EMPossible

Advanced Computation:
Computational Electromagnetics

Finite-Difference Approximation of Maxwell's Equations on a Yee Grid

Outline

- The Starting Point
- Finite-Difference Approximation of $\nabla \times \vec{E} = \mu_{\mathrm{r}}\vec{H}$
- Finite-Difference Approximation of $\nabla \times \vec{H} = \varepsilon_{\mathrm{r}}\vec{E}$
- Summary of Finite-Difference Equations on a Yee Grid

EMPossible

Slide 2

The Starting Point

Normalize the Grid Coordinates x, y and z

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = k_0 \mu_{xx} \tilde{H}_x$$
$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = k_0 \mu_{yy} \tilde{H}_y$$
$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = k_0 \mu_{zz} \tilde{H}_z$$

$$\frac{\partial \tilde{H}_z}{\partial y} - \frac{\partial \tilde{H}_y}{\partial z} = k_0 \varepsilon_{xx} E_x$$
$$\frac{\partial \tilde{H}_x}{\partial z} - \frac{\partial \tilde{H}_z}{\partial x} = k_0 \varepsilon_{yy} E_y$$
$$\frac{\partial \tilde{H}_y}{\partial x} - \frac{\partial \tilde{H}_x}{\partial y} = k_0 \varepsilon_{zz} E_z$$

Normalize the grid

$$x' = k_0 x$$
$$y' = k_0 y$$
$$z' = k_0 z$$

$$\frac{\partial E_z}{\partial y'} - \frac{\partial E_y}{\partial z'} = \mu_{xx} \tilde{H}_x$$
$$\frac{\partial E_x}{\partial z'} - \frac{\partial E_z}{\partial x'} = \mu_{yy} \tilde{H}_y$$
$$\frac{\partial E_y}{\partial x'} - \frac{\partial E_x}{\partial y'} = \mu_{zz} \tilde{H}_z$$

$$\frac{\partial \tilde{H}_z}{\partial y'} - \frac{\partial \tilde{H}_y}{\partial z'} = \varepsilon_{xx} E_x$$
$$\frac{\partial \tilde{H}_x}{\partial z'} - \frac{\partial \tilde{H}_z}{\partial x'} = \varepsilon_{yy} E_y$$
$$\frac{\partial \tilde{H}_y}{\partial x'} - \frac{\partial \tilde{H}_x}{\partial y'} = \varepsilon_{zz} E_z$$

This “absorbs” the k_0 term into the spatial derivatives and simplifies Maxwell’s equations.

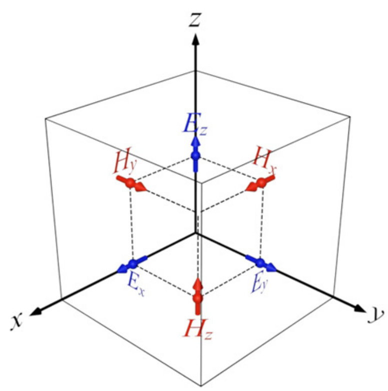
EMPossible

The Starting Point

$$\frac{\partial E_z}{\partial y'} - \frac{\partial E_y}{\partial z'} = \mu_{xx}\tilde{H}_x \qquad \frac{\partial \tilde{H}_z}{\partial y'} - \frac{\partial \tilde{H}_y}{\partial z'} = \varepsilon_{xx}E_x$$

$$\frac{\partial E_x}{\partial z'} - \frac{\partial E_z}{\partial x'} = \mu_{yy}\tilde{H}_y \qquad \frac{\partial \tilde{H}_x}{\partial z'} - \frac{\partial \tilde{H}_z}{\partial x'} = \varepsilon_{yy}E_y$$

$$\frac{\partial E_y}{\partial x'} - \frac{\partial E_x}{\partial y'} = \mu_{zz}\tilde{H}_z \qquad \frac{\partial \tilde{H}_y}{\partial x'} - \frac{\partial \tilde{H}_x}{\partial y'} = \varepsilon_{zz}E_z$$



Here, only diagonally anisotropic material tensors were retained. This will be needed to incorporate a uniaxial perfectly matched layer (UPML) absorbing boundary condition.

