an infinite amount of information.

In order to store $V(x, y)$ on a computer, it is stored in an array where function values are only known at discrete points.

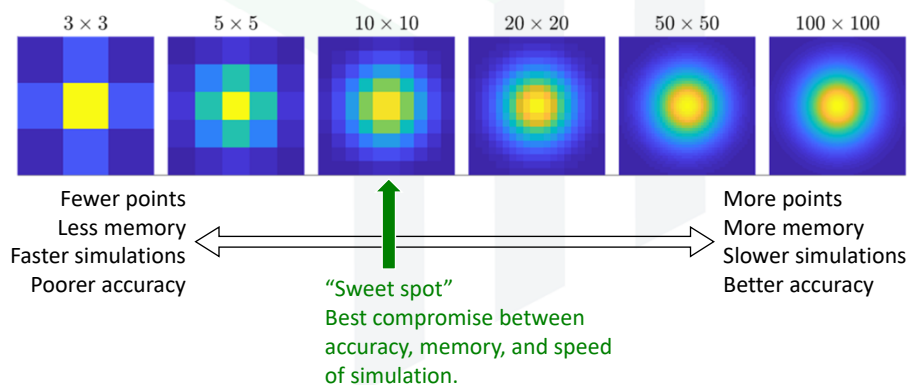
The discrete function $V(i, j)$ is usually visualized this way.

$$V(i, j) \approx V(x, y)$$



A Fundamental Tradeoff for Discrete Functions

There is always a fundamental trade-off between accuracy and speed of simulation. The purpose of almost all work in computational methods is to get improved accuracy with fewer points.



Finite-Difference Approximations

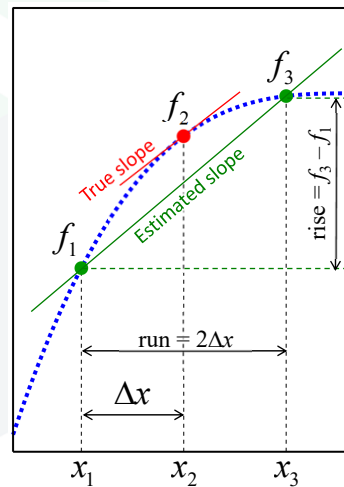
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Finite-Difference Approximations (1 of 2)

Suppose the first-order derivative of $f(x)$ is to be numerically calculated at $x = x_2$.

The first-order derivative is slope. The slope can be estimated as rise÷run using information from surrounding points.

$$f'(x_2) \approx \frac{\text{rise}}{\text{run}} = \frac{f_3 - f_1}{2\Delta x}$$



EMPossible

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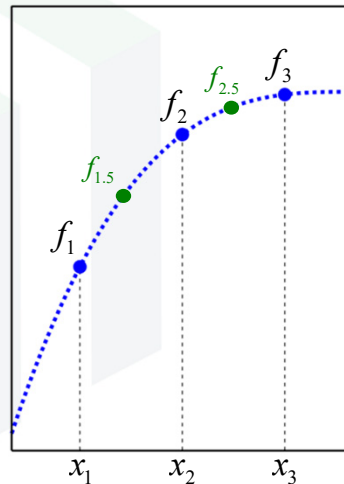
Finite-Difference Approximations (2 of 2)

The derivative at the midpoints between data points can be estimated.

$$f'(x_{1.5}) = \frac{f_2 - f_1}{\Delta x} \quad f'(x_{2.5}) = \frac{f_3 - f_2}{\Delta x}$$

The second-order derivative is the slope of the slope.

$$\begin{aligned} f''(x_2) &= \frac{f'(x_{2.5}) - f'(x_{1.5})}{\Delta x} \\ &= \frac{\frac{f_3 - f_2}{\Delta x} - \frac{f_2 - f_1}{\Delta x}}{\Delta x} \\ &= \frac{f_3 - 2f_2 + f_1}{\Delta x^2} \end{aligned}$$



