Computational Science:
Computational Methods in Engineering

Curve Fitting to Polynomials & Interpolation/Extrapolation

https://empossible.net/academics/emp4301_5301/

Outline

- Exact Fit Methods – Fitting Polynomials
- Interpolation & Extrapolation
- MATLAB Implementation
  https://empossible.net/academics/emp4301_5301/

2

Fitting Polynomials

https://empossible.net/academics/emp4301_5301/

3

Statement of the Problem

Suppose it is desired to fit the following Nth-order polynomial to a set of points.

$$f(x) \approx a_0 + a_1 x + a_2 x^2 + \cdots + a_N x^N$$

There exists $N+1$ unknown coefficients so $N+1$ points are needed to calculate the coefficients exactly.

EMPossible

4

Calculating the Unknown Polynomial Coefficients

First, write the polynomial at the $N+1$ points.

$$
\begin{aligned}
f(x_1) &= a_0 + a_1 x_1 + a_2 x_1^2 + \cdots + a_N x_1^N \\
f(x_2) &= a_0 + a_1 x_2 + a_2 x_2^2 + \cdots + a_N x_2^N \\
&\vdots \\
f(x_{N+1}) &= a_0 + a_1 x_{N+1} + a_2 x_{N+1}^2 + \cdots + a_N x_{N+1}^N
\end{aligned}
$$

Since this is an exact fit, there are no residual terms.

EMPossible 5

Calculating the Unknown Polynomial Coefficients

Second, put the equations into matrix form.

$$
\begin{aligned}
f(x_1) &= a_0 + a_1 x_1 + a_2 x_1^2 + \cdots + a_N x_1^N \\
f(x_2) &= a_0 + a_1 x_2 + a_2 x_2^2 + \cdots + a_N x_2^N \\
&\vdots \\
f(x_{N+1}) &= a_0 + a_1 x_{N+1} + a_2 x_{N+1}^2 + \cdots + a_N x_{N+1}^N
\end{aligned}
\Rightarrow
\begin{bmatrix} f(x_1) \\ f(x_2) \\ f(x_3) \\ \vdots \\ f(x_{N+1}) \end{bmatrix}
=
\begin{bmatrix}
1 & x_1 & x_1^2 & & x_1^N \\
1 & x_2 & x_2^2 & \cdots & x_2^N \\
1 & x_3 & x_3^2 & & x_3^N \\
 & \vdots & & \ddots & \\
1 & x_{N+1} & x_{N+1}^2 & & x_{N+1}^N
\end{bmatrix}
\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_N \end{bmatrix}
$$

This can be written more compactly as

$$[f] = [X][a]$$

Aside:
This is a Vandermonde matrix and is usually ill-conditioned for large matrices. See Lagrange interpolation.

EMPossible 6

Calculating the Unknown Polynomial Coefficients

Last, the matrix equation is solved for $[a]$ to find the polynomial coefficients.

$$[a]=[X]^{-1}[f]$$

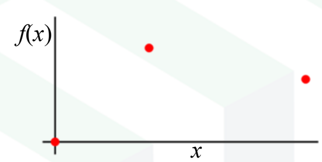
$$[a]=\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_N \end{bmatrix}$$

EMPossible 7

Example 1 (1 of 2)

Fit the following points to a polynomial..

$$f(0)=0$$
$$f(1.5)=1.5$$
$$f(4.0)=1.0$$



Step 1 – Determine order of polynomial

Since there are three points, a quadratic polynomial can be fit.

$$f(x)\approx a_0+a_1x+a_2x^2$$

EMPossible 8

Example 1 (2 of 2)

Step 2 – The matrix equation is

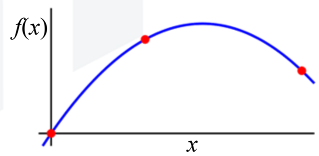
$$\begin{aligned} f_1 &= a_0 + a_1 x_1 + a_2 x_1^2 \\ f_2 &= a_0 + a_1 x_2 + a_2 x_2^2 \\ f_3 &= a_0 + a_1 x_3 + a_2 x_3^2 \end{aligned} \rightarrow \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} \rightarrow \begin{bmatrix} 0 \\ 1.5 \\ 1.0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1.5 & 2.25 \\ 1 & 4 & 16 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$$

Step 3 – Calculate polynomial coefficients.

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1.5 & 2.25 \\ 1 & 4 & 16 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1.5 \\ 1.0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1.45 \\ -0.3 \end{bmatrix}$$

Step 4 – Write the final polynomial.

$$f(x) \approx 1.45x - 0.3x^2$$



EMPossible

9

Interpolation & Extrapolation

https://empossible.net/academics/emp4301_5301/

10

What is Interpolation & Extrapolation?

Interpolation – Sometimes an intermediate value located between measured values is needed.

Extrapolation – Sometimes a value located outside of the measured values is needed.

Why?
Measuring the new value may be difficult, expensive, time-consuming or impossible.

Interpolation and extrapolation can be thought of as two steps:

Step 1 – Fit data to a curve.
Step 2 – Use the curve fit to calculate the new value(s).

