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Advanced Electromagnetics:
21st Century Electromagnetics

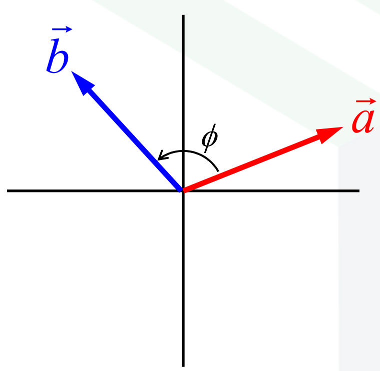
Tensor Math

Lecture Outline

- Rotation matrices
- Tensor rotation
- Tensor diagonalization (i.e. "unrotation")
- More on rotation matrices
  - Rotation matrix for rotating $\vec{a}$ onto $\vec{b}$
  - Rotation matrix for rotating $\left(\vec{a}, \vec{b}, \vec{c}\right)$ onto $\left(\vec{a}', \vec{b}', \vec{c}'\right)$.

Rotation Matrices

Definition of a Rotation Matrix



Rotation matrix $[R]$ is defined as

$$\vec{b} = \left[R(\phi)\right]\vec{a}$$

The rotation matrix should not change the amplitude. This implies that $[R]$ is *unitary*.

$$\left|\vec{b}\right| = \left|\vec{a}\right|$$

$$[R]^H = [R]^{-1}$$

$$[R][R]^H = [R]^H[R] = \mathbf{I}$$

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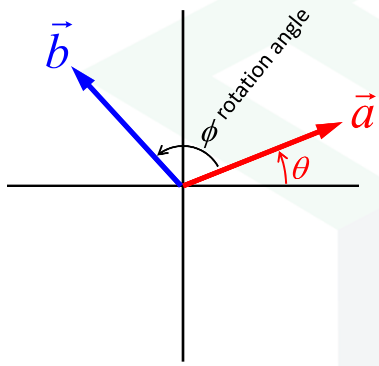
Derivation of a 2D Rotation Matrix



Start with vector $\vec{a}$ at angle θ.

$$\vec{a} = a_x\hat{x} + a_y\hat{y}$$
$$= a\left[\cos\theta\,\hat{x} + \sin\theta\,\hat{y}\right]$$

Add angle ϕ to rotate the vector

$$\vec{b} = a\left[\cos(\theta+\phi)\,\hat{x} + \sin(\theta+\phi)\,\hat{y}\right]$$

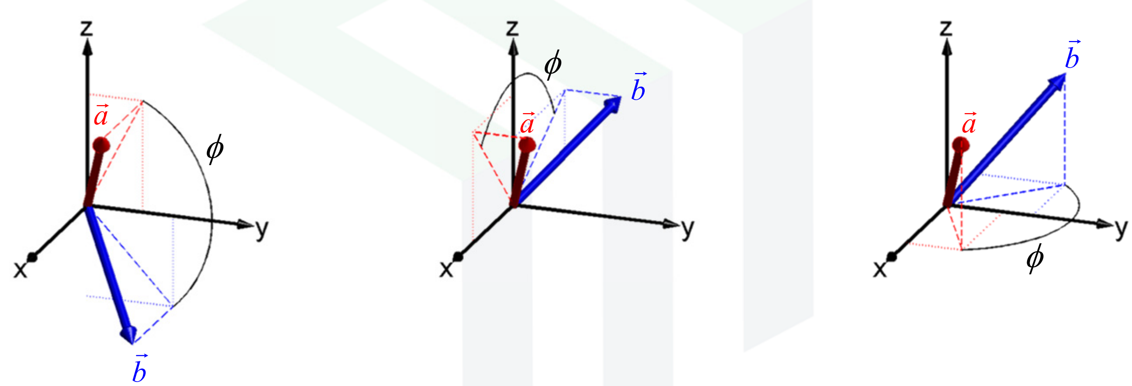
Apply trig identities

$$\vec{b} = a\begin{bmatrix} (\cos\theta\cos\phi - \sin\theta\sin\phi)\,\hat{x} \\ +(\sin\theta\cos\phi + \cos\theta\sin\phi)\,\hat{y} \end{bmatrix}$$
$$= \begin{bmatrix} (a_x\cos\phi - a_y\sin\phi)\,\hat{x} \\ +(a_y\cos\phi + a_x\sin\phi)\,\hat{y} \end{bmatrix}$$

$$\begin{bmatrix} b_x \\ b_y \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}}_{[R(\phi)]}\begin{bmatrix} a_x \\ a_y \end{bmatrix}$$