Finite-Difference Approximation of Laplace's Equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

$$\frac{\partial^2 V}{\partial x^2} \cong \frac{V(i+1, j) - 2V(i, j) + V(i-1, j)}{(\Delta x)^2}$$

$$\frac{\partial^2 V}{\partial y^2} \cong \frac{V(i, j+1) - 2V(i, j) + V(i, j-1)}{(\Delta y)^2}$$

$$\frac{V(i+1, j) - 2V(i, j) + V(i-1, j)}{(\Delta x)^2} + \frac{V(i, j+1) - 2V(i, j) + V(i, j-1)}{(\Delta y)^2} = 0$$

Rearrange Terms in the Finite-Difference Equation

$$\frac{V(i+1, j) - 2V(i, j) + V(i-1, j)}{(\Delta x)^2} + \frac{V(i, j+1) - 2V(i, j) + V(i, j-1)}{(\Delta y)^2} = 0$$

$$\frac{V(i+1, j)}{(\Delta x)^2} - \frac{2V(i, j)}{(\Delta x)^2} + \frac{V(i-1, j)}{(\Delta x)^2} + \frac{V(i, j+1)}{(\Delta y)^2} - \frac{2V(i, j)}{(\Delta y)^2} + \frac{V(i, j-1)}{(\Delta y)^2} = 0$$

$$\left[\frac{1}{(\Delta x)^2} \right] V(i-1, j) + \left[\frac{1}{(\Delta x)^2} \right] V(i+1, j) - \left[\frac{2}{(\Delta x)^2} + \frac{2}{(\Delta y)^2} \right] V(i, j) + \left[\frac{1}{(\Delta y)^2} \right] V(i, j-1) + \left[\frac{1}{(\Delta y)^2} \right] V(i, j+1) = 0$$

Matrix Solution of Electric Potential $V(x, y)$

Write Large Set of Equations

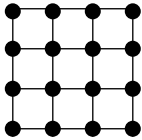
The final form of the finite-difference equation is

$$\left[\frac{1}{(\Delta x)^2} \right] V(i-1, j) + \left[\frac{1}{(\Delta x)^2} \right] V(i+1, j) - \left[\frac{2}{(\Delta x)^2} + \frac{2}{(\Delta y)^2} \right] V(i, j) + \left[\frac{1}{(\Delta y)^2} \right] V(i, j-1) + \left[\frac{1}{(\Delta y)^2} \right] V(i, j+1) = 0$$

This equation is written once for every point on the grid.