EMPossible 11

Linear Interpolation

Given two data points, a line can be fit from which it is possible to interpolate or extrapolate anything else.

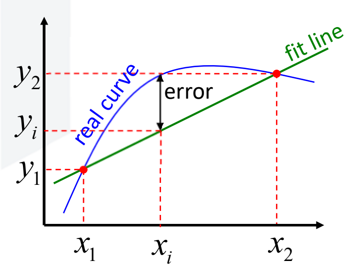
Given two data points (x_1, y_1) and (x_2, y_2), the equation for the line is

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

slope

This equation can be used to interpolate y_i from any desired position x_i.

$$y_i = y_1 + \frac{y_2 - y_1}{x_2 - x_1}(x_i - x_1)$$

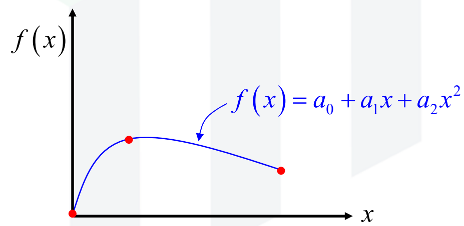


EMPossible 12

Polynomial Interpolation (1 of 3)

Interpolation using a quadratic polynomial is likely the most common method of interpolation.

This requires three points.



Polynomial Interpolation (2 of 3)

Write the matrix equation $[f] = [X][a]$.

$$\begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$$

Solve for the polynomial coefficients $[a]$.

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{bmatrix}^{-1} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} \frac{x_2 x_3}{(x_1 - x_2)(x_1 - x_3)} & -\frac{x_1 x_3}{(x_1 - x_2)(x_2 - x_3)} & \frac{x_1 x_2}{(x_1 - x_3)(x_2 - x_3)} \\ -\frac{x_2 + x_3}{(x_1 - x_2)(x_1 - x_3)} & \frac{x_1 + x_3}{(x_1 - x_2)(x_2 - x_3)} & -\frac{x_1 + x_2}{(x_1 - x_3)(x_2 - x_3)} \\ \frac{1}{(x_1 - x_2)(x_1 - x_3)} & -\frac{1}{(x_1 - x_2)(x_2 - x_3)} & \frac{1}{(x_1 - x_3)(x_2 - x_3)} \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$$

Polynomial Interpolation (3 of 3)

The final equations are

$$a_0 = \frac{x_2 x_3}{(x_1 - x_2)(x_1 - x_3)} f_1 - \frac{x_1 x_3}{(x_1 - x_2)(x_2 - x_3)} f_2 + \frac{x_1 x_2}{(x_1 - x_3)(x_2 - x_3)} f_3$$

$$a_1 = -\frac{x_2 + x_3}{(x_1 - x_2)(x_1 - x_3)} f_1 + \frac{x_1 + x_3}{(x_1 - x_2)(x_2 - x_3)} f_2 - \frac{x_1 + x_2}{(x_1 - x_3)(x_2 - x_3)} f_3$$

$$a_2 = \frac{1}{(x_1 - x_2)(x_1 - x_3)} f_1 - \frac{1}{(x_1 - x_2)(x_2 - x_3)} f_2 + \frac{1}{(x_1 - x_3)(x_2 - x_3)} f_3$$

Typically, these are not calculated using these equations. Instead, the matrix inversion is performed numerically.

EMPossible

15

Example 1 – Quadratic Interpolation

Given the following points, interpolate the value at $x = 3$.

$$f(0) = 0$$
$$f(1.5) = 1.5$$
$$f(4.0) = 1.0$$

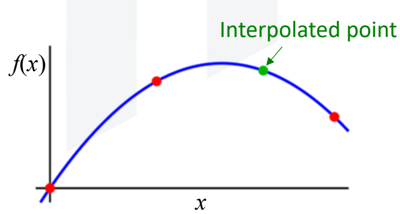
Solution

Step 1 – These points were previously fit to a quadratic polynomial. The answer was

$$f(x) = 1.45x - 0.3x^2$$

Step 2 – This is evaluated at $x = 3$.

$$f(3) = 1.45(3) - 0.3(3)^2 = \boxed{1.65}$$



EMPossible

16

Example 2 – Quadratic Extrapolation

Given the following points, extrapolate the value at $x = -0.5$.

$$f(0) = 0$$
$$f(1.5) = 1.5$$
$$f(4.0) = 1.0$$

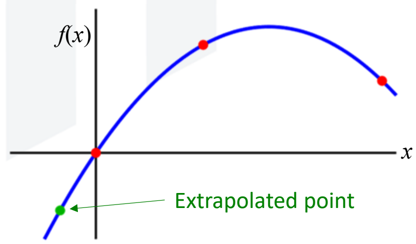
Solution

Step 1 – These points were previously fit to a quadratic polynomial. The answer was

$$f(x) = 1.45x - 0.3x^2$$

Step 2 – Evaluate this at $x = -0.5$.

$$f(-0.5) = 1.45(-0.5) - 0.3(-0.5)^2 = \boxed{-0.8}$$



EMPossible

17