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Computational Science:

Computational Methods in Engineering

The Family of Fourier Transforms

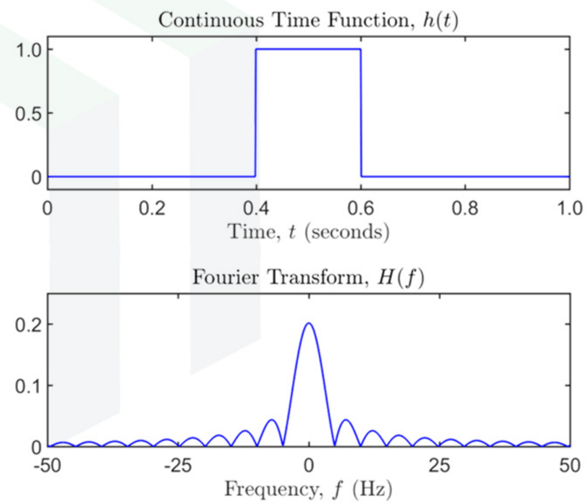
The Temporal Fourier Transform

The general Fourier transform is defined as

$$F(s)=\int_{-\infty}^{\infty} f(u)e^{-j2\pi us}du$$

The *temporal Fourier transform* calculates the frequency content of a time-domain signal.

$$F(\omega)=\int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$$



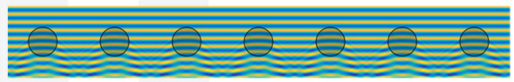
The Spatial Fourier Transform

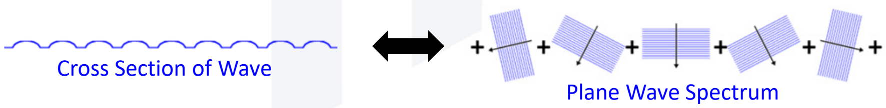
The general Fourier transform is defined as

$$F(s)=\int_{-\infty}^{\infty} f(u)e^{-j2\pi us}\,du$$

The *spatial Fourier transform* calculates the "spatial waves" comprising a signal.

$$F(k)=\int_{-\infty}^{\infty} f(x)e^{jkx}\,dx$$





Sign Convention

EQUATION(S)	ENGINEERING (Negative Sign Convention)	$-j \leftrightarrow i$	PHYSICS / SCIENCE (Positive Sign Convention)	
	Temporal	Spatial	Temporal	Spatial
Fourier Transform	$F(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$ $f(t)=\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega$	$F(k)=\frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{jkx}dx$ $f(x)=\int_{-\infty}^{\infty} F(k)e^{-jkx}dk$	$F(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)e^{i\omega t}dt$ $f(t)=\int_{-\infty}^{\infty} F(\omega)e^{-i\omega t}d\omega$	$F(k)=\frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-ikx}dx$ $f(x)=\int_{-\infty}^{\infty} F(k)e^{ikx}dk$
Fourier Series	$a_n=\frac{1}{\tau}\int_{-\tau/2}^{\tau/2} f(t)e^{-j\omega t}dt$ $f(t)=\sum_{n=-\infty}^{\infty} a_n e^{j\omega t}$	$a_n=\frac{1}{\Lambda}\int_{-\Lambda/2}^{\Lambda/2} f(x)e^{j\frac{2\pi nx}{\Lambda}}dx$ $f(x)=\sum_{n=-\infty}^{\infty} a_n e^{-j\frac{2\pi nx}{\Lambda}}$	$a_n=\frac{1}{\tau}\int_{-\tau/2}^{\tau/2} f(t)e^{i\omega t}dt$ $f(t)=\sum_{n=-\infty}^{\infty} a_n e^{-i\omega t}$	$a_n=\frac{1}{\Lambda}\int_{-\Lambda/2}^{\Lambda/2} f(x)e^{-i\frac{2\pi nx}{\Lambda}}dx$ $f(x)=\sum_{n=-\infty}^{\infty} a_n e^{i\frac{2\pi nx}{\Lambda}}$

The Fourier Series

If a signal is periodic with period τ, the Fourier transform is a series of impulses. Therefore, the Fourier transform can be written a series of discrete sines and cosines, or complex exponentials. This is called the *Fourier series*.

