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Computational Science:
Computational Methods in Engineering

Newton's Method for Multiple Variables

Setup for Method Formulation

Newton's method can be generalized to multiple variables using the Hessian.

A second-order Taylor series can be written for $f(\mathbf{x})$ near $\mathbf{x} = \mathbf{x}_i$.

$$f(\mathbf{x}) \cong f(\mathbf{x}_i) + \left[\nabla f(\mathbf{x}_i)\right]^T (\mathbf{x}-\mathbf{x}_i) + \frac{1}{2}(\mathbf{x}-\mathbf{x}_i)^T [H_i](\mathbf{x}-\mathbf{x}_i)$$

At an extremum, $\nabla f(\mathbf{x}) = 0$. To find this point, derive the gradient of the above expression.

$$\nabla f(\mathbf{x}) \cong \nabla f(\mathbf{x}_i) + [H_i](\mathbf{x}-\mathbf{x}_i)$$

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2

Formulation of the Update Equation

Set the gradient to zero and solve for $\mathbf{x}$.

$\nabla f(\mathbf{x}) \cong \nabla f(\mathbf{x}_i) + [H_i](\mathbf{x} - \mathbf{x}_i) = 0$	Start with equation from previous slide.
$\nabla f(\mathbf{x}_i) + [H_i](\mathbf{x} - \mathbf{x}_i) = 0$	Drop the gradient definition $\nabla f(\mathbf{x}) \cong$.
$[H_i](\mathbf{x} - \mathbf{x}_i) = -\nabla f(\mathbf{x}_i)$	Move gradient term $\nabla f(\mathbf{x}_i)$ to right-hand side.
$\mathbf{x} - \mathbf{x}_i = -[H_i]^{-1} \nabla f(\mathbf{x}_i)$	Predivide by sides by $[H_i]$.
$\mathbf{x} = \mathbf{x}_i - [H_i]^{-1} \nabla f(\mathbf{x}_i)$	Solve equation for $\mathbf{x}$.

From this, the update equation for the multivariable Newton's method is

$$\boxed{\mathbf{x}_{i+1} = \mathbf{x}_i - [H_i]^{-1} \nabla f(\mathbf{x}_i)}$$

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3