Electromagnetics:

Microwave Engineering

The Scattering Matrix

Lecture Outline

- The Scattering Matrix
- Translation of Terminal Planes
- Scattering Matrix Example

The Scattering Matrix

Scattering Matrices

The scattering matrix $[S]$ relates the voltage waves incident on the ports to those reflected from the ports.

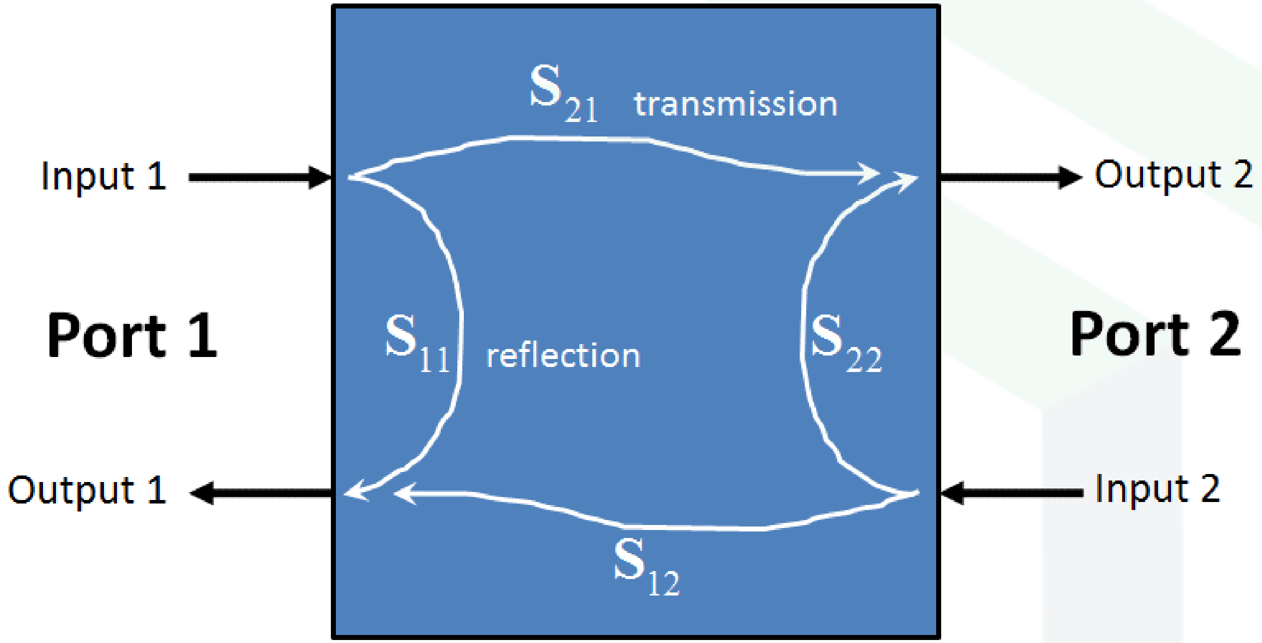
$$\begin{bmatrix} V_1^- \\ V_2^- \\ \vdots \\ V_N^- \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & \cdots & S_{1N} \\ S_{21} & S_{22} & \cdots & S_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ S_{N1} & S_{N2} & \cdots & S_{NN} \end{bmatrix} \begin{bmatrix} V_1^+ \\ V_2^+ \\ \vdots \\ V_N^+ \end{bmatrix}$$

$$[V^-] = [S][V^+]$$

$$S_{ij} = \left. \frac{V_i^-}{V_j^+} \right|_{\text{No other applied voltages}}$$

S_{ij} is found by driving port j with incident wave of voltage V_j^+ and measuring the reflected wave amplitude V_i^- coming out of port i. All other ports should be terminated with matched loads to avoid reflections.