These equations are valid independent of the chosen sign convention.

EMPossible

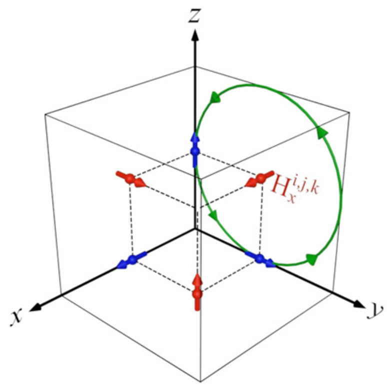
Slide 5

Finite-Difference Approximation of $\nabla \times \vec{E} = \mu_r \vec{H}$

Slide 6

Finite-Difference Equation for H_x

$$\frac{\partial E_z}{\partial y'} - \frac{\partial E_y}{\partial z'} = \mu_{xx} \tilde{H}_x$$



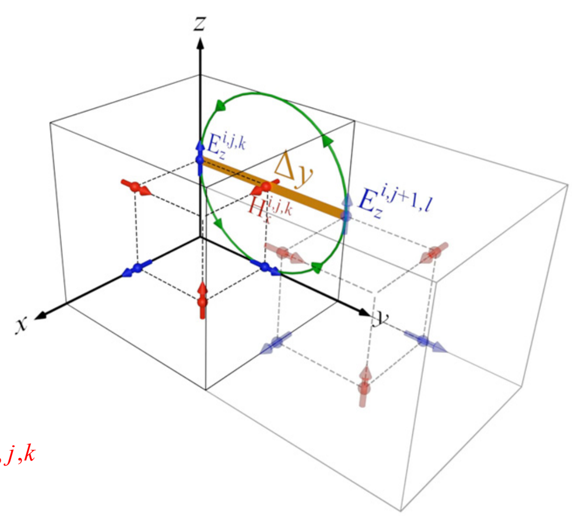
$$= \mu_{xx}^{i,j,k} \tilde{H}_x^{i,j,k}$$

EMPossible

Slide 7

Finite-Difference Equation for H_x

$$\frac{\partial E_z}{\partial y'} - \frac{\partial E_y}{\partial z'} = \mu_{xx} \tilde{H}_x$$

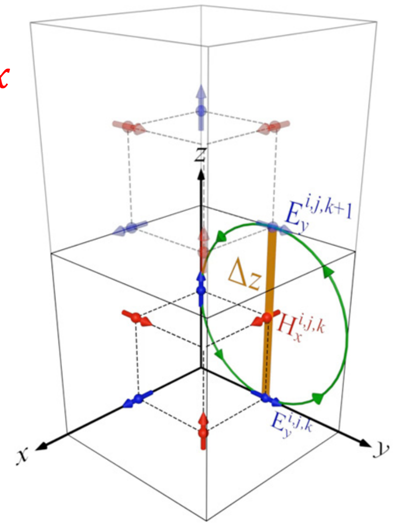


$$\frac{E_z^{i,j+1,k} - E_z^{i,j,k}}{\Delta y'} \qquad = \mu_{xx}^{i,j,k} \tilde{H}_x^{i,j,k}$$

EMPossible

Slide 8

Finite-Difference Equation for H_x

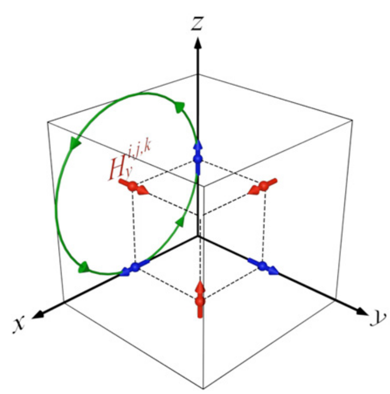


$$\frac{\partial E_z}{\partial y'} - \frac{\partial E_y}{\partial z'} = \mu_{xx} \tilde{H}_x$$

$$\frac{E_z^{i,j+1,k} - E_z^{i,j,k}}{\Delta y'} - \frac{E_y^{i,j,k+1} - E_y^{i,j,k}}{\Delta z'} = \mu_{xx}^{i,j,k} \tilde{H}_x^{i,j,k}$$

