Computational Science:
Computational Methods in Engineering

Golden Section Search

Outline

• Description of the Method
• Method Summary
• Derivation of the Golden Ratio $R$
Description of the Method

Define Starting Interval

Choose a lower bound $x_L$ and an upper bound $x_U$ that span a single maximum.
Evaluate the Function at the Bounds

\[ f_L = f(x_L) \]
\[ f_U = f(x_U) \]
\[ x_L \leq x \leq x_U \]

Calculate Two Very Special Intermediate Points

Calculate the Golden ratio \( R \).
\[ R = \frac{\sqrt{5} - 1}{2} \approx 0.6180... \]

Calculate distance \( d \).
\[ d = R(x_U - x_L) \]
Calculate Two Very Special Intermediate Points

Calculate the Golden ratio R.

\[ R = \frac{\sqrt{5} - 1}{2} \approx 0.6180... \]

Calculate distance d.

\[ d = R (x_U - x_L) \]

Calculate special intermediate points \( x_1 \) and \( x_2 \).

\[ x_1 = x_U - d \]
\[ x_2 = x_L + d \]

Evaluate the Function at the Intermediate Points

\[ f_1 = f(x_1) \]
\[ f_2 = f(x_2) \]
Determine Position of the Maximum

Is $f_1 > f_2$? → Max to Left

Is $f_1 < f_2$? → Max to Right
Adjust Points to Close in on Maximum

Old intermediate point $x_1$ becomes new lower bound $x_L$.

$x_L = x_1$  $f_L = f_1$

Old intermediate point $x_2$ becomes new intermediate point $x_1$.

$x_1 = x_2$  $f_1 = f_2$

Upper bound $x_U$ remains the same.

Calculate New Intermediate Point

Calculate distance $d$.

$d = R(x_U - x_L)$

Calculate new intermediate point $x_2$.

$x_2 = x_L + d$
Evaluate Function at New Intermediate Point

\[ f_2 = f(x_2) \]

Observe that the entire iteration only involved calculating one new point and one new evaluation of the function.

Determine Position of the Maximum

Is \( f_1 > f_2 \)? \( \Rightarrow \) Max to Left

Is \( f_1 < f_2 \)? \( \Rightarrow \) Max to Right
Determine Position of the Maximum

Is $f_1 > f_2$? $\rightarrow$ Max to Left
Is $f_1 < f_2$? $\rightarrow$ Max to Right

Adjust Points to Close in on Maximum

Old intermediate point $x_2$ becomes new upper bound $x_U$.
$$x_U = x_2 \quad f_U = f_2$$

Old intermediate point $x_1$ becomes new intermediate point $x_2$.
$$x_2 = x_1 \quad f_2 = f_1$$

Lower bound $x_L$ remains the same.
Calculate New Intermediate Point

Calculate distance $d$. 
$$d = R(x_U - x_L)$$

Calculate new intermediate point $x_1$. 
$$x_1 = x_U - d$$

Evaluate Function at New Intermediate Point

$$f_1 = f(x_1)$$

Observe that the entire iteration only involved calculating one new point and one new evaluation of the function.
Determine Position of the Maximum

Is $f_1 > f_2$? $\rightarrow$ Max to Left

Is $f_1 < f_2$? $\rightarrow$ Max to Right

...and so on.
Repeat Until Convergence

Converged if:
\[ 2 \left| \frac{x_U - x_L}{x_U + x_L} \right| < \text{tolerance} \]

Calculate Final Answer

Estimate the final maximum to be at the midpoint of the last interval.

\[ x_e \approx \frac{x_U + x_L}{2} \]

\[ f_e = f(x_e) \]
**Method Summary**

**Block Diagram of Golden Section Search**

1. **Define Starting Bounds**
   \[ x_L \leq x \leq x_U \]

2. **Evaluate Function at Bounds**
   \[ f_L = f(x_L) \quad \text{and} \quad f_U = f(x_U) \]

