Computational Science:
Computational Methods in Engineering

Golden Section Search

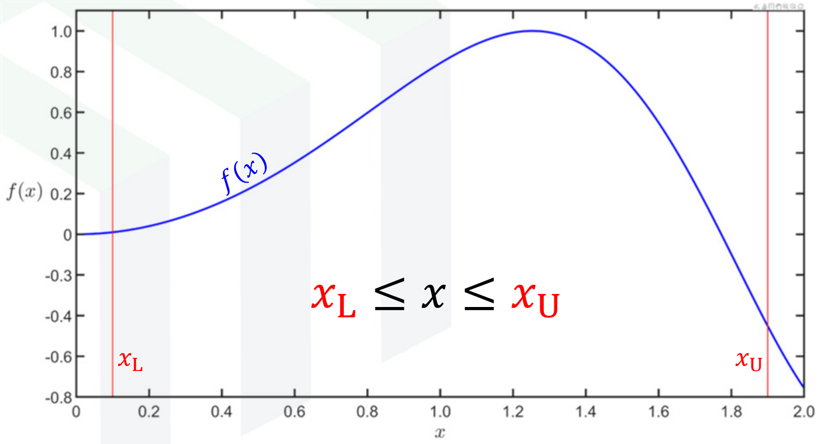
Outline

- Description of the Method
- Method Summary
- Derivation of the Golden Ratio R

Description of the Method

Define Starting Interval

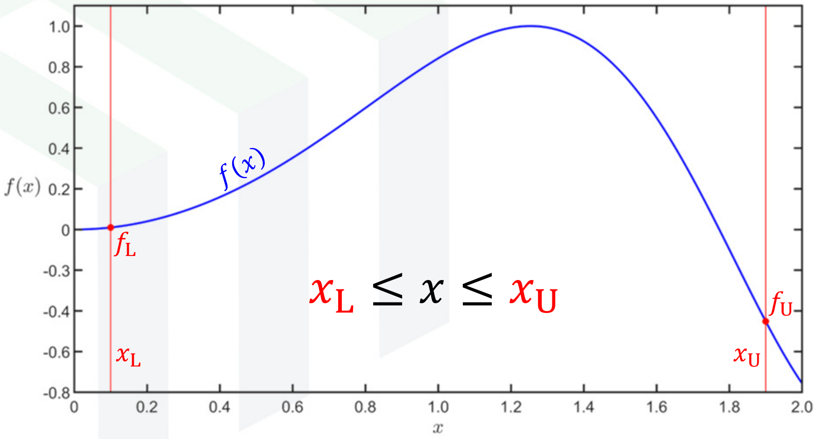
Choose a lower bound x_L and an upper bound x_U that span a single maximum.



Evaluate the Function at the Bounds

$f_\mathrm{L} = f(x_\mathrm{L})$

$f_\mathrm{U} = f(x_\mathrm{U})$



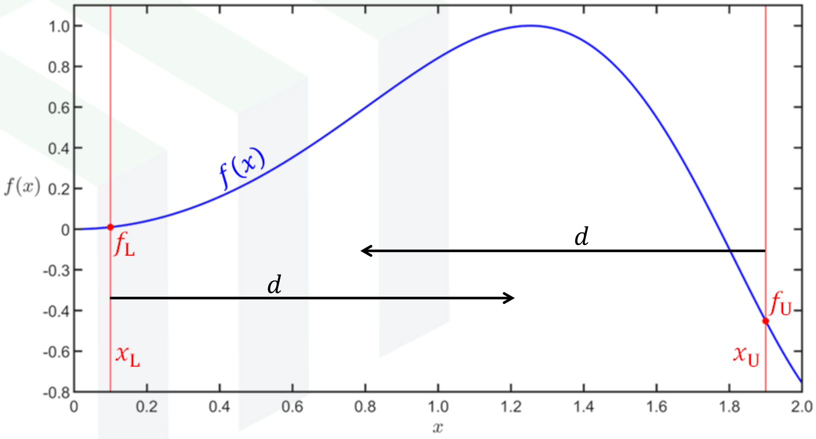
Calculate Two Very Special Intermediate Points

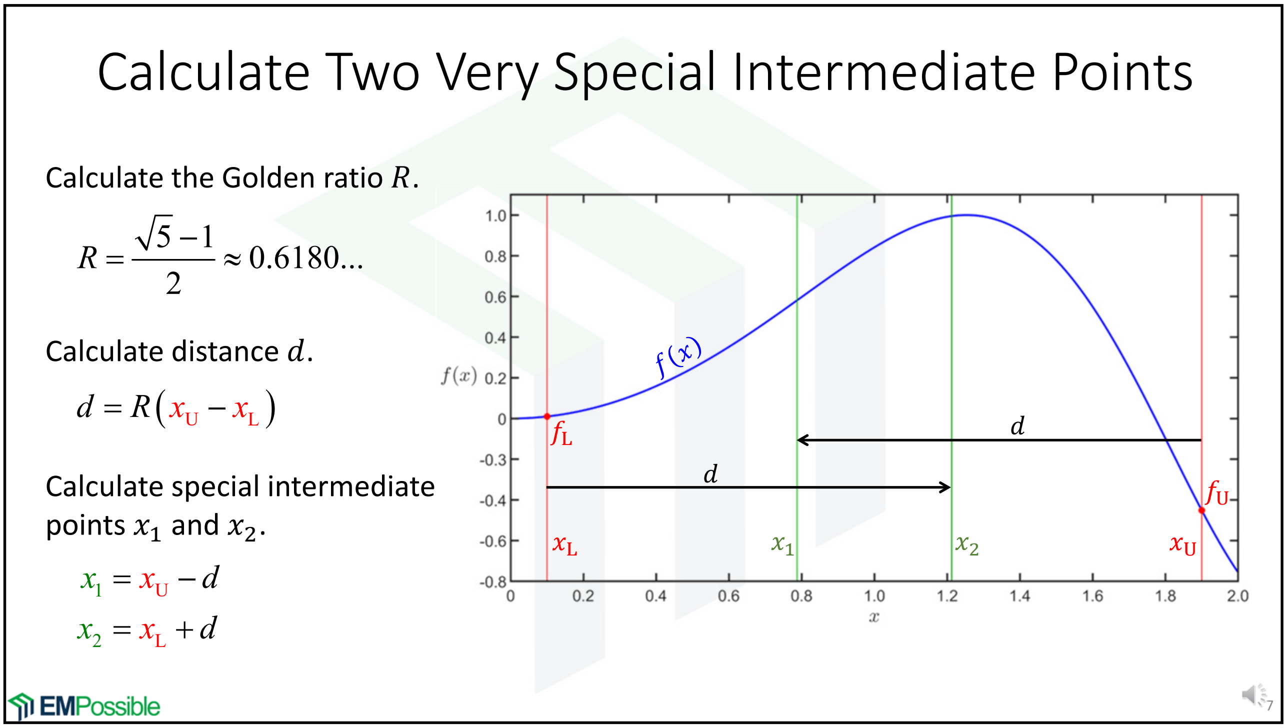
Calculate the Golden ratio R.

$$R = \frac{\sqrt{5}-1}{2} \approx 0.6180\ldots$$

Calculate distance d.

$$d = R\left(x_\mathrm{U} - x_\mathrm{L}\right)$$





Calculate Two Very Special Intermediate Points

Calculate the Golden ratio R.