Interpretation of a Scattering Matrix for a Two-Port Network



$S_{11} \equiv$ forward reflection (e.g. r and Γ)

$S_{12} \equiv$ backward transmission

$S_{21} \equiv$ forward transmission (e.g. t)

$S_{22} \equiv$ backward reflection

$$\begin{bmatrix} V_1^- \\ V_2^- \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} V_1^+ \\ V_2^+ \end{bmatrix}$$

Reciprocal and Lossless Networks

Reciprocal

In reciprocal networks, $[S]$ is symmetric.

$$[S] = [S]^T$$

Lossless

In lossless networks, $[S]$ is unitary.

$$[S] = [S]^*$$

$$[S]^* = \left([S]^T\right)^{-1} \rightarrow \underbrace{|S_{11}|^2 + |S_{21}|^2 = 1}$$

For 2 port networks

Relation between $[Z]$ and $[S]$

We must assume that all ports have the same characteristic impedance Z_0
We start with the definition of the $[Z]$ matrix.

$$[V] = [Z][I]$$

We then expand $[V]$ and $[I]$ in terms of forward and backward waves.

$$[V] = [V^+] + [V^-] \qquad [I] = [I^+] + [I^-]$$

$$[Z]([I^+] + [I^-]) = [V^+] + [V^-]$$

Relation between $[Z]$ and $[S]$

$$[Z]([I^+]+[I^-])=[V^+]+[V^-]$$

$$[Z]\left(\frac{[V^+]}{Z_0}-\frac{[V^-]}{Z_0}\right)=[V^+]+[V^-]$$

$$[1]=\begin{bmatrix}1 & 0 & \cdots & 0\\ 0 & 1 & & \vdots\\ 0 & & \ddots & \vdots\\ 0 & \cdots & \cdots & 1\end{bmatrix}$$

$$[Z]([V^+]+[V^-])=Z_0[V^+]+Z_0[V^-]$$

$$[Z][V^-]+Z_0[V^-]\ =[Z][V^+]+Z_0[V^+]$$

$$([Z^-]+Z_0[1])[V^-]\ =([Z^-]+Z_0[1])[V^+]$$

$$[V^-]\ =\underbrace{([Z^-]+Z_0[1])^{-1}([Z^-]+Z_0[1])}_{[S]}[V^+]$$

Relation between $[Z]$ and $[S]$

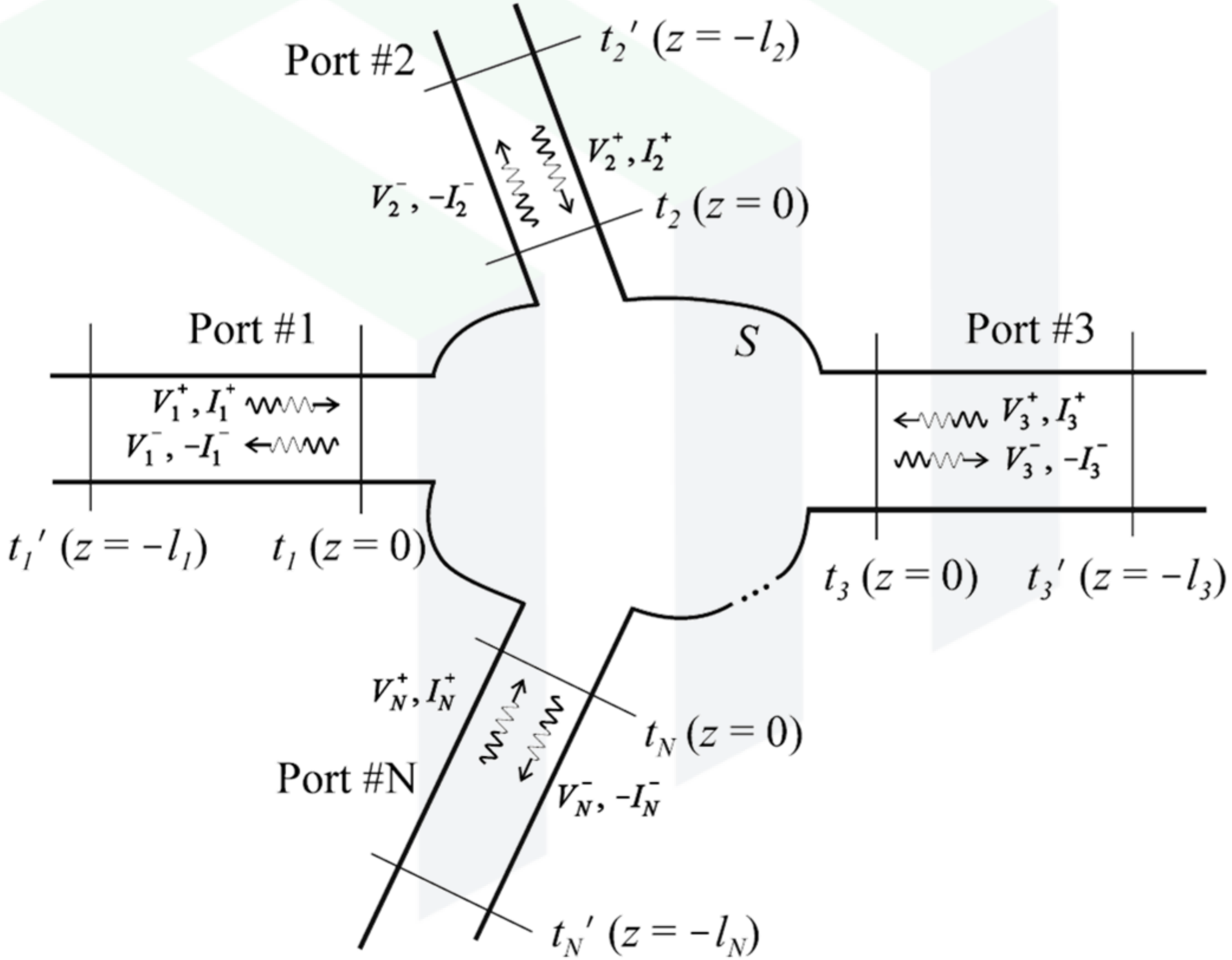
$$[S] = ([Z^{-}] + Z_0[1])^{-1}([Z^{-}] + Z_0[1])$$