3. **Calculate Two Intermediate Points**
   \[ x_1 = x_U - d \]
   \[ x_2 = x_L + d \]
   \[ d = R(x_U - x_L) \quad R = \left(\sqrt{5} - 1\right)/2 \]

4. **Evaluate Function at Intermediate Points**
   \[ f_1 = f(x_1) \quad \text{and} \quad f_2 = f(x_2) \]

5. **Adjust Points for Max on Left**
   \[ x_1 = x_i \]
   \[ x_2 = x_i - R(x_u - x_i) \]
   \[ f_1 = f_i \quad f_2 = f_i \]

6. **Adjust Points for Max on Right**
   \[ x_1 = x_i \]
   \[ x_2 = x_i + R(x_u - x_i) \]
   \[ f_1 = f_i \quad f_2 = f_i \]

7. **Calculate Final Answer**
   \[ x_e = x_L + x_U \quad f_e = f(x_e) \]

8. **Calculate Final Answer**
   \[ x_e = \frac{x_L + x_U}{2} \quad f_e = \frac{f_1 + f_2}{2} \]

9. **Unusual Case**

10. **Final Answer**
    \[ 2 \left| \frac{x_U - x_L}{x_U + x_L} \right| < \text{tolerance?} \]
The Magic of the Golden-Section Search

Three of the four points line up from one iteration to the next. This means the function only has to be evaluated at one new point each iteration.

The interval is reduced by 38.2% each iteration.
Derivation of the Golden Ratio $R$

Derivation of Golden Ratio (1 of 3)

From this picture, define three length parameters $\ell_0$, $\ell_1$ and $\ell_2$ as

$$
\ell_0 = x_U - x_L \\
\ell_1 = x_2 - x_L \\
\ell_2 = x_U - x_2
$$

Recognizing that the points of the next iteration should lie on top of points from the previous iteration, define two conditions to ensure this.

$$
\ell_0 = \ell_1 + \ell_2 \\
\frac{\ell_1}{\ell_0} = \frac{\ell_2}{\ell_1}
$$

Condition 1 – ensures $\ell_1 + \ell_2$ covers entire span.

Condition 2 – ensures the next iteration has the same proportional spacing as the current iteration.
Derivation of Golden Ratio (2 of 3)

\[ \ell_0 = \ell_1 + \ell_2 \quad \text{Condition 1 – ensures } \ell_1 + \ell_2 \text{ covers entire span.} \]

\[ \frac{\ell_1}{\ell_0} = \frac{\ell_2}{\ell_1} \quad \text{Condition 2 – ensures the next iteration has the same proportional spacing as the current iteration.} \]

Substitute Condition 1 into Condition 2 to eliminate \( \ell_0 \).

\[ \frac{\ell_1}{\ell_1 + \ell_2} = \frac{\ell_2}{\ell_1} \]

Define the Golden ratio as \( R = \ell_2/\ell_1 \).

\[ \frac{\ell_1}{\ell_1 + \ell_2} = \frac{\ell_2}{\ell_1} \rightarrow \frac{\ell_1 + \ell_2}{\ell_1} = \frac{\ell_1}{\ell_2} \rightarrow 1 + \frac{\ell_2}{\ell_1} = \frac{\ell_1}{\ell_2} \rightarrow 1 + R = \frac{1}{R} \rightarrow R^2 + R - 1 = 0 \]

invert equation expand left side Substitute \( R = \ell_2/\ell_1 \) Simplify

Derivation of Golden Ratio (3 of 3)

Recall the quadratic formula.

\[ ax^2 + bx + c = 0 \quad \rightarrow \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]

Solve for \( R \) using the quadratic formula.

\[ R^2 + R - 1 = 0 \quad \rightarrow \quad R = \frac{-1 \pm \sqrt{1^2 - 4(1)(-1)}}{2(1)} = \frac{-1 \pm \sqrt{5}}{2} \]

Pick the positive root to keep \( R \) positive. \[ R = \frac{\sqrt{5} - 1}{2} \approx 0.618033988749895... \]