$$R = \frac{\sqrt{5}-1}{2} \approx 0.6180\ldots$$

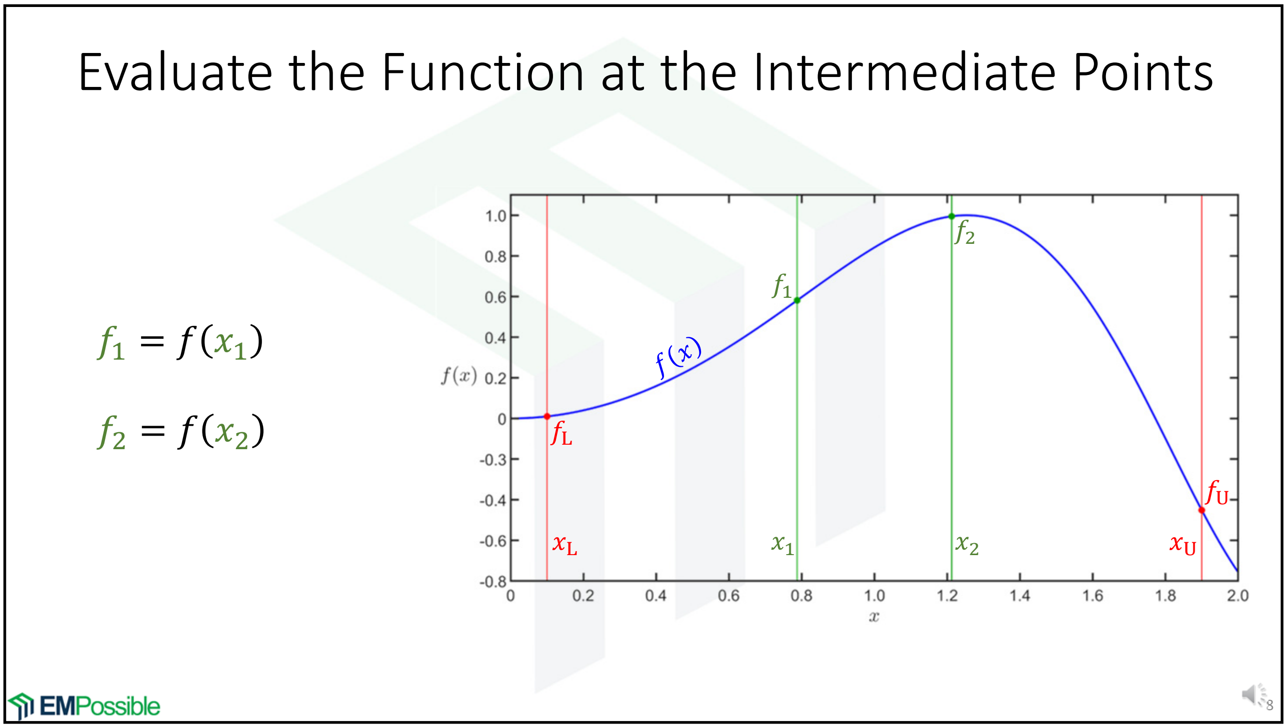
Calculate distance d.

$$d = R\left(x_U - x_L\right)$$

Calculate special intermediate points x_1 and x_2.

$$x_1 = x_U - d$$

$$x_2 = x_L + d$$



Evaluate the Function at the Intermediate Points

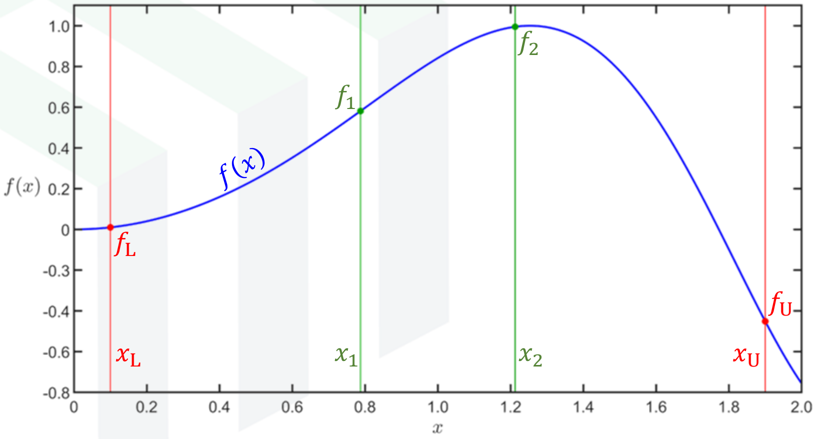
$$f_1 = f(x_1)$$

$$f_2 = f(x_2)$$

Determine Position of the Maximum

Is $f_1 > f_2$? → Max to Left

Is $f_1 < f_2$? → Max to Right



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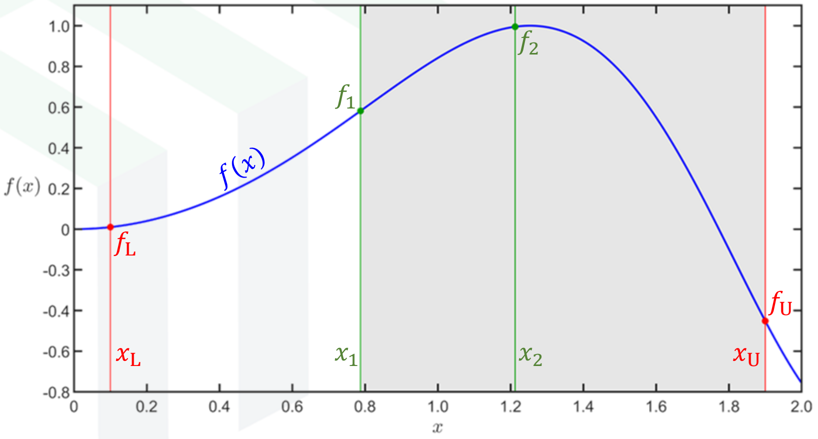
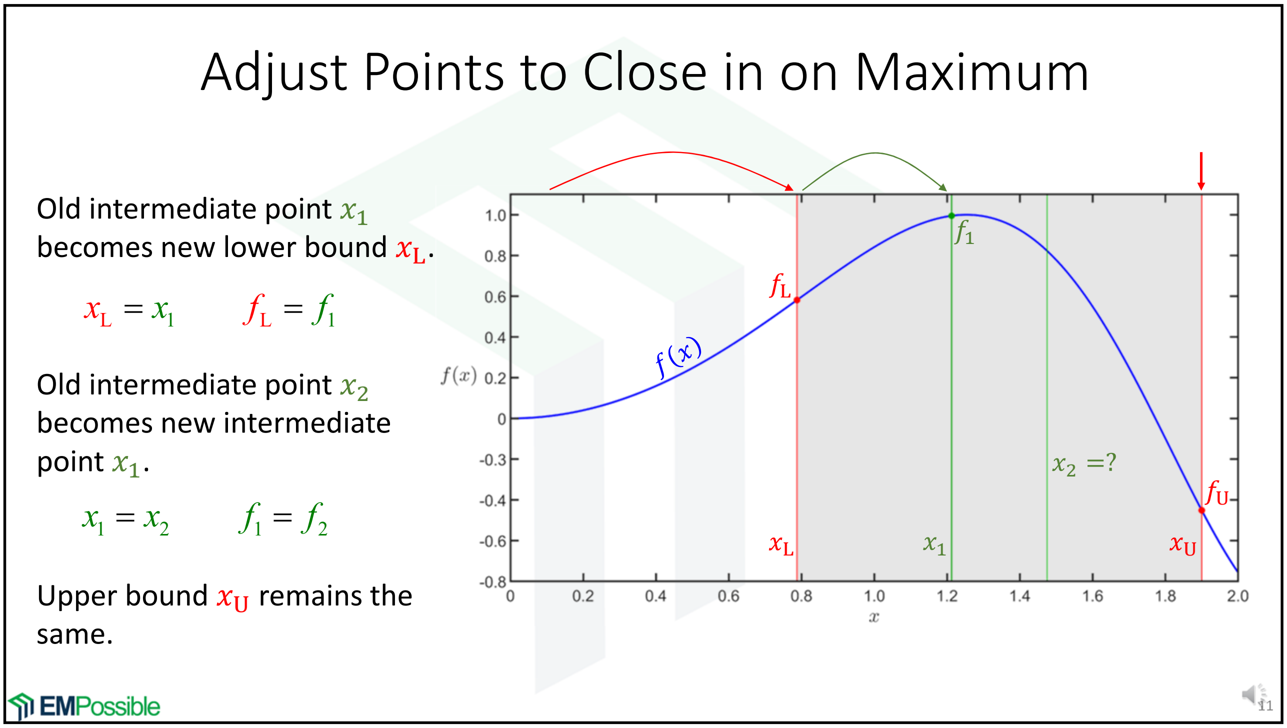




Adjust Points to Close in on Maximum

Old intermediate point x_1 becomes new lower bound x_L.