$$[Z] = Z_0([1] - [S])^{-1}([1] + [S])$$

Translation of Terminal Planes

Translation of Terminal Planes

The voltages and currents at different terminal planes differ only by phase (assuming lossless lines)



EMPossible

Translation of Terminal Planes

The voltages and currents at different terminal planes differ only by phase (assuming lossless lines)

$$\begin{aligned} V_1'^{+} &= e^{-j\theta_1} V_1^{+} \\ V_2'^{+} &= e^{-j\theta_2} V_2^{+} \\ &\vdots \\ V_N'^{+} &= e^{-j\theta_N} V_N^{+} \end{aligned} \rightarrow \begin{bmatrix} V_1'^{+} \\ V_2'^{+} \\ \vdots \\ V_N'^{+} \end{bmatrix} = \begin{bmatrix} e^{-j\theta_1} & & & \\ & e^{-j\theta_2} & & \\ & & \ddots & \\ & & & e^{-j\theta_N} \end{bmatrix} \begin{bmatrix} V_1^{+} \\ V_2^{+} \\ \vdots \\ V_N^{+} \end{bmatrix}$$

$$\begin{aligned} V_1'^{-} &= e^{-j\theta_1} V_1^{-} \\ V_2'^{-} &= e^{-j\theta_2} V_2^{-} \\ &\vdots \\ V_N'^{-} &= e^{-j\theta_N} V_N^{-} \end{aligned} \rightarrow \begin{bmatrix} V_1'^{-} \\ V_2'^{-} \\ \vdots \\ V_N'^{-} \end{bmatrix} = \begin{bmatrix} e^{-j\theta_1} & & & \\ & e^{-j\theta_2} & & \\ & & \ddots & \\ & & & e^{-j\theta_N} \end{bmatrix} \begin{bmatrix} V_1^{-} \\ V_2^{-} \\ \vdots \\ V_N^{-} \end{bmatrix}$$

$$\theta_i = \beta_i l_i$$

$V_i^{\pm} \equiv$ voltage at exactly Port i

$V_i'^{\pm} \equiv$ voltage at distance l_i away from Port i

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Scattering Matrix of a Length of Lossless Transmission Line

$$\begin{bmatrix} V_1^- \\ V_2^- \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} V_1^+ \\ V_2^+ \end{bmatrix}$$