EMPossible Slide 9

Finite-Difference Equation for H_y

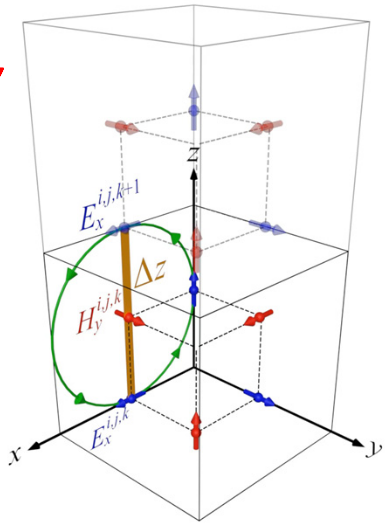


$$\frac{\partial E_x}{\partial z'} - \frac{\partial E_z}{\partial x'} = \mu_{yy} \tilde{H}_y$$

$$= \mu_{yy}^{i,j,k} \tilde{H}_y^{i,j,k}$$

EMPossible Slide 10

Finite-Difference Equation for H_y



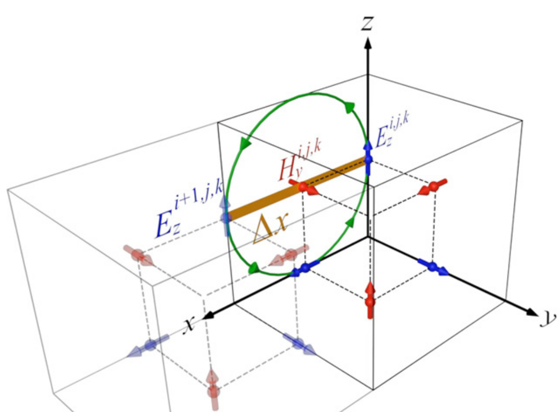
$$\boxed{\frac{\partial E_x}{\partial z'}} - \frac{\partial E_z}{\partial x'} = \mu_{yy} \tilde{H}_y$$

$$\frac{E_x^{i,j,k+1} - E_x^{i,j,k}}{\Delta z'} \qquad\qquad = \mu_{yy}^{i,j,k} \tilde{H}_y^{i,j,k}$$

EMPossible

Slide 11

Finite-Difference Equation for H_y



$$\frac{\partial E_x}{\partial z'} \boxed{- \frac{\partial E_z}{\partial x'}} = \mu_{yy} \tilde{H}_y$$

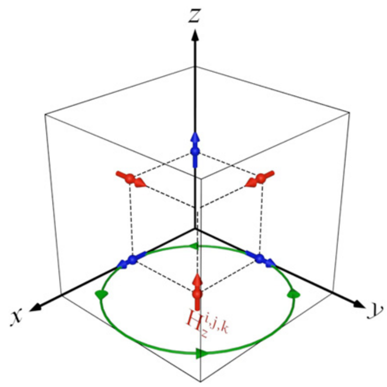
$$\frac{E_x^{i,j,k+1} - E_x^{i,j,k}}{\Delta z'} - \frac{E_z^{i+1,j,k} - E_z^{i,j,k}}{\Delta x'} = \mu_{yy}^{i,j,k} \tilde{H}_y^{i,j,k}$$

EMPossible

Slide 12

Finite-Difference Equation for H_z

$$\frac{\partial E_y}{\partial x'} - \frac{\partial E_x}{\partial y'} = \mu_{zz} \tilde{H}_z$$