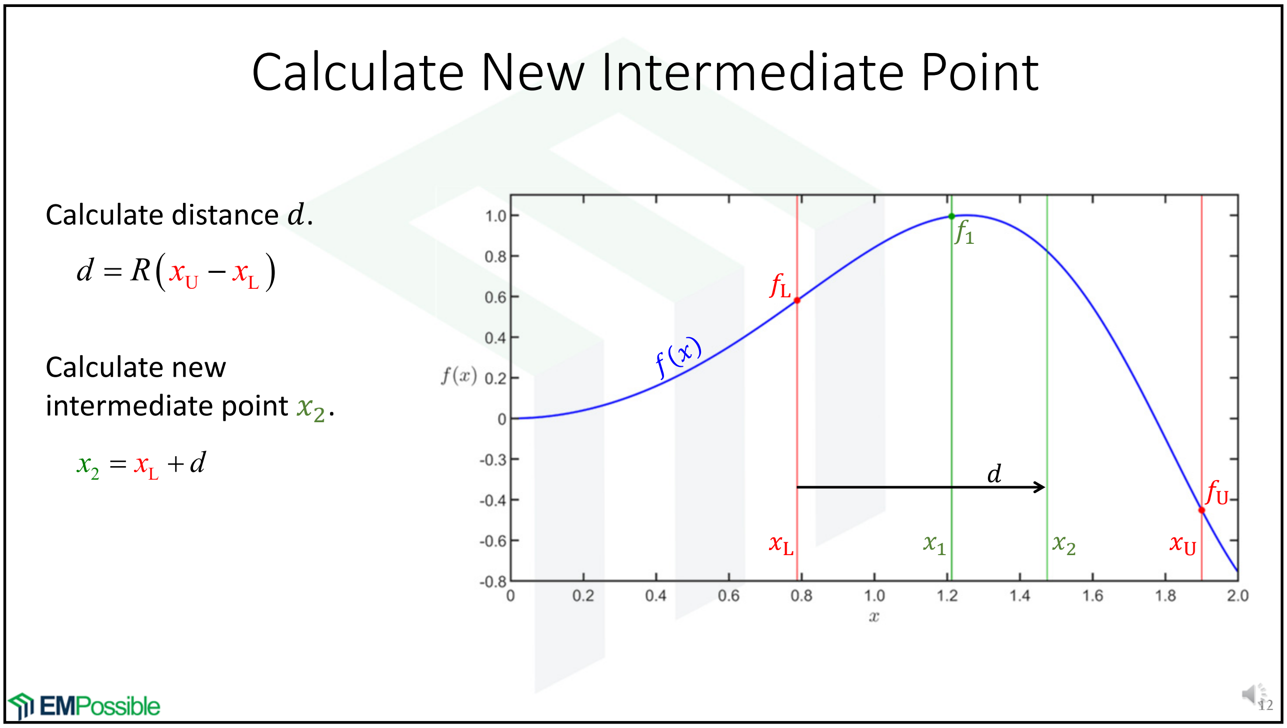
$$x_L = x_1 \qquad f_L = f_1$$

Old intermediate point x_2 becomes new intermediate point x_1.

$$x_1 = x_2 \qquad f_1 = f_2$$

Upper bound x_U remains the same.

EMPossible



Calculate New Intermediate Point

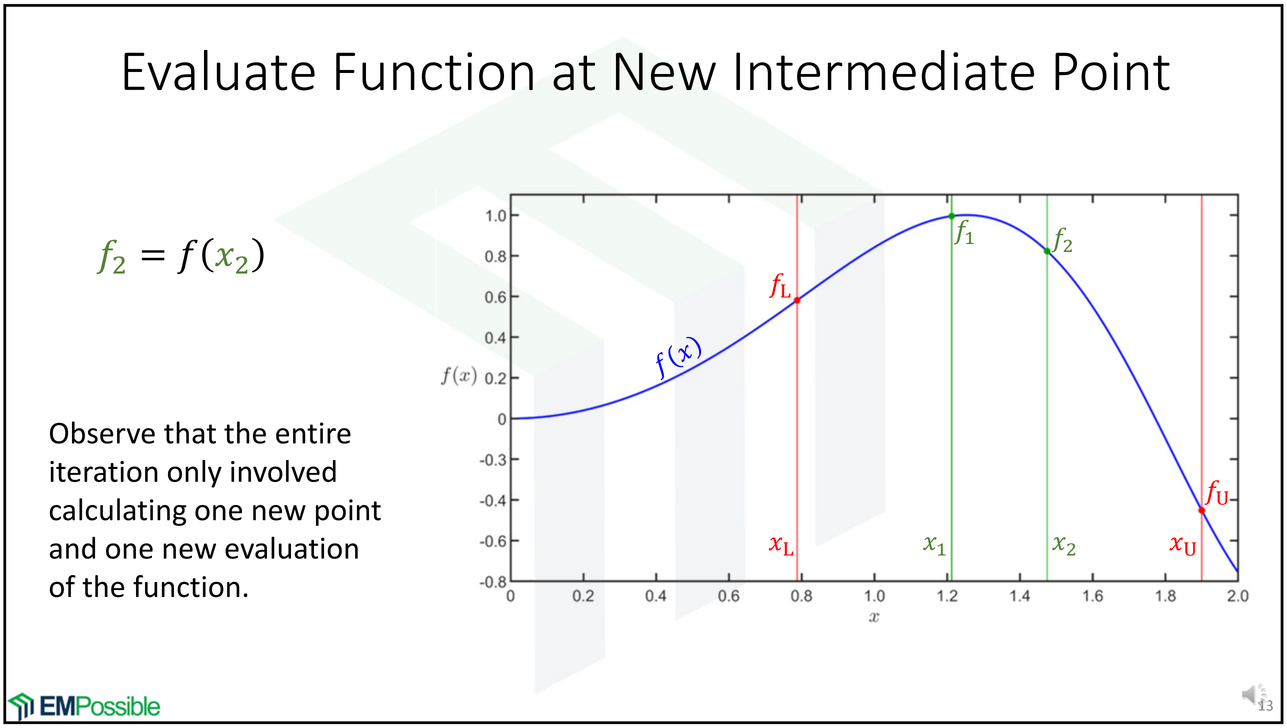
Calculate distance d.

$$d = R\left(x_U - x_L\right)$$

Calculate new intermediate point x_2.