S_{11} - zero forward reflection
S_{12} - 100% backward transmission
S_{21} - 100% forward transmission
S_{22} - zero backward reflection

When the transmission line has length l, the scattering matrix is

$$\begin{bmatrix} V_1^- \\ V_2^- \end{bmatrix} = \begin{bmatrix} 0 & e^{-j\theta} \\ e^{-j\theta} & 0 \end{bmatrix} \begin{bmatrix} V_1^+ \\ V_2^+ \end{bmatrix} \qquad \theta = \beta l$$

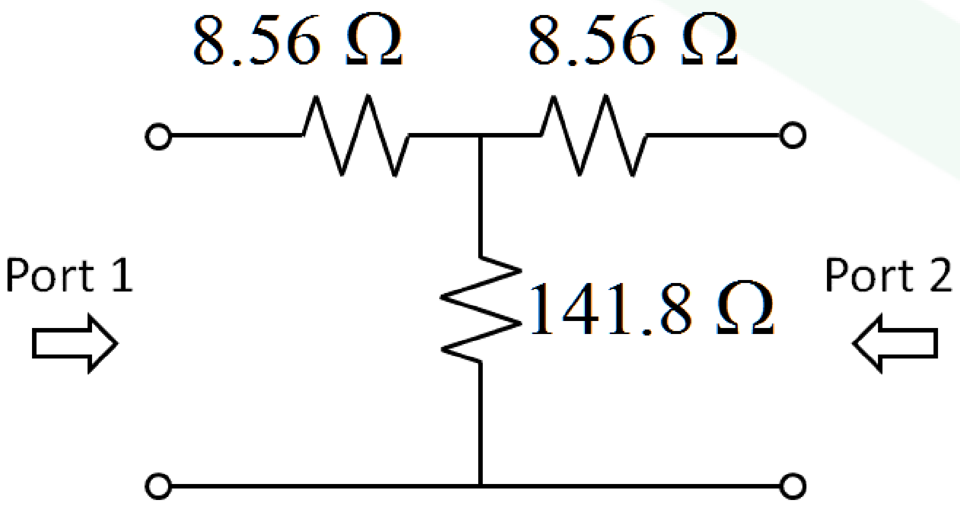
S_{11} - zero reflection
S_{12} - accumulates phase
S_{21} - accumulates phase
S_{22} - zero reflection

We can use this to model large cirtuits by separating circuit elements with lengths of transmission line.

Scattering Matrix Example

Scattering Matrix Example

Given the characteristic impedance of the transmission line is $50\ \Omega$, find the scattering matrix $[S]$. Assume size $\ll$ wavelength



$$S_{11} = \left.\frac{V_1^-}{V_1^+}\right|_{V_2^+=0}$$

We can write S_{11} in terms of the reflection coefficient. We match impedance at Port 2 to isolate S_{11} to the reflection at Port 1

$$S_{11} = \Gamma^{(1)}\Big|_{V_2^+=0} = \left.\frac{Z_{in}^{(1)} - Z_0}{Z_{in}^{(1)} + Z_0}\right|_{Z_0 \text{ on Port } 2}$$

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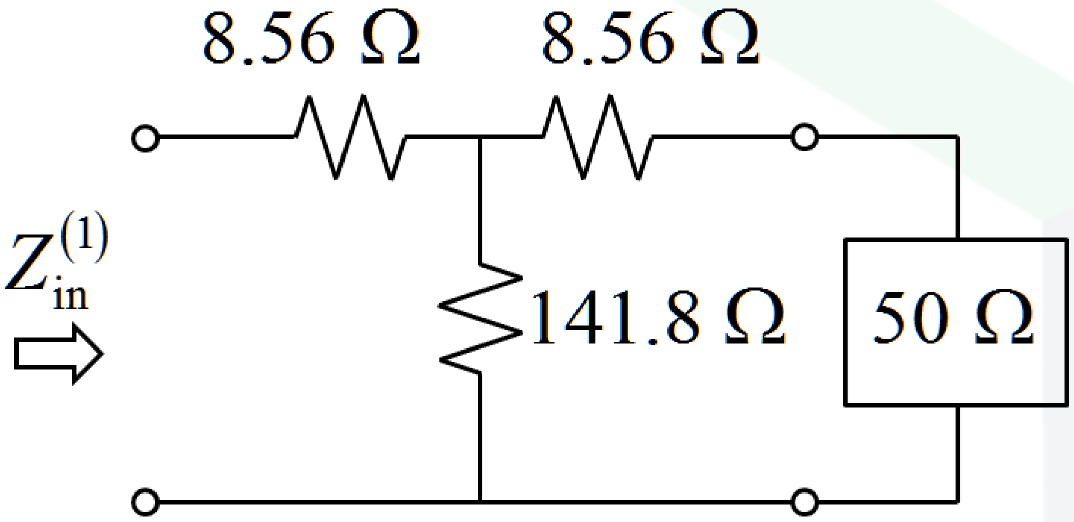
Scattering Matrix Example