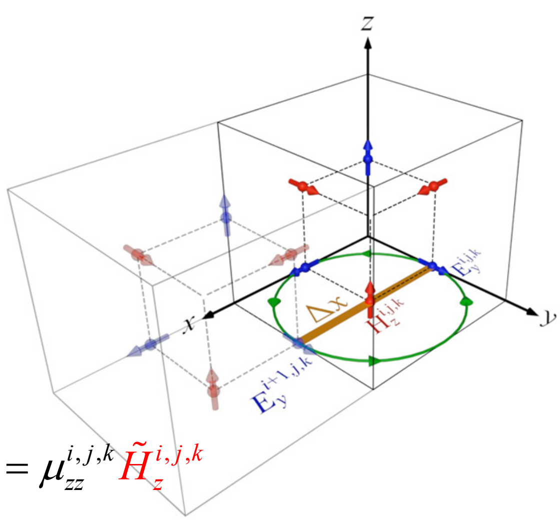
$$= \mu_{zz}^{i,j,k} \tilde{H}_z^{i,j,k}$$

EMPossible

Slide 13

Finite-Difference Equation for H_z

$$\frac{\partial E_y}{\partial x'} - \frac{\partial E_x}{\partial y'} = \mu_{zz} \tilde{H}_z$$



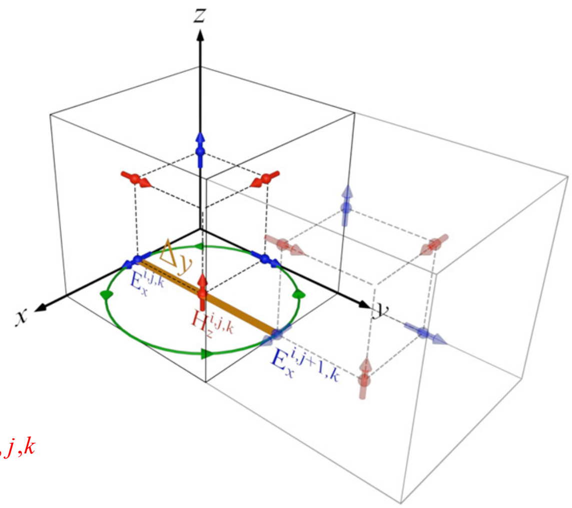
$$\frac{E_y^{i+1,j,k} - E_y^{i,j,k}}{\Delta x'} = \mu_{zz}^{i,j,k} \tilde{H}_z^{i,j,k}$$

EMPossible

Slide 14

Finite-Difference Equation for H_z

$$\frac{\partial E_y}{\partial x'} - \boxed{\frac{\partial E_x}{\partial y'}} = \mu_{zz}\tilde{H}_z$$



$$\frac{E_y^{i+1,j,k} - E_y^{i,j,k}}{\Delta x'} - \frac{E_x^{i,j+1,k} - E_x^{i,j,k}}{\Delta y'} = \mu_{zz}^{i,j,k}\tilde{H}_z^{i,j,k}$$

EMPossible

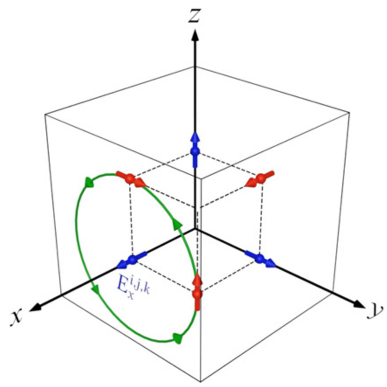
Slide 15

Finite-Difference Approximation of $\nabla \times \vec{H} = \varepsilon_r \vec{E}$

Slide 16

Finite-Difference Equation for E_x

$$\frac{\partial \tilde{H}_z}{\partial y'} - \frac{\partial \tilde{H}_y}{\partial z'} = \varepsilon_{xx} E_x$$