$$x_2 = x_L + d$$

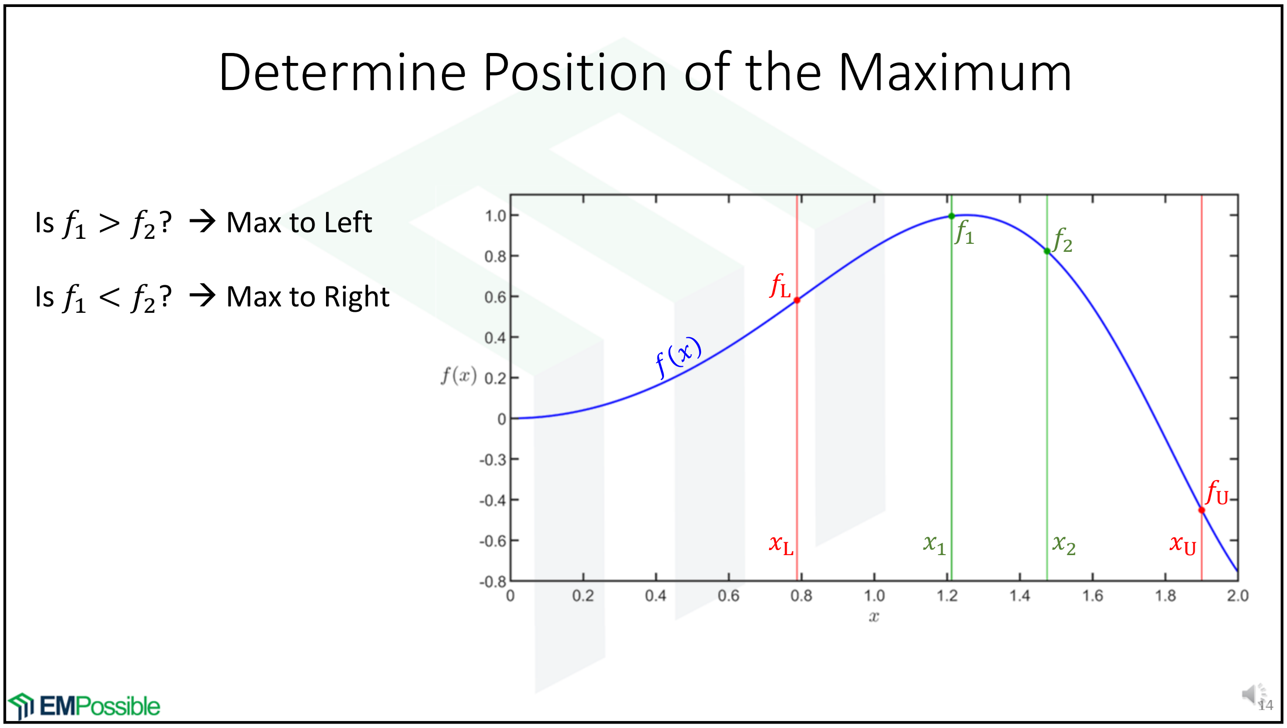
EMPossible



Evaluate Function at New Intermediate Point

$f_2 = f(x_2)$

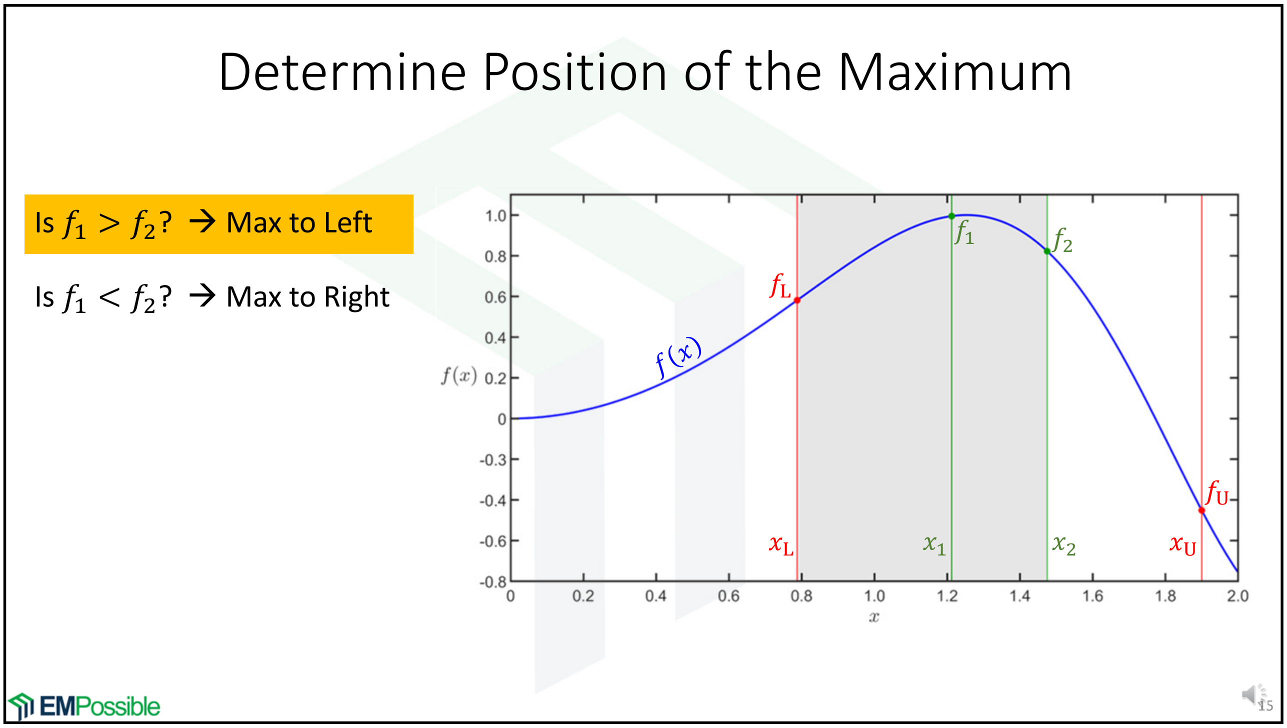
Observe that the entire iteration only involved calculating one new point and one new evaluation of the function.



Determine Position of the Maximum

Is $f_1 > f_2$? → Max to Left

Is $f_1 < f_2$? → Max to Right



Determine Position of the Maximum

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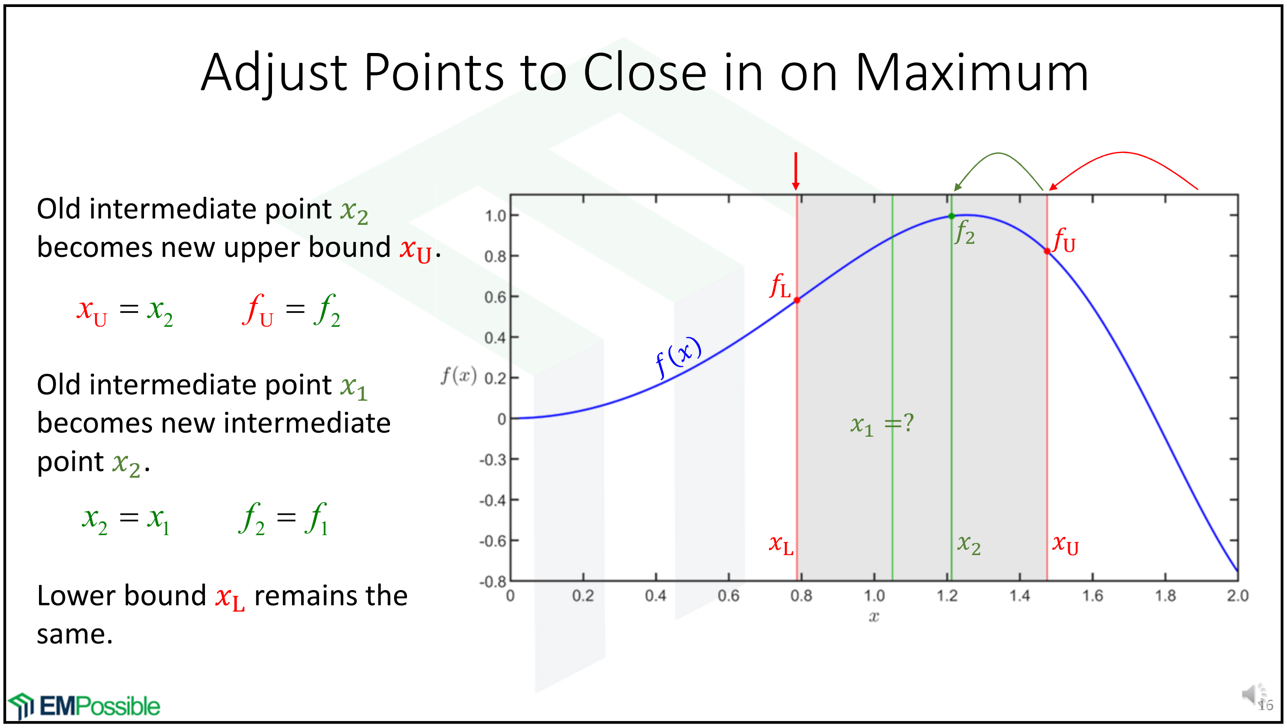


Adjust Points to Close in on Maximum

Old intermediate point x_2 becomes new upper bound x_U.

$$x_U = x_2 \qquad f_U = f_2$$

Old intermediate point x_1 becomes new intermediate point x_2.

$$x_2 = x_1 \qquad f_2 = f_1$$

Lower bound x_L remains the same.