To determine Z_{in}, we write our circuit as

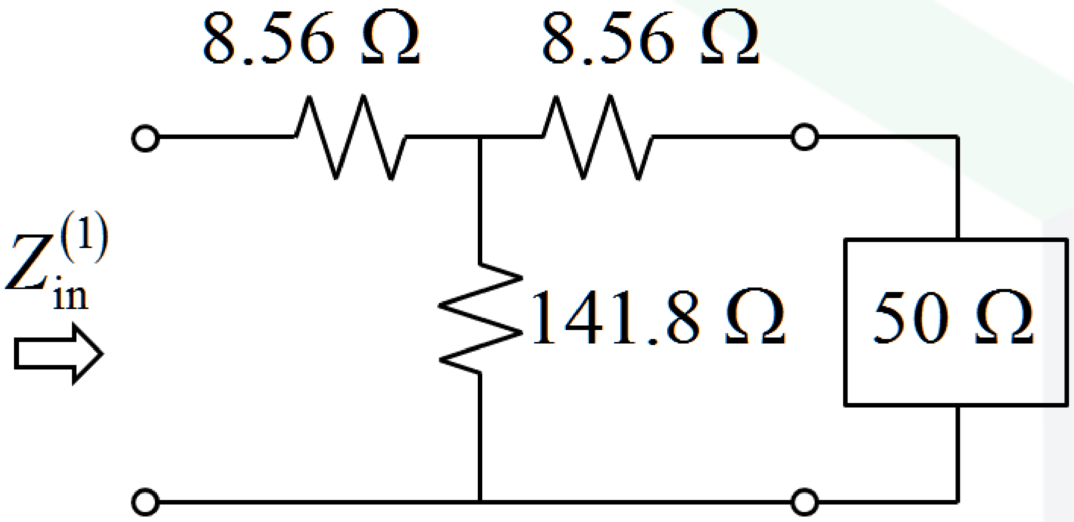


$$Z_{in}^{(1)} = 8.56 + 141.8||(8.56 + 50) = 50$$

Therefore, we obtain

$$S_{11} = \left. \frac{Z_{in}^{(1)} - Z_0}{Z_{in}^{(1)} - Z_0} \right|_{Z_0 \text{ on Port 2}} = \frac{50 - 50}{50 + 50} = \boxed{0}$$