3D Rotation Matrices

Rotation matrices for 3D coordinates can be written directly from the previous result.



$$[R_x(\phi)] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix}$$

$$[R_y(\phi)] = \begin{bmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{bmatrix}$$

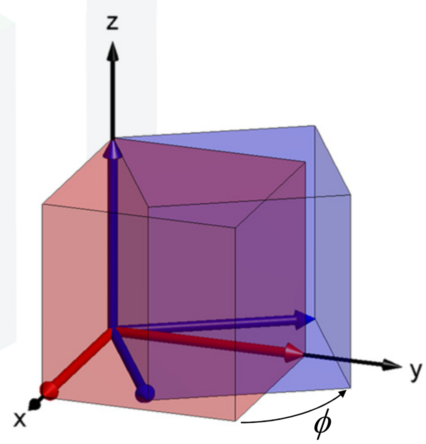
$$[R_z(\phi)] = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Tensor Rotation

Tensor Rotation

Tensors are rotated using the same rotation matrices, but in a slightly different manner than vectors.

$$\left[\varepsilon^{(R)}\right]=\left[R_z(\phi)\right]\left[\varepsilon\right]\left[R_z(\phi)\right]^{-1}$$



Combinations of Rotations

Suppose it is desired to first rotate about the x-axis by some angle, second rotate about the y-axis by some angle, and third rotate about the z-axis by some angle.

For vectors, this would be done as…

$$\vec{b} = [R_z][R_y][R_x]\vec{a}$$

For tensors, this would be done as…

$$\left[\varepsilon^{(R)}\right] = [R_z][R_y][R_x][\varepsilon][R_x]^{-1}[R_y]^{-1}[R_z]^{-1}$$

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Composite Rotation Matrix

Multiple rotation matrices can be combined into a single composite rotation matrix $[R]$.

$$[R] = [R_y][R_z][R_x]$$

This equation implies we will rotate first about the x-axis, second about the z-axis, and third about the y-axis.

Vector rotation using the composite rotation matrix is

$$\vec{b} = [R]\vec{a}$$

Tensor rotation using the composite rotation matrix is

$$\left[\varepsilon^{(R)}\right] = [R][\varepsilon][R]^{-1}$$

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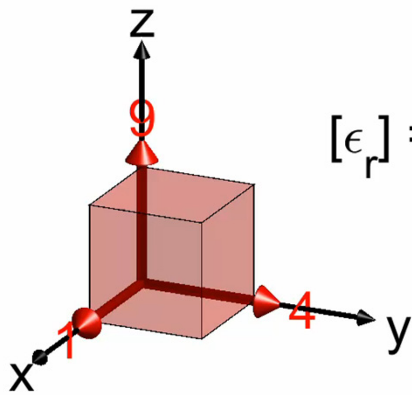
Animation of Tensor Rotation



$$[\epsilon_r] = \begin{bmatrix} 1.0 & 0.0 & 0.0 \\ 0.0 & 4.0 & 0.0 \\ 0.0 & 0.0 & 9.0 \end{bmatrix}$$

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Order of Rotations Matter

The order that the rotation matrices are multiplied controls the order that the rotations are performed.

$$\vec{b} = \left[R_z\left(-45°\right)\right]\left[R_y\left(220°\right)\right]\left[R_x\left(10°\right)\right]\vec{a}$$

1. First rotates about the x-axis by 10°.
2. Second rotates about the y-axis by 220°.
3. Third rotates about the z-axis by -45°.

In general, different results are obtained when the order of rotation is changed.

$$\left[R_y\right]\left[R_x\right] \neq \left[R_x\right]\left[R_y\right]$$

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Numerical Examples (1 of 2)

Given: $\left[\varepsilon_r\right]=\begin{bmatrix}1&0&0\\0&2&0\\0&0&3\end{bmatrix}$

Rotate about x-axis by 20°

$$\left[R_x(20°)\right]=\begin{bmatrix}1&0&0\\0&0.9397&-0.3420\\0&0.3420&0.9397\end{bmatrix}\qquad \left[R_x(20°)\right]\left[\varepsilon_r\right]=\begin{bmatrix}1&0&0\\0&2.1170&-0.3214\\0&-0.3214&2.8830\end{bmatrix}$$