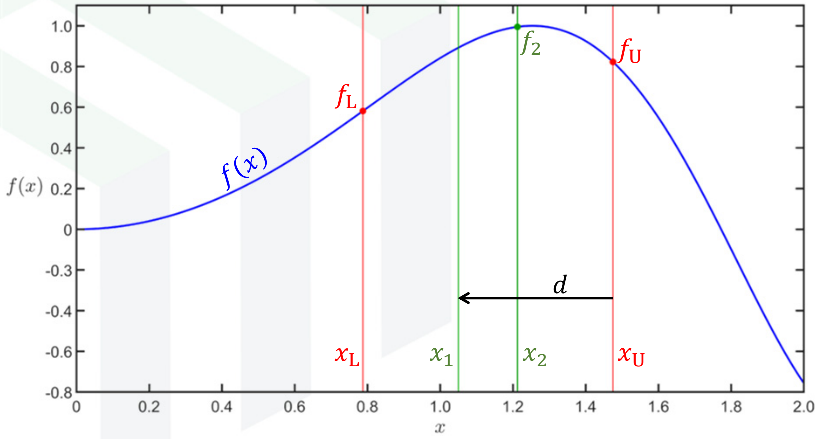
Calculate New Intermediate Point

Calculate distance d.

$$d = R\left(x_U - x_L\right)$$

Calculate new intermediate point x_1.

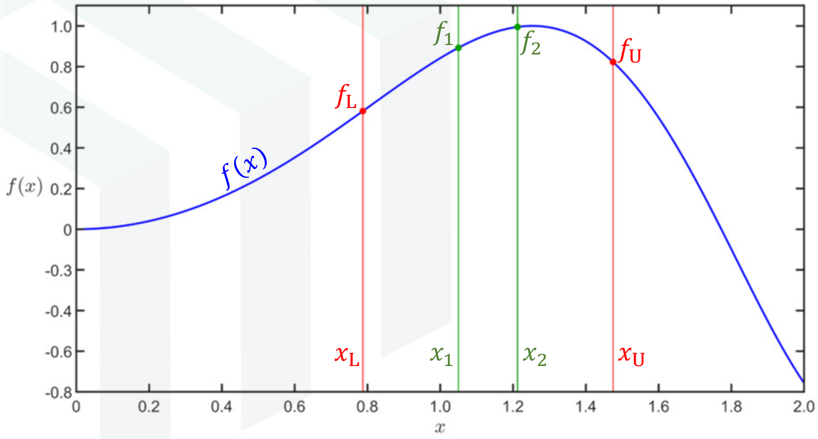
$$x_1 = x_U - d$$

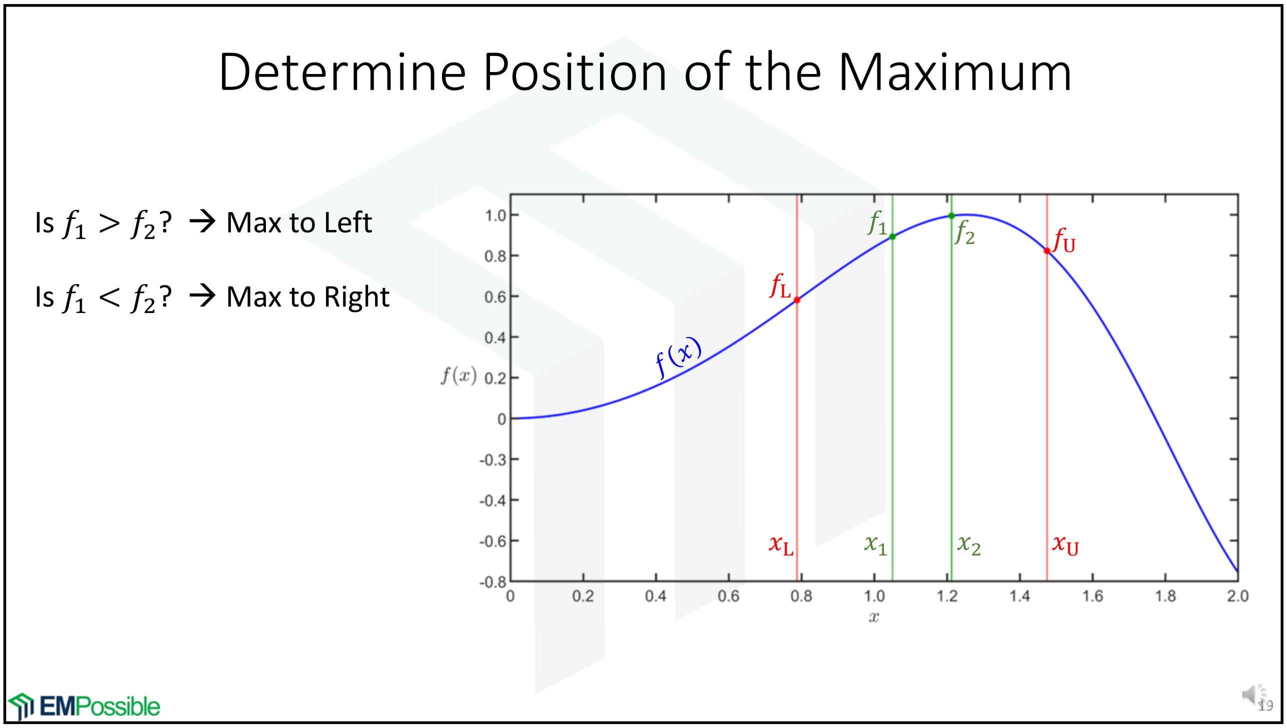


Evaluate Function at New Intermediate Point

$$f_1 = f(x_1)$$

Observe that the entire iteration only involved calculating one new point and one new evaluation of the function.