The Fourier coefficients are the amplitudes of the sines and cosines, or complex exponentials.

$$f(t) = \sum_{k=-\infty}^{\infty} a_k e^{-j\frac{2\pi k t}{\tau}} \qquad a_k = \frac{1}{\tau}\int_{-\tau/2}^{\tau/2} f(t) e^{-j\frac{2\pi k t}{\tau}} dt$$

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Slide 5

The Discrete-Time Fourier Transform (DTFT)

The ordinary Fourier transform of a signal that has been sampled $f(n)$ gives what is called a *discrete-time Fourier transform* (DTFT).

$$\begin{aligned} F(\omega) &= \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt \\ &= \int_{-\infty}^{\infty} \left[\sum_{n=-\infty}^{\infty} f(n) e^{-j\omega n} \right] dt \\ &= \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} f(n) e^{-j\omega n} dt \\ &= \sum_{n=-\infty}^{\infty} f(n) e^{-j\omega n} \end{aligned}$$

$$F(\omega) = \sum_{n=-\infty}^{\infty} f(n) e^{-j\omega n}$$

Alternatively, a DTFT is the reconstruction of a Fourier series in the frequency domain.

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The Discrete Fourier Transform (DFT)

The DTFT for a finite-length sequence $f(n)$ of length N is uniquely defined from only N points in the DTFT. This set of N points is called the *discrete Fourier transform* (DFT).

$$F(k) = F(\omega)\Big|_{\omega=\frac{2\pi k}{N}} = \sum_{n=0}^{N-1} f(n) e^{-j\frac{2\pi nk}{N}} \qquad k = 0,1,2,\ldots,N-1$$

A common notation for the DFT is

$$F(k) = \sum_{n=0}^{N-1} f(n) W_N^{kn}$$

$$f(n) = \frac{1}{N}\sum_{k=0}^{N-1} F(k) W_N^{-kn}$$

$$W_N = e^{-j\frac{2\pi}{N}}$$

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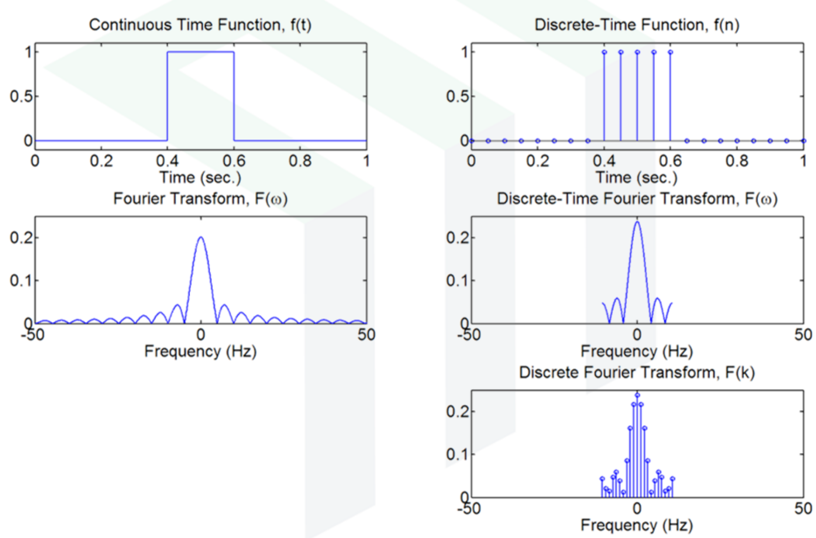
Slide 7

Comparison and Visualization of Various Fourier Transforms



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Slide 8