$$N_x = 4 \quad \Delta x = 0.5$$

$$N_y = 4 \quad \Delta y = 0.5$$

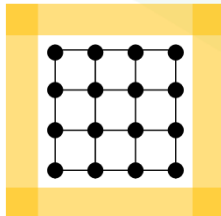


$$\begin{aligned} 4 \cdot V(0,1) + 4 \cdot V(2,1) - 16 \cdot V(1,1) + 4 \cdot V(1,0) + 4 \cdot V(1,2) &= 0 \\ 4 \cdot V(1,1) + 4 \cdot V(3,1) - 16 \cdot V(2,1) + 4 \cdot V(2,0) + 4 \cdot V(2,2) &= 0 \\ 4 \cdot V(2,1) + 4 \cdot V(4,1) - 16 \cdot V(3,1) + 4 \cdot V(3,0) + 4 \cdot V(3,2) &= 0 \\ 4 \cdot V(3,1) + 4 \cdot V(5,1) - 16 \cdot V(4,1) + 4 \cdot V(4,0) + 4 \cdot V(4,2) &= 0 \\ 4 \cdot V(0,2) + 4 \cdot V(2,2) - 16 \cdot V(1,2) + 4 \cdot V(1,1) + 4 \cdot V(1,3) &= 0 \\ 4 \cdot V(1,2) + 4 \cdot V(3,2) - 16 \cdot V(2,2) + 4 \cdot V(2,1) + 4 \cdot V(2,3) &= 0 \\ 4 \cdot V(2,2) + 4 \cdot V(4,2) - 16 \cdot V(3,2) + 4 \cdot V(3,1) + 4 \cdot V(3,3) &= 0 \\ 4 \cdot V(3,2) + 4 \cdot V(5,2) - 16 \cdot V(4,2) + 4 \cdot V(4,1) + 4 \cdot V(4,3) &= 0 \\ 4 \cdot V(0,3) + 4 \cdot V(2,3) - 16 \cdot V(1,3) + 4 \cdot V(1,2) + 4 \cdot V(1,4) &= 0 \\ 4 \cdot V(1,3) + 4 \cdot V(3,3) - 16 \cdot V(2,3) + 4 \cdot V(2,2) + 4 \cdot V(2,4) &= 0 \\ 4 \cdot V(2,3) + 4 \cdot V(4,3) - 16 \cdot V(3,3) + 4 \cdot V(3,2) + 4 \cdot V(3,4) &= 0 \\ 4 \cdot V(3,3) + 4 \cdot V(5,3) - 16 \cdot V(4,3) + 4 \cdot V(4,2) + 4 \cdot V(4,4) &= 0 \\ 4 \cdot V(0,4) + 4 \cdot V(2,4) - 16 \cdot V(1,4) + 4 \cdot V(1,3) + 4 \cdot V(1,5) &= 0 \\ 4 \cdot V(1,4) + 4 \cdot V(3,4) - 16 \cdot V(2,4) + 4 \cdot V(2,3) + 4 \cdot V(2,5) &= 0 \\ 4 \cdot V(2,4) + 4 \cdot V(4,4) - 16 \cdot V(3,4) + 4 \cdot V(3,3) + 4 \cdot V(3,5) &= 0 \\ 4 \cdot V(3,4) + 4 \cdot V(5,4) - 16 \cdot V(4,4) + 4 \cdot V(4,3) + 4 \cdot V(4,5) &= 0 \end{aligned}$$

Consider Boundary Conditions

$$N_x = 4 \quad \Delta x = 0.5$$

$$N_y = 4 \quad \Delta y = 0.5$$



$$\begin{aligned} 4 \cdot V(0,1) + 4 \cdot V(2,1) - 16 \cdot V(1,1) + 4 \cdot V(1,0) + 4 \cdot V(1,2) &= 0 \\ 4 \cdot V(1,1) + 4 \cdot V(3,1) - 16 \cdot V(2,1) + 4 \cdot V(2,0) + 4 \cdot V(2,2) &= 0 \\ 4 \cdot V(2,1) + 4 \cdot V(4,1) - 16 \cdot V(3,1) + 4 \cdot V(3,0) + 4 \cdot V(3,2) &= 0 \\ 4 \cdot V(3,1) + 4 \cdot V(5,1) - 16 \cdot V(4,1) + 4 \cdot V(4,0) + 4 \cdot V(4,2) &= 0 \\ 4 \cdot V(0,2) + 4 \cdot V(2,2) - 16 \cdot V(1,2) + 4 \cdot V(1,1) + 4 \cdot V(1,3) &= 0 \\ 4 \cdot V(1,2) + 4 \cdot V(3,2) - 16 \cdot V(2,2) + 4 \cdot V(2,1) + 4 \cdot V(2,3) &= 0 \\ 4 \cdot V(2,2) + 4 \cdot V(4,2) - 16 \cdot V(3,2) + 4 \cdot V(3,1) + 4 \cdot V(3,3) &= 0 \\ 4 \cdot V(3,2) + 4 \cdot V(5,2) - 16 \cdot V(4,2) + 4 \cdot V(4,1) + 4 \cdot V(4,3) &= 0 \\ 4 \cdot V(0,3) + 4 \cdot V(2,3) - 16 \cdot V(1,3) + 4 \cdot V(1,2) + 4 \cdot V(1,4) &= 0 \\ 4 \cdot V(1,3) + 4 \cdot V(3,3) - 16 \cdot V(2,3) + 4 \cdot V(2,2) + 4 \cdot V(2,4) &= 0 \\ 4 \cdot V(2,3) + 4 \cdot V(4,3) - 16 \cdot V(3,3) + 4 \cdot V(3,2) + 4 \cdot V(3,4) &= 0 \\ 4 \cdot V(3,3) + 4 \cdot V(5,3) - 16 \cdot V(4,3) + 4 \cdot V(4,2) + 4 \cdot V(4,4) &= 0 \\ 4 \cdot V(0,4) + 4 \cdot V(2,4) - 16 \cdot V(1,4) + 4 \cdot V(1,3) + 4 \cdot V(1,5) &= 0 \\ 4 \cdot V(1,4) + 4 \cdot V(3,4) - 16 \cdot V(2,4) + 4 \cdot V(2,3) + 4 \cdot V(2,5) &= 0 \\ 4 \cdot V(2,4) + 4 \cdot V(4,4) - 16 \cdot V(3,4) + 4 \cdot V(3,3) + 4 \cdot V(3,5) &= 0 \\ 4 \cdot V(3,4) + 4 \cdot V(5,4) - 16 \cdot V(4,4) + 4 \cdot V(4,3) + 4 \cdot V(4,5) &= 0 \end{aligned}$$