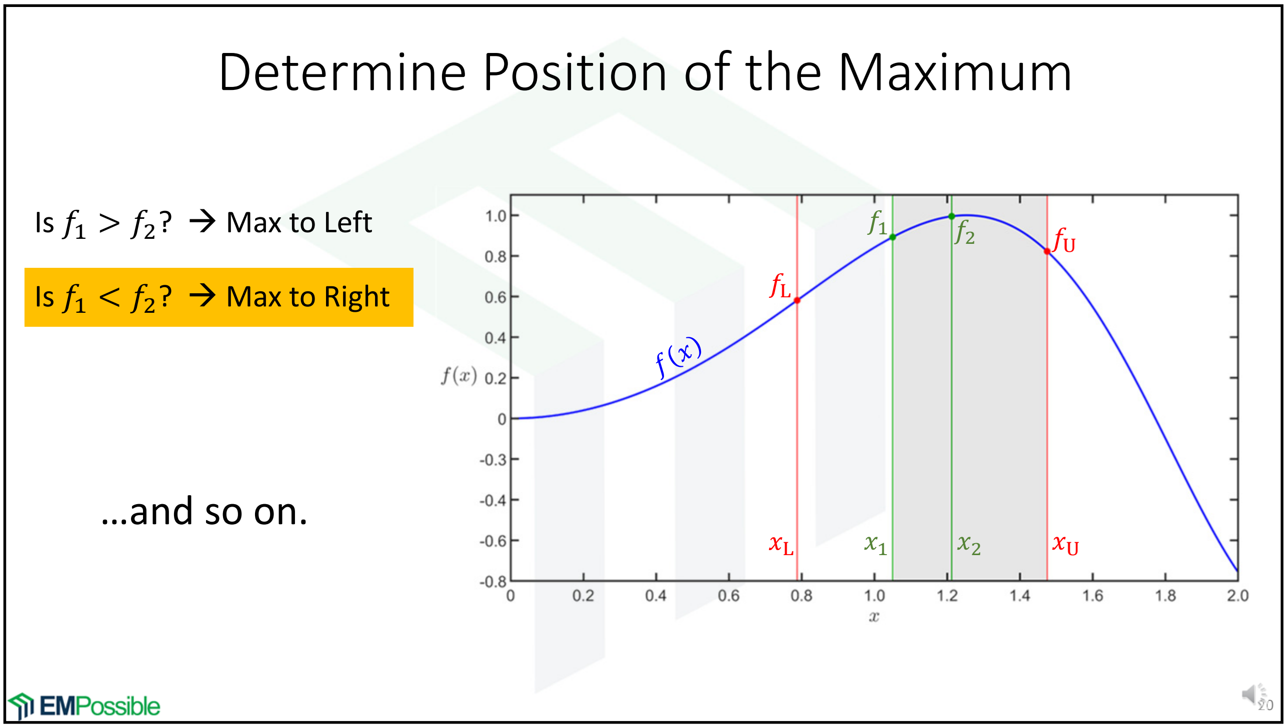




Determine Position of the Maximum

Is $f_1 > f_2$? → Max to Left

Is $f_1 < f_2$? → Max to Right



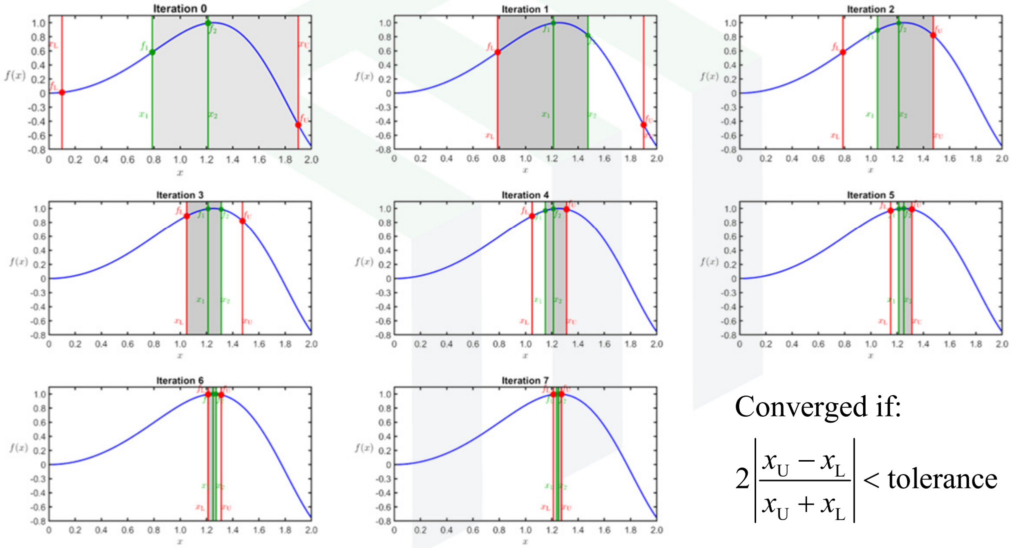
Determine Position of the Maximum

Is $f_1 > f_2$? → Max to Left

Is $f_1 < f_2$? → Max to Right

…and so on.

Repeat Until Convergence



Converged if:

$$2\left|\frac{x_\mathrm{U} - x_\mathrm{L}}{x_\mathrm{U} + x_\mathrm{L}}\right| < \text{tolerance}$$

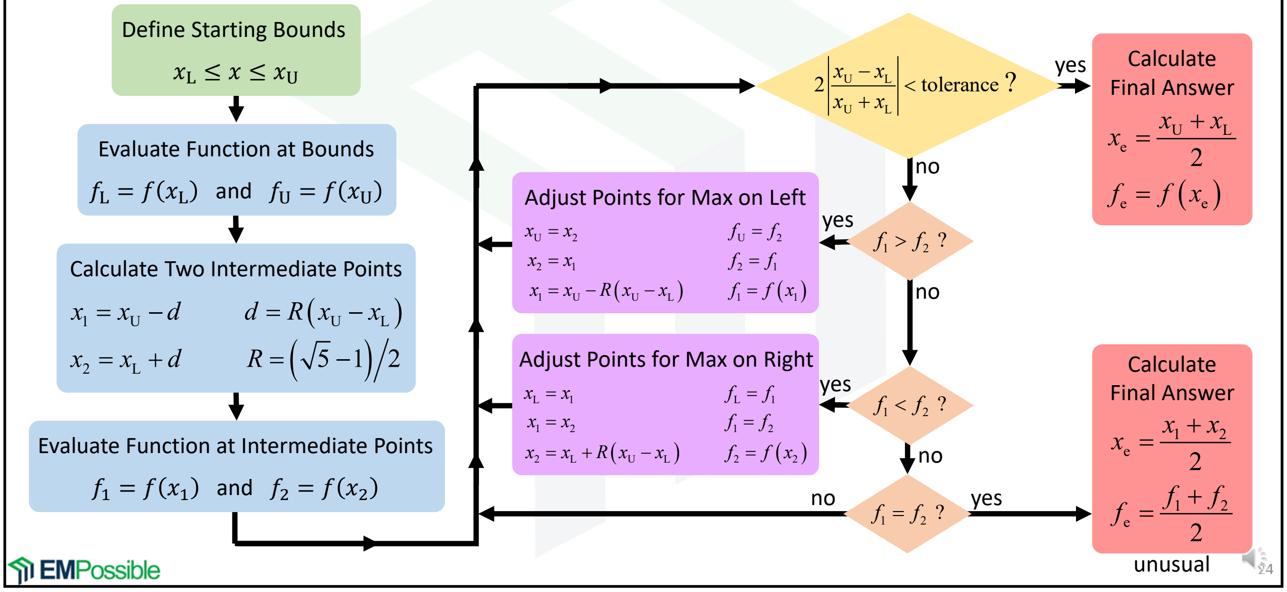
Calculate Final Answer

Estimate the final maximum to be at the midpoint of the last interval.

$$x_\mathrm{e} \approx \frac{x_\mathrm{U} + x_\mathrm{L}}{2} \qquad f_\mathrm{e} = f\left(x_\mathrm{e}\right)$$

Method Summary

Block Diagram of Golden Section Search to Find a Maximum



Define Starting Bounds

$x_L \le x \le x_U$

Evaluate Function at Bounds

$f_L = f(x_L)$ and $f_U = f(x_U)$

Calculate Two Intermediate Points

$x_1 = x_U - d \qquad d = R\left(x_U - x_L\right)$

$x_2 = x_L + d \qquad R = \left(\sqrt{5}-1\right)/2$

Evaluate Function at Intermediate Points

$f_1 = f(x_1)$ and $f_2 = f(x_2)$

$$2\left|\frac{x_U - x_L}{x_U + x_L}\right| < \text{tolerance ?}$$

yes

Calculate Final Answer

$$x_e = \frac{x_U + x_L}{2}$$

$$f_e = f\left(x_e\right)$$

no

$f_1 > f_2$?

yes

Adjust Points for Max on Left

$x_U = x_2 \qquad f_U = f_2$

$x_2 = x_1 \qquad f_2 = f_1$

$x_1 = x_U - R\left(x_U - x_L\right) \qquad f_1 = f\left(x_1\right)$

no

$f_1 < f_2$?

yes

Adjust Points for Max on Right

$x_L = x_1 \qquad f_L = f_1$

$x_1 = x_2 \qquad f_1 = f_2$

$x_2 = x_L + R\left(x_U - x_L\right) \qquad f_2 = f\left(x_2\right)$

no

$f_1 = f_2$?

