The Family of Fourier Transforms

The Temporal Fourier Transform

The general Fourier transform is defined as

\[ F(s) = \int_{-\infty}^{\infty} f(u) e^{-j2\pi su} du \]

The temporal Fourier transform calculates the frequency content of a time-domain signal.

\[ F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt \]
The Spatial Fourier Transform

The general Fourier transform is defined as

$$F(s) = \int_{-\infty}^{\infty} f(u) e^{-j2\pi su} \, du$$

The spatial Fourier transform calculates the “spatial waves” comprising a signal.

$$F(k) = \int_{-\infty}^{\infty} f(x) e^{jkx} \, dx$$

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**Sign Convention**

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<th>EQUATION(S)</th>
<th>ENGINEERING (Negative Sign Convention)</th>
<th>PHYSICS / SCIENCE (Positive Sign Convention)</th>
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<tr>
<td>Fourier Transform</td>
<td>Temporal: $F(\omega) = \frac{1}{2\pi} \int f(t) e^{j\omega t} , dt$ Spatial: $F(k) = \frac{1}{2\pi} \int f(x) e^{jkx} , dx$</td>
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<td>Fourier Series</td>
<td>Temporal: $a_n = \frac{1}{\tau} \int_{-\tau/2}^{\tau/2} f(t) e^{j2\pi nt/\tau} , dt$ Spatial: $a_n = \frac{1}{2\pi \Lambda} \int_{-\Lambda/2}^{\Lambda/2} f(x) e^{j2\pi nx/\Lambda} , dx$</td>
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The Fourier Series

If a signal is periodic with period $\tau$, the Fourier transform is a series of impulses. Therefore, the Fourier transform can be written a series of discrete sines and cosines, or complex exponentials. This is called the Fourier series.

The Fourier coefficients are the amplitudes of the sines and cosines, or complex exponentials.

$$f(t) = \sum_{k=-\infty}^{\infty} a_k e^{-j\frac{2\pi kt}{\tau}} \quad a_k = \frac{1}{\tau} \int_{-\tau/2}^{\tau/2} f(t) e^{-j\frac{2\pi kt}{\tau}} dt$$

The Discrete-Time Fourier Transform (DTFT)

The ordinary Fourier transform of a signal that has been sampled $f(n)$ gives what is called a discrete-time Fourier transform (DTFT).

$$F(\omega) = \sum_{n=-\infty}^{\infty} f(n)e^{-j\omega n}$$

Alternatively, a DTFT is the reconstruction of a Fourier series in the frequency domain.
The Discrete Fourier Transform (DFT)

The DTFT for a finite-length sequence $f(n)$ of length $N$ is uniquely defined from only $N$ points in the DTFT. This set of $N$ points is called the discrete Fourier transform (DFT).

$$F(k) = F(\omega) \bigg|_{\omega = \frac{2\pi k}{N}} = \sum_{n=0}^{N-1} f(n) e^{-j\frac{2\pi n k}{N}} \quad k = 0, 1, 2, \ldots, N-1$$

A common notation for the DFT is

$$F(k) = \sum_{n=0}^{N-1} f(n) W_N^{kn}$$

$$f(n) = \frac{1}{N} \sum_{k=0}^{N-1} F(k) W_N^{-kn}$$

$$W_N = e^{-j\frac{2\pi}{N}}$$

Comparison and Visualization of Various Fourier Transforms