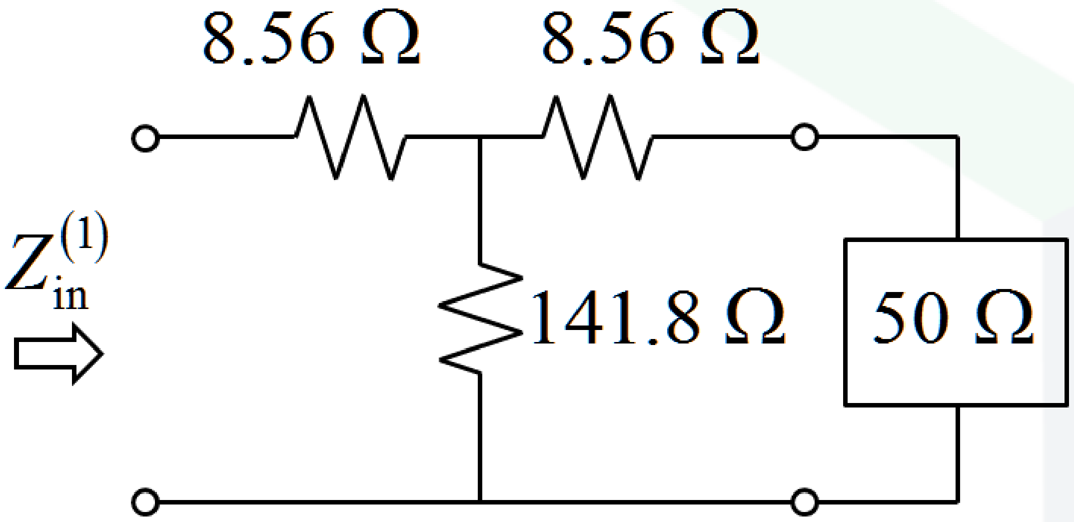
Scattering Matrix Example



$$S_{21} = \left. \frac{V_2^-}{V_1^+} \right|_{V_2^+=0}$$

In order to isolate S_{21} to the transmission coefficient of just the network, we match the impedance at Port 2. This reduces to the circuit analyzed for S_{11}

Scattering Matrix Example



Applying V_1^+ to Port 1, the voltage across the 141.8 Ω is calculated by voltage division.

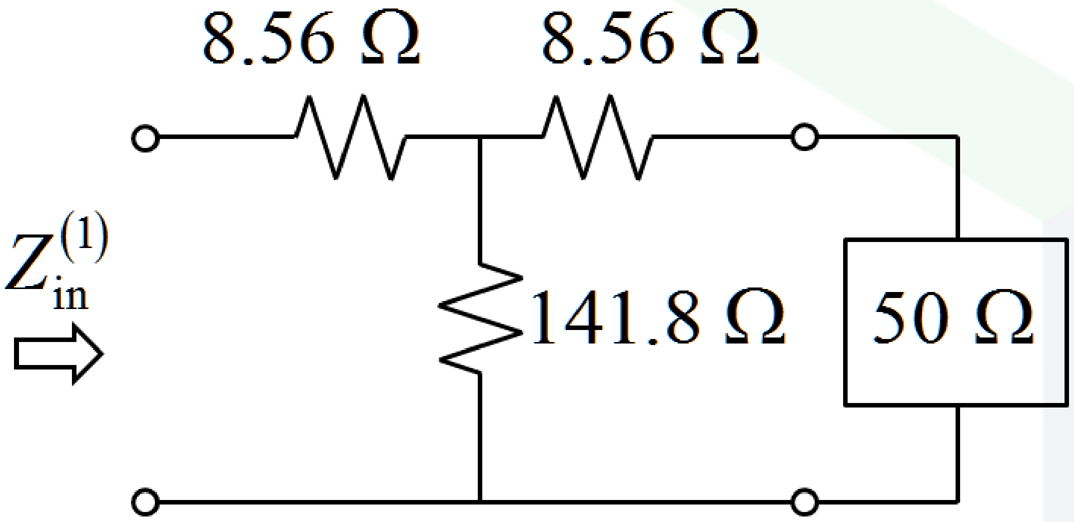
$$R' = 141.8||(8.56 + 50) = 41.44$$

$$V' = V_1^+ \frac{R'}{8.56 + R'} = 0.8288V_1^+$$

The voltage across the load is calculated by applying voltage division again.