no

yes

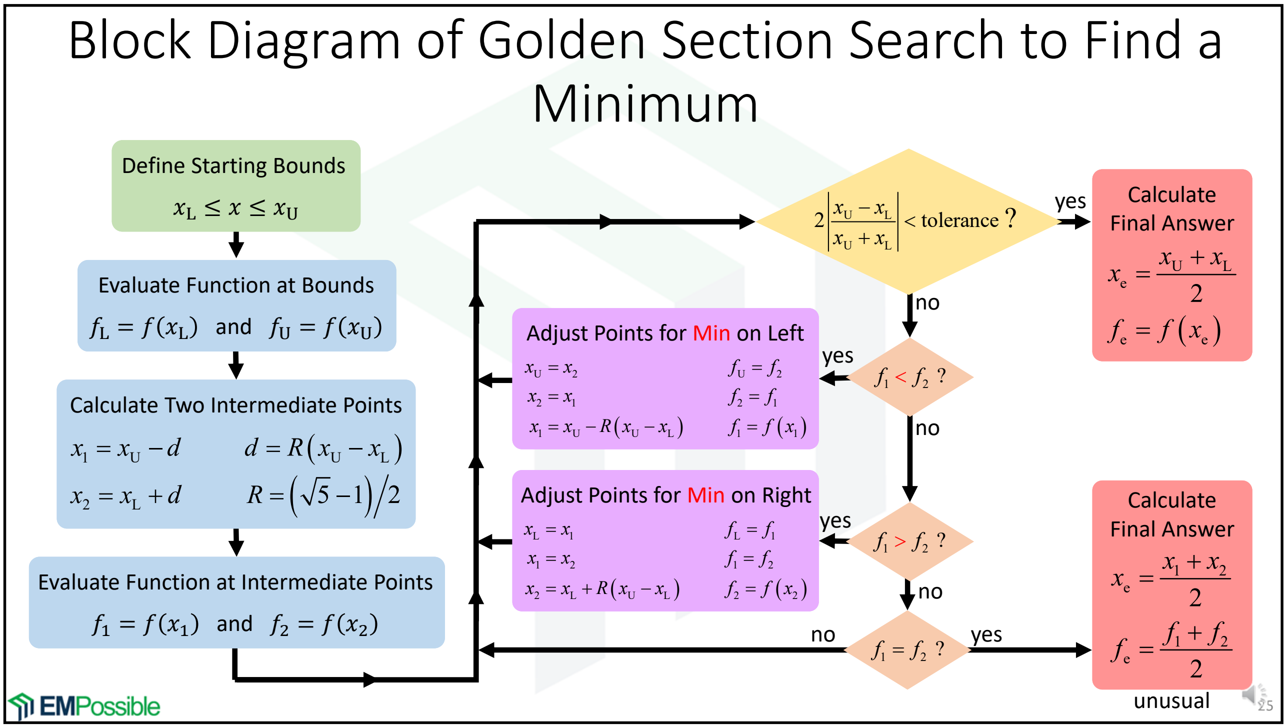
Calculate Final Answer

$$x_e = \frac{x_1 + x_2}{2}$$

$$f_e = \frac{f_1 + f_2}{2}$$

unusual

EMPossible



Block Diagram of Golden Section Search to Find a Minimum

Define Starting Bounds

$x_L \le x \le x_U$

Evaluate Function at Bounds

$f_L = f(x_L)$ and $f_U = f(x_U)$

Calculate Two Intermediate Points

$x_1 = x_U - d \qquad d = R(x_U - x_L)$

$x_2 = x_L + d \qquad R = (\sqrt{5} - 1)/2$

Evaluate Function at Intermediate Points

$f_1 = f(x_1)$ and $f_2 = f(x_2)$

$2\left|\dfrac{x_U - x_L}{x_U + x_L}\right| < \text{tolerance}\ ?$

yes

Calculate Final Answer

$x_e = \dfrac{x_U + x_L}{2}$

$f_e = f(x_e)$

no

$f_1 < f_2\ ?$

yes

Adjust Points for Min on Left

$x_U = x_2 \qquad f_U = f_2$

$x_2 = x_1 \qquad f_2 = f_1$

$x_1 = x_U - R(x_U - x_L) \qquad f_1 = f(x_1)$

no

$f_1 > f_2\ ?$

yes

Adjust Points for Min on Right

$x_L = x_1 \qquad f_L = f_1$

$x_1 = x_2 \qquad f_1 = f_2$

$x_2 = x_L + R(x_U - x_L) \qquad f_2 = f(x_2)$

no

$f_1 = f_2\ ?$

no

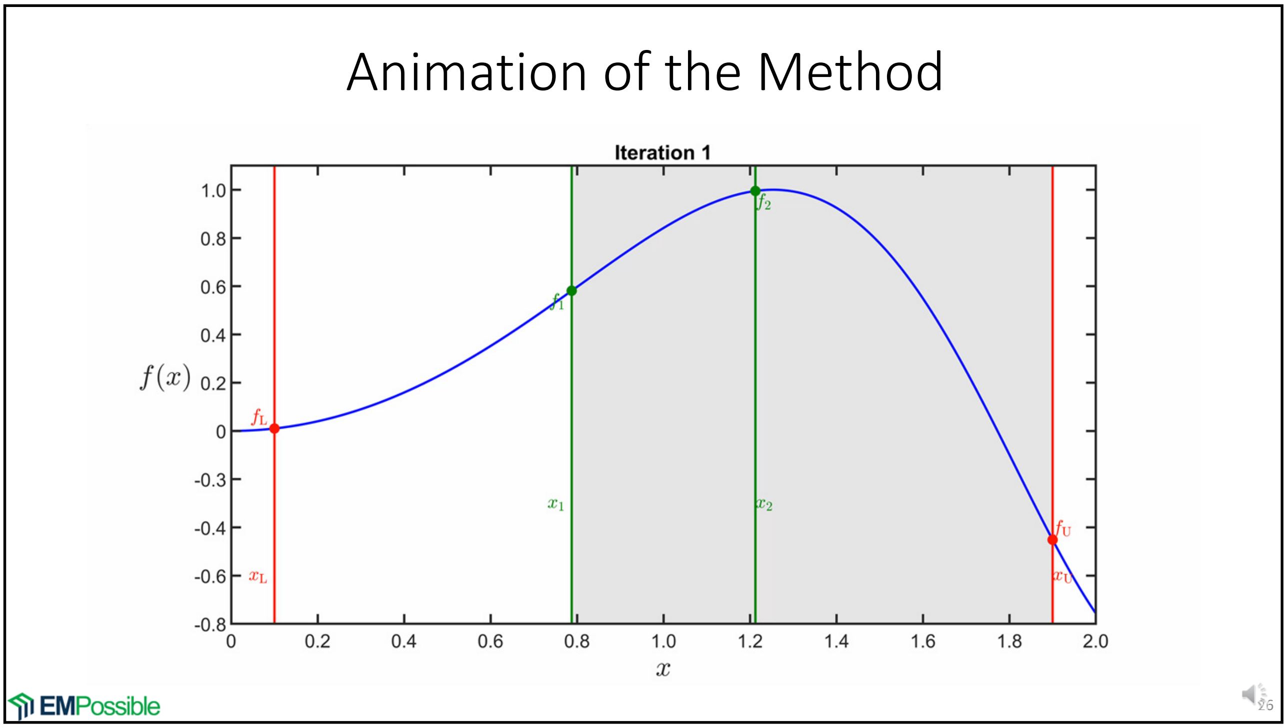
yes

Calculate Final Answer

$x_e = \dfrac{x_1 + x_2}{2}$

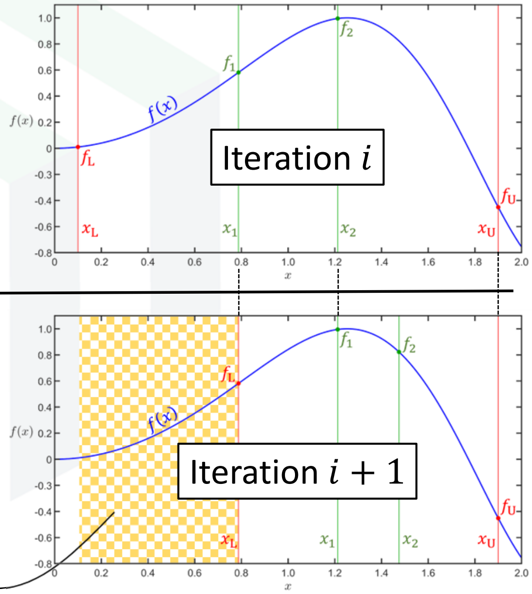
$f_e = \dfrac{f_1 + f_2}{2}$

unusual



Animation of the Method

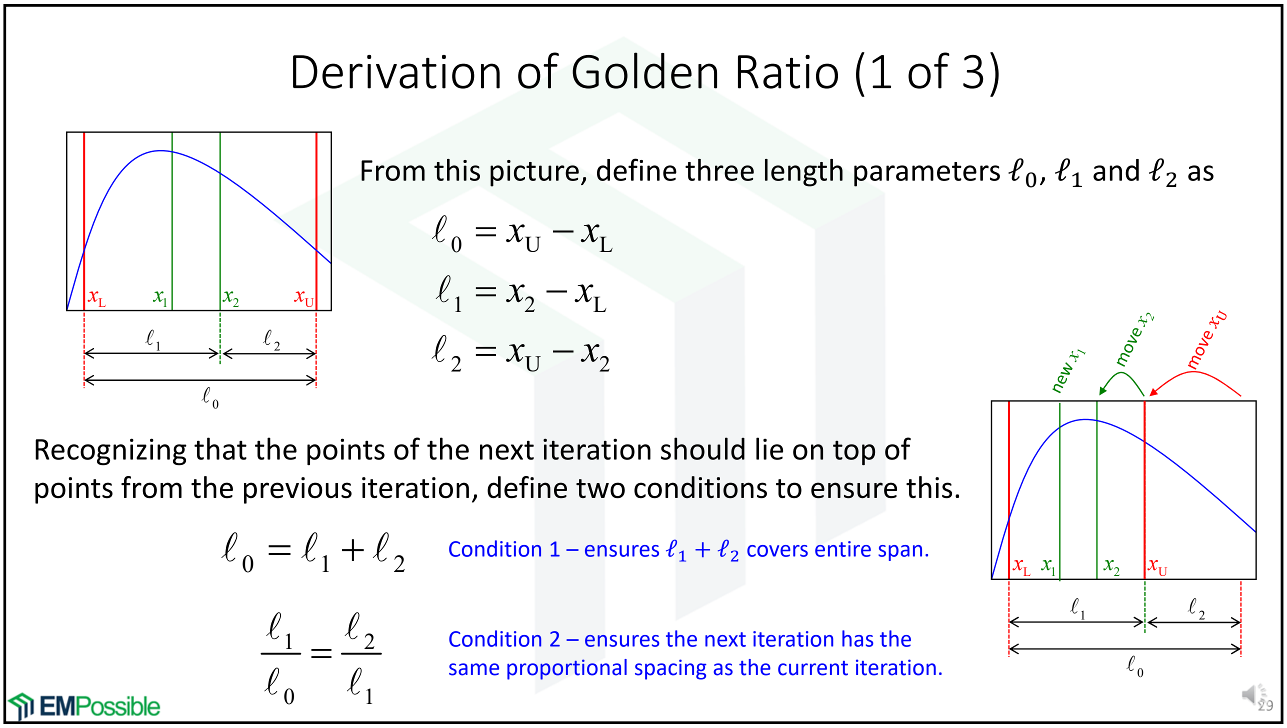
The Magic of the Golden-Section Search



Three of the four points line up from one iteration to the next. This means the function only has to be evaluated at one new point each iteration.

The interval is reduced by 38.2% each iteration.

Derivation of the Golden Ratio R



Derivation of Golden Ratio (1 of 3)

From this picture, define three length parameters ℓ_0, ℓ_1 and ℓ_2 as

$$\ell_0 = x_U - x_L$$
$$\ell_1 = x_2 - x_L$$
$$\ell_2 = x_U - x_2$$

Recognizing that the points of the next iteration should lie on top of points from the previous iteration, define two conditions to ensure this.