All of the highlighted terms will be set to zero.

→ Dirichlet Boundary Conditions

Calculating the Transmission Line Parameters

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Calculating the Electric Fields

Once the scalar potential $V(x, y)$ is found, the electric field intensity $\vec{E}(x, y)$ is

$$\vec{E}(x, y) = -\nabla V(x, y)$$

It follows that the electric flux density $\vec{D}(x, y)$ is

$$\vec{D}(x, y) = \epsilon_r(x, y) \vec{E}(x, y)$$

Distributed Capacitance C

In the electrostatic approximation, the transmission line is a capacitor. The total energy U stored in a capacitor is

$$U = \frac{1}{2} \iint_A (\vec{D} \cdot \vec{E}) dA$$

The capacitance C is related to the total stored energy U through

$$U = \frac{CV_0^2}{2} \quad V_0 \text{ is the voltage across the capacitor.}$$

If the above equations are set equal and solved for C , the answer is

$$C = \frac{1}{V_0^2} \iint_A (\vec{D} \cdot \vec{E}) dA$$

Distributed Inductance L

The voltage signal v_V along the transmission line travels at the same velocity as the electromagnetic field v_E . This means

$$v_V = v_E \rightarrow \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{\mu\epsilon}} \rightarrow LC = \frac{\mu_r \epsilon_r}{c_0^2}$$

Solving this for L gives

$$L = \frac{\mu_r \epsilon_r}{c_0^2 C} \quad \text{This means the distributed inductance } L \text{ can be calculated directly from the distributed capacitance } C.$$

Dielectric materials should not alter the inductance. However if the value of C calculated on the previous slide is used, it will. This is incorrect. The solution is to calculate the distributed capacitance C_h with a homogeneous dielectric and then calculate the distributed inductance L from this.

$$L = \frac{1}{c_0^2 C_h}$$

Calculating the Transmission Line Parameters

The characteristic impedance Z_c is calculated from the distributed inductance L and distributed capacitance C through

$$Z_c = \sqrt{\frac{L}{C}}$$

It follows that the phase constant is

$$\beta = \omega\sqrt{LC} = k_0 n_{\text{eff}}$$

$$k_0 = \frac{2\pi}{\lambda_0}$$

Recall, both Z_c and β are needed to analyze transmission line circuits.

Note: This simple model did not consider loss, so $R = G = 0$ (lossless transmission line)