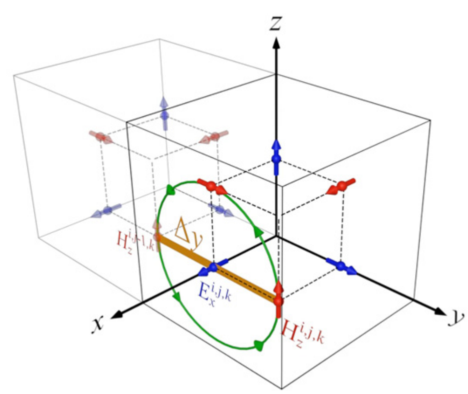
$$= \varepsilon_{xx}^{i,j,k} E_x^{i,j,k}$$

EMPossible

Slide 17

Finite-Difference Equation for E_x

$$\frac{\partial \tilde{H}_z}{\partial y'} - \frac{\partial \tilde{H}_y}{\partial z'} = \varepsilon_{xx} E_x$$



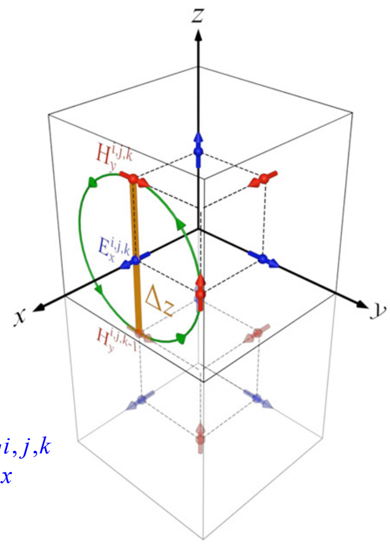
$$\frac{\tilde{H}_z^{i,j,k} - \tilde{H}_z^{i,j-1,k}}{\Delta y'} = \varepsilon_{xx}^{i,j,k} E_x^{i,j,k}$$

EMPossible

Slide 18

Finite-Difference Equation for E_x

$$\frac{\partial \tilde{H}_z}{\partial y'} - \frac{\partial \tilde{H}_y}{\partial z'} = \varepsilon_{xx} E_x$$



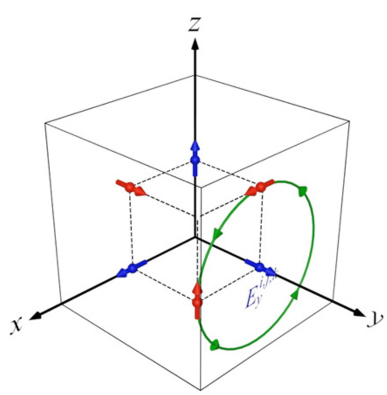
$$\frac{\tilde{H}_z^{i,j,k} - \tilde{H}_z^{i,j-1,k}}{\Delta y'} - \frac{\tilde{H}_y^{i,j,k} - \tilde{H}_y^{i,j,k-1}}{\Delta z'} = \varepsilon_{xx}^{i,j,k} E_x^{i,j,k}$$

EMPossible

Slide 19

Finite-Difference Equation for E_y

$$\frac{\partial \tilde{H}_x}{\partial z'} - \frac{\partial \tilde{H}_z}{\partial x'} = \varepsilon_{yy} E_y$$



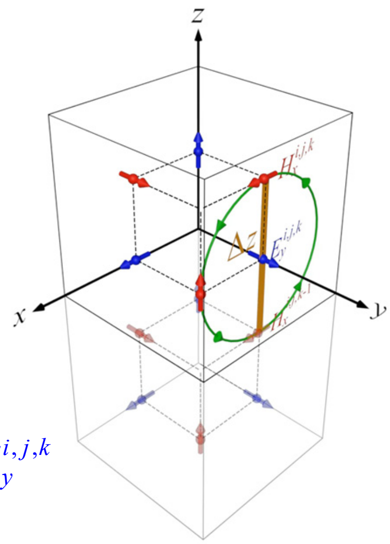
$$= \varepsilon_{yy}^{i,j,k} E_y^{i,j,k}$$