$$\ell_0 = \ell_1 + \ell_2$$

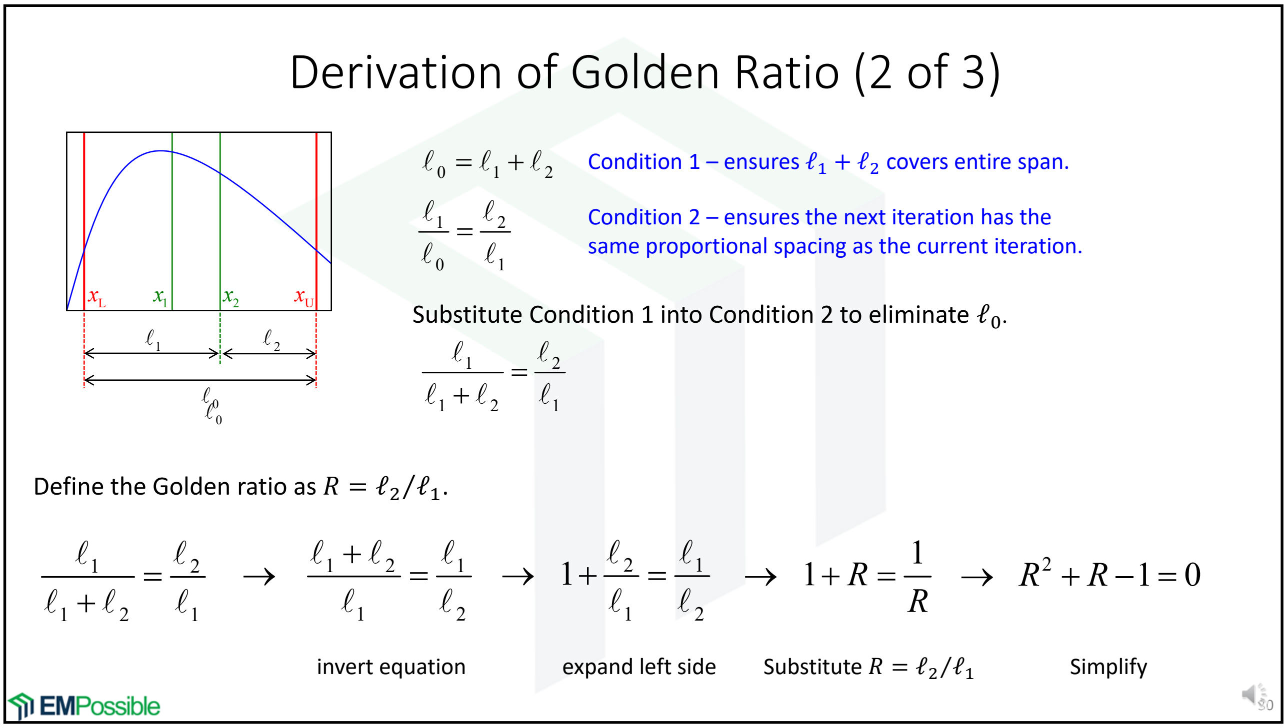
Condition 1 – ensures $\ell_1 + \ell_2$ covers entire span.

$$\frac{\ell_1}{\ell_0} = \frac{\ell_2}{\ell_1}$$

Condition 2 – ensures the next iteration has the same proportional spacing as the current iteration.

EMPossible

29



Derivation of Golden Ratio (2 of 3)

$$\ell_0 = \ell_1 + \ell_2$$

Condition 1 – ensures $\ell_1 + \ell_2$ covers entire span.

$$\frac{\ell_1}{\ell_0} = \frac{\ell_2}{\ell_1}$$

Condition 2 – ensures the next iteration has the same proportional spacing as the current iteration.

Substitute Condition 1 into Condition 2 to eliminate ℓ_0.

$$\frac{\ell_1}{\ell_1 + \ell_2} = \frac{\ell_2}{\ell_1}$$

Define the Golden ratio as $R = \ell_2/\ell_1$.

$$\frac{\ell_1}{\ell_1 + \ell_2} = \frac{\ell_2}{\ell_1} \rightarrow \frac{\ell_1 + \ell_2}{\ell_1} = \frac{\ell_1}{\ell_2} \rightarrow 1 + \frac{\ell_2}{\ell_1} = \frac{\ell_1}{\ell_2} \rightarrow 1 + R = \frac{1}{R} \rightarrow R^2 + R - 1 = 0$$

invert equation

expand left side

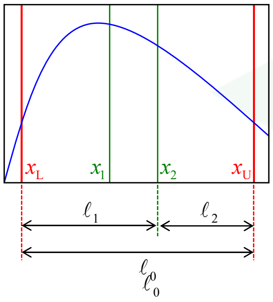
Substitute $R = \ell_2/\ell_1$

Simplify

EMPossible

30

Derivation of Golden Ratio (3 of 3)



$\ell_0 = \ell_1 + \ell_2$ Condition 1 – ensures $\ell_1 + \ell_2$ covers entire span.

$\frac{\ell_1}{\ell_0} = \frac{\ell_2}{\ell_1}$ Condition 2 – ensures the next iteration has the same proportional spacing as the current iteration.

Recall the quadratic formula.

$$ax^2 + bx + c = 0 \quad \rightarrow \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Solve for R using the quadratic formula.

$$R^2 + R - 1 = 0 \quad \rightarrow \quad R = \frac{-1 \pm \sqrt{1^2 - 4(1)(-1)}}{2(1)} = \frac{-1 \pm \sqrt{5}}{2}$$

Pick the positive root to keep R positive.

$$\boxed{R = \frac{\sqrt{5} - 1}{2} \approx 0.618033988749895...}$$

EMPossible