Rotate about y-axis by 45°

$$\left[R_y(45°)\right]=\begin{bmatrix}0.7071&0&0.7071\\0&1&0\\-0.7071&0&0.7071\end{bmatrix}\qquad \left[R_y(45°)\right]\left[\varepsilon_r\right]=\begin{bmatrix}2&0&1\\0&2&0\\1&0&2\end{bmatrix}$$

Rotate about z-axis by 60°

$$\left[R_z(60°)\right]=\begin{bmatrix}0.5000&-0.8660&0\\0.8660&0.5000&0\\0&0&1\end{bmatrix}\qquad \left[R_z(60°)\right]\left[\varepsilon_r\right]=\begin{bmatrix}1.7500&-0.4330&0\\-0.4330&1.2500&0\\0&0&3\end{bmatrix}$$

Note: the tensor element along the axis of rotation remains unchanged.

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Numerical Examples (2 of 2)

Given: $\left[\varepsilon_r\right]=\begin{bmatrix}1&0&0\\0&2&0\\0&0&3\end{bmatrix}$

Rotate first about x-axis by 20° and second about the y-axis by 45°

$$\left[R_y(45°)R_x(20°)\right]=\begin{bmatrix}0.7071&0.2418&0.6645\\0&0.9397&-0.3420\\-0.7071&0.2418&0.6645\end{bmatrix}\qquad \left[R_y(45°)R_x(20°)\right]\left[\varepsilon_r\right]=\begin{bmatrix}1.9415&-0.2273&0.9415\\-0.2273&2.1170&-0.2273\\0.9415&-0.2273&1.9415\end{bmatrix}$$

Rotate first about z-axis by 60° and second about the y-axis by 45°

$$\left[R_y(45°)R_z(60°)\right]=\begin{bmatrix}0.3536&-0.6124&0.7071\\0.8660&0.5000&0\\-0.3536&0.6124&0.7071\end{bmatrix}\qquad \left[R_y(45°)R_z(60°)\right]\left[\varepsilon_r\right]=\begin{bmatrix}2.3750&-0.3062&0.6250\\-0.3062&1.2500&0.3062\\0.6250&0.3062&2.3750\end{bmatrix}$$

Rotate first about x-axis by 20°, second about the y-axis by 45°, and third about the z-axis by 60°

$$\left[R_z(60°)R_y(45°)R_x(20°)\right]=\begin{bmatrix}0.3536&-0.6929&0.6284\\0.6124&0.6793&0.4044\\-0.7071&0.2418&0.6645\end{bmatrix}\qquad \left[R_z(60°)R_y(45°)R_x(20°)\right]\left[\varepsilon_r\right]=\begin{bmatrix}2.2699&0.0377&0.6676\\0.0377&1.7886&0.7017\\0.6676&0.7017&1.9415\end{bmatrix}$$

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Tensor Diagonalization ("Unrotation")

Tensor Diagonalization (1 of 2)

A tensor can always be diagonalized along its principle axes $\hat{a}$, $\hat{b}$, and $\hat{c}$.

$$[\varepsilon]=\begin{bmatrix}\varepsilon_a & 0 & 0\\ 0 & \varepsilon_b & 0\\ 0 & 0 & \varepsilon_c\end{bmatrix}$$

Principle Axes:

ε_a is along $\hat{a}$

ε_b is along $\hat{b}$

ε_c is along $\hat{c}$

Convention:

$\varepsilon_a \le \varepsilon_b \le \varepsilon_c$

But suppose a general nine-element tensor is given.

$$[\varepsilon_{\text{rot}}]=\begin{bmatrix}\varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz}\\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz}\\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz}\end{bmatrix}$$

How are the diagonal elements ε_a, ε_b, ε_c and the principle axes $\hat{a}$, $\hat{b}$, and $\hat{c}$ determined?

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Tensor Diagonalization (2 of 2)

The diagonalized tensor $[\varepsilon]$ is related to the rotated tensor $[\varepsilon_{\text{rot}}]$ through the composite rotation matrix $[R]$.

$$[\varepsilon_{\text{rot}}] = [R][\varepsilon][R]^{-1}$$

Calculate the eigen-vectors and eigen-values of $[\varepsilon_{\text{rot}}]$.

$$[\varepsilon_{\text{rot}}] \rightarrow \begin{array}{l} [R] \text{ is the eigen-vector matrix of } [\varepsilon_{\text{rot}}] \\ [\varepsilon] \text{ is the eigen-value matrix of } [\varepsilon_{\text{rot}}] \end{array}$$

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Determining Principle Axes (1 of 2)

What are the principle axes of $[\varepsilon_{\text{rot}}]$?