$$V_2^- = V' \frac{50}{8.56 + 50} = 0.7077V_1^+$$

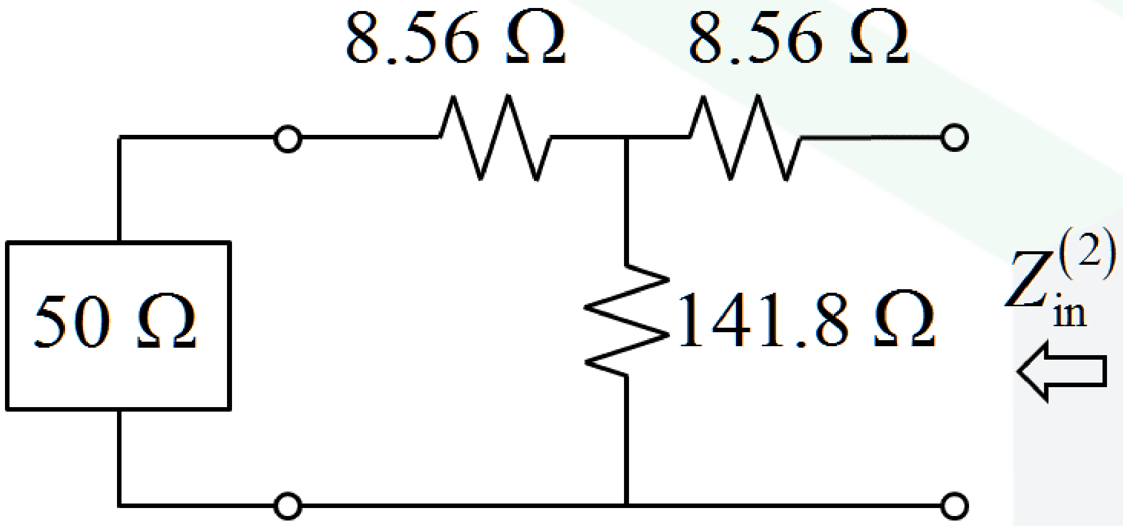
Scattering Matrix Example



The scattering parameter is then

$$S_{21} = \left.\frac{V_2^-}{V_1^+}\right|_{V_2^+=0} = \frac{0.7077V_1^+}{V_1^+} = \boxed{0.7077}$$

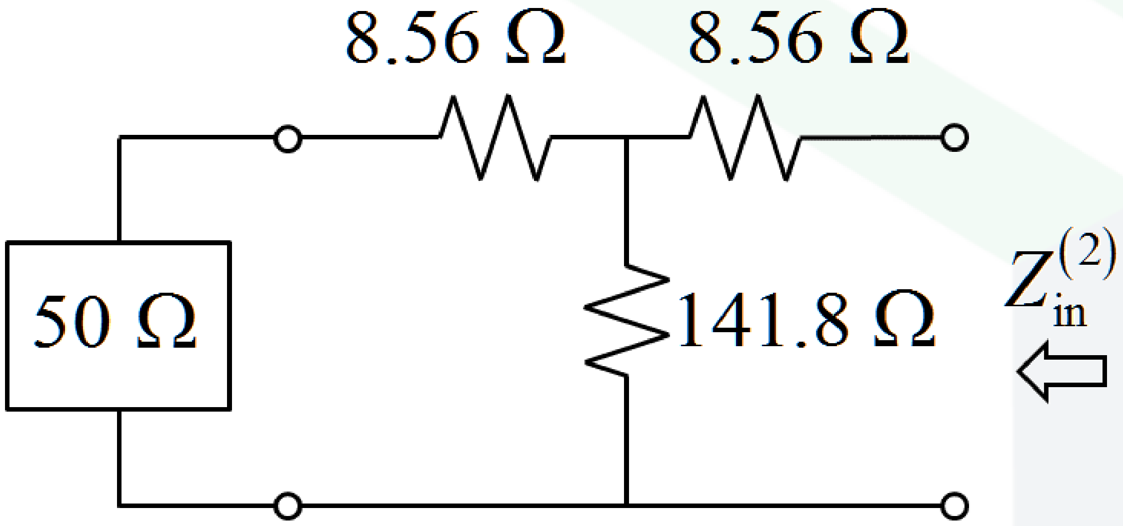
Scattering Matrix Example



$$S_{12} = \left.\frac{V_1^-}{V_2^+}\right|_{V_1^+=0}$$

In order to isolate S_{12} to the transmission coefficient of just the network, we match the impedance at Port 1.

Scattering Matrix Example



Applying V_2^+ to Port 2, the voltage across the 141.8 Ω is calculated by voltage division.