EMPossible

Slide 20

Finite-Difference Equation for E_y

$$\frac{\partial \tilde{H}_x}{\partial z'} - \frac{\partial \tilde{H}_z}{\partial x'} = \varepsilon_{yy} E_y$$



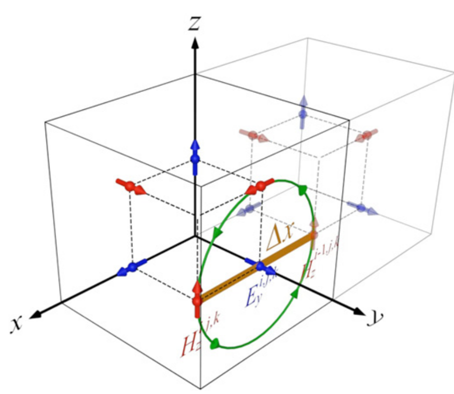
$$\frac{\tilde{H}_x^{i,j,k} - \tilde{H}_x^{i,j,k-1}}{\Delta z'} \qquad = \varepsilon_{yy}^{i,j,k} E_y^{i,j,k}$$

EMPossible

Slide 21

Finite-Difference Equation for E_y

$$\frac{\partial \tilde{H}_x}{\partial z'} - \frac{\partial \tilde{H}_z}{\partial x'} = \varepsilon_{yy} E_y$$



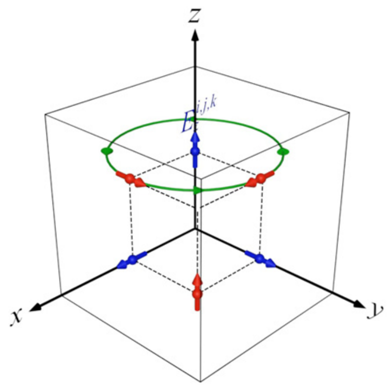
$$\frac{\tilde{H}_x^{i,j,k} - \tilde{H}_x^{i,j,k-1}}{\Delta z'} - \frac{\tilde{H}_z^{i,j,k} - \tilde{H}_z^{i-1,j,k}}{\Delta x'} = \varepsilon_{yy}^{i,j,k} E_y^{i,j,k}$$

EMPossible

Slide 22

Finite-Difference Equation for E_z

$$\frac{\partial \tilde{H}_y}{\partial x'} - \frac{\partial \tilde{H}_x}{\partial y'} = \varepsilon_{zz} E_z$$



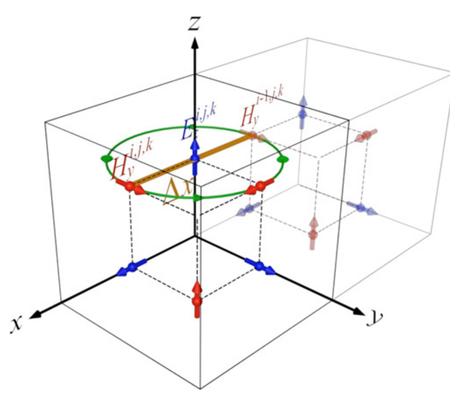
$$= \varepsilon_{zz}^{i,j,k} E_z^{i,j,k}$$

EMPossible

Slide 23

Finite-Difference Equation for E_z

$$\frac{\partial \tilde{H}_y}{\partial x'} - \frac{\partial \tilde{H}_x}{\partial y'} = \varepsilon_{zz} E_z$$



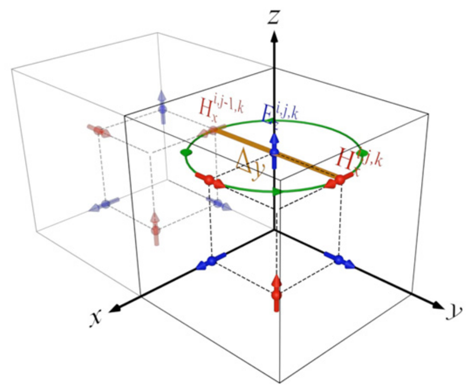
$$\frac{\tilde{H}_y^{i,j,k} - \tilde{H}_y^{i-1,j,k}}{\Delta x'} = \varepsilon_{zz}^{i,j,k} E_z^{i,j,k}$$