To find out, put the original principal axes $\hat{a}$, $\hat{b}$, and $\hat{c}$ into a matrix.

$$[\vec{P}] = \begin{bmatrix} a_x & b_x & c_x \\ a_y & b_y & c_y \\ a_z & b_z & c_z \end{bmatrix}$$

$$\begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \hat{a} & \hat{b} & \hat{c} \end{array}$$

Now rotate them according to $[R]$ so they correspond to that of $[\varepsilon_{\text{rot}}]$.

$$[\vec{P}_{\text{rot}}] = [R][\vec{P}]$$

A tensor is not being rotated here so rotation is not performed as $[R][P][R]^{-1}$.

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Determining Principle Axes (2 of 2)

For the special case of <u>cubic symmetry</u>, the principle axes start off as

$$[\vec{P}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{matrix} \uparrow & \uparrow & \uparrow \\ \hat{a} & \hat{b} & \hat{c} \end{matrix}$$

$[\vec{P}]$ is just the identity matrix here. The principle axes of the rotated tensor are then

$$[\vec{P}_{\text{rot}}] = [R][I] = [R]$$

Here is a second interpretation of the eigen-vector matrix $[R]$. These are the principle axes of the rotated tensor.

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Directly Setting the Principle Axes

The orientation may be directly known in terms of the desired crystal axes $\hat{a}$, $\hat{b}$, and $\hat{c}$.

Place these principal directions into a matrix and interpret it as the composite rotation matrix.

$$[R] = \begin{bmatrix} a_x & b_x & c_x \\ a_y & b_y & c_y \\ a_z & b_z & c_z \end{bmatrix}$$

$$\begin{matrix} \uparrow & \uparrow & \uparrow \\ \hat{a} & \hat{b} & \hat{c} \end{matrix}$$

The rotated tensor can be directly calculated from $[R]$.

$$[\varepsilon_{\text{rot}}] = [R][\varepsilon][R]^{-1}$$

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More on Rotation Matrices

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Rotation Matrix $[R]$ that Rotates $\vec{a}$ Onto $\vec{b}$

Normalize Vectors

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} \qquad \hat{b} = \frac{\vec{b}}{|\vec{b}|}$$

Algorithm

$$\vec{v} = \hat{a} \times \hat{b}$$

$$s = |\vec{v}| \qquad \text{i.e. } \sin\theta_{ab}$$

$$c = \hat{a} \bullet \hat{b} \qquad \text{i.e. } \cos\theta_{ab}$$

$$[\vec{v}\times] = \begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix}$$

$$[R] = [I] - [\vec{v}\times] - \left(\frac{1-c}{s^2}\right)[\vec{v}\times]^2$$

$[R]$ is such that

$$[R]\hat{a} = \hat{b}$$

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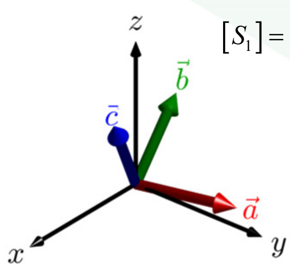
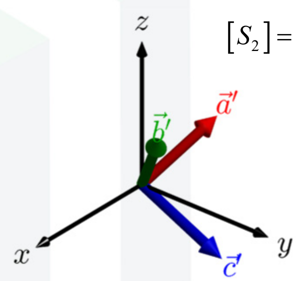
Rotation Matrix $[R]$ that Rotates $\left(\vec{a},\vec{b},\vec{c}\right)$ Onto $\left(\vec{a}',\vec{b}',\vec{c}'\right)$



$$[S_1]=\begin{bmatrix} a_x & b_x & c_x \\ a_y & b_y & c_y \\ a_z & b_z & c_z \end{bmatrix}$$



$$[S_2]=\begin{bmatrix} a'_x & b'_x & c'_x \\ a'_y & b'_y & c'_y \\ a'_z & b'_z & c'_z \end{bmatrix}$$

$$[R]=[S_2][S_1]^{-1}$$

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