EMPossible

Slide 24

Finite-Difference Equation for E_z

$$\frac{\partial \tilde{H}_y}{\partial x'} - \frac{\partial \tilde{H}_x}{\partial y'} = \varepsilon_{zz} E_z$$



$$\frac{\tilde{H}_y^{i,j,k} - \tilde{H}_y^{i-1,j,k}}{\Delta x'} - \frac{\tilde{H}_x^{i,j,k} - \tilde{H}_x^{i,j-1,k}}{\Delta y'} = \varepsilon_{zz}^{i,j,k} E_z^{i,j,k}$$

EMPossible

Slide 25

Summary of Finite-Difference Equations on a Yee Grid

Slide 26

Summary of Finite-Difference Approximations of Maxwell's Equations

$$\frac{\partial E_z}{\partial y'} - \frac{\partial E_y}{\partial z'} = \mu_{xx} \tilde{H}_x \quad \Rightarrow \quad \frac{E_z^{i,j+1,k} - E_z^{i,j,k}}{\Delta y'} - \frac{E_y^{i,j,k+1} - E_y^{i,j,k}}{\Delta z'} = \mu_{xx}^{i,j,k} \tilde{H}_x^{i,j,k}$$

$$\frac{\partial E_x}{\partial z'} - \frac{\partial E_z}{\partial x'} = \mu_{yy} \tilde{H}_y \quad \Rightarrow \quad \frac{E_x^{i,j,k+1} - E_x^{i,j,k}}{\Delta z'} - \frac{E_z^{i+1,j,k} - E_z^{i,j,k}}{\Delta x'} = \mu_{yy}^{i,j,k} \tilde{H}_y^{i,j,k}$$

$$\frac{\partial E_y}{\partial x'} - \frac{\partial E_x}{\partial y'} = \mu_{zz} \tilde{H}_z \quad \Rightarrow \quad \frac{E_y^{i+1,j,k} - E_y^{i,j,k}}{\Delta x'} - \frac{E_x^{i,j+1,k} - E_x^{i,j,k}}{\Delta y'} = \mu_{zz}^{i,j,k} \tilde{H}_z^{i,j,k}$$

$$\frac{\partial \tilde{H}_z}{\partial y'} - \frac{\partial \tilde{H}_y}{\partial z'} = \varepsilon_{xx} E_x \quad \Rightarrow \quad \frac{\tilde{H}_z^{i,j,k} - \tilde{H}_z^{i,j-1,k}}{\Delta y'} - \frac{\tilde{H}_y^{i,j,k} - \tilde{H}_y^{i,j,k-1}}{\Delta z'} = \varepsilon_{xx}^{i,j,k} E_x^{i,j,k}$$

$$\frac{\partial \tilde{H}_x}{\partial z'} - \frac{\partial \tilde{H}_z}{\partial x'} = \varepsilon_{yy} E_y \quad \Rightarrow \quad \frac{\tilde{H}_x^{i,j,k} - \tilde{H}_x^{i,j,k-1}}{\Delta z'} - \frac{\tilde{H}_z^{i,j,k} - \tilde{H}_z^{i-1,j,k}}{\Delta x'} = \varepsilon_{yy}^{i,j,k} E_y^{i,j,k}$$

$$\frac{\partial \tilde{H}_y}{\partial x'} - \frac{\partial \tilde{H}_x}{\partial y'} = \varepsilon_{zz} E_z \quad \Rightarrow \quad \frac{\tilde{H}_y^{i,j,k} - \tilde{H}_y^{i-1,j,k}}{\Delta x'} - \frac{\tilde{H}_x^{i,j,k} - \tilde{H}_x^{i,j-1,k}}{\Delta y'} = \varepsilon_{zz}^{i,j,k} E_z^{i,j,k}$$

EMPossible

Slide 27