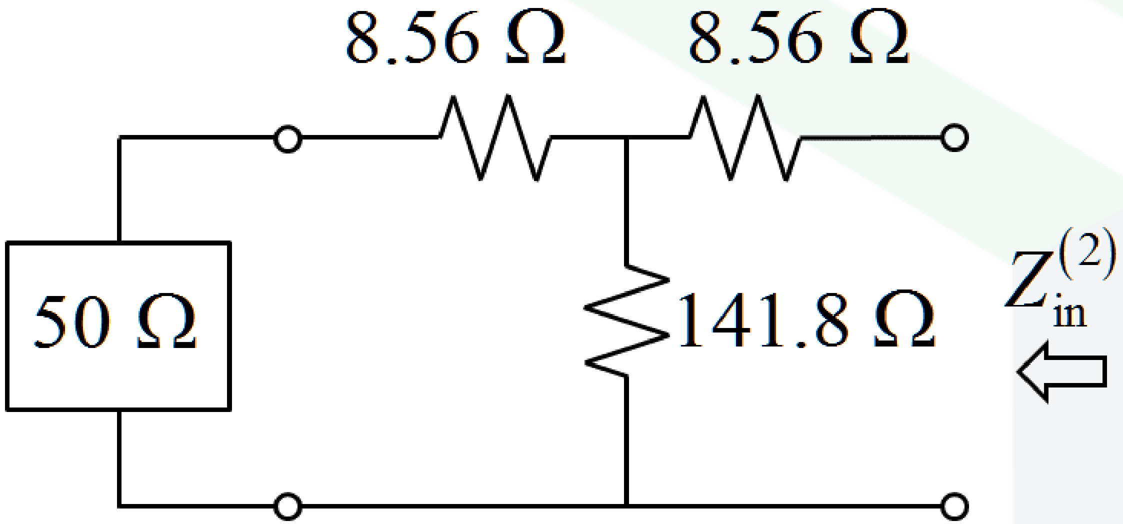
$$R' = 141.8||(8.56 + 50) = 41.44$$

$$V' = V_2^+ \frac{R'}{8.56 + R'} = 0.8288V_2^+$$

The voltage across the load is calculated by applying voltage division again.

$$V_1^- = V' \frac{50}{8.56 + 50} = 0.7077V_2^+$$

Scattering Matrix Example

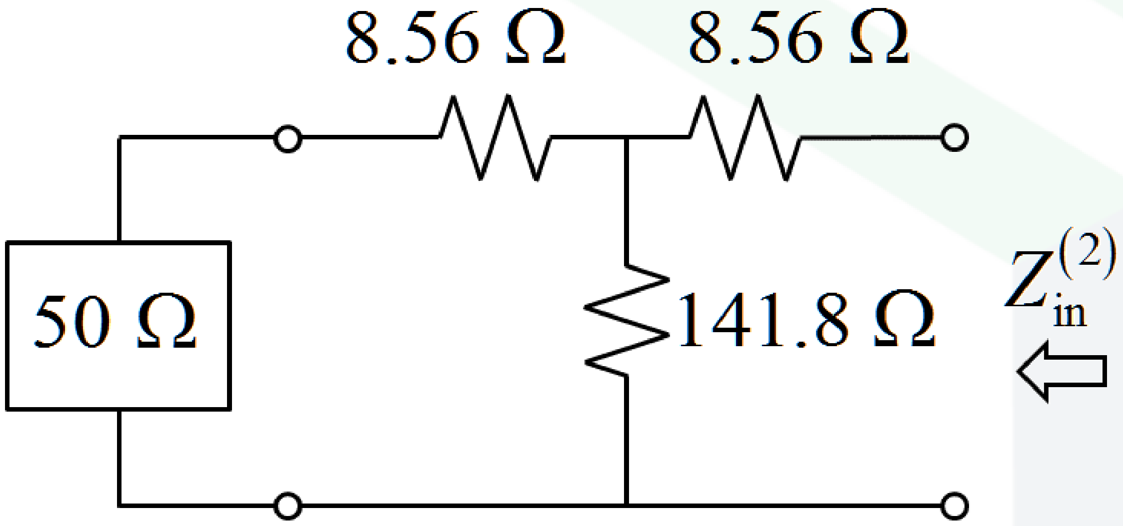


The scattering parameter is then

$$S_{12} = \left.\frac{V_1^-}{V_2^+}\right|_{V_1^+=0} = \frac{0.7077V_2^+}{V_2^+} = \boxed{0.7077}$$

Scattering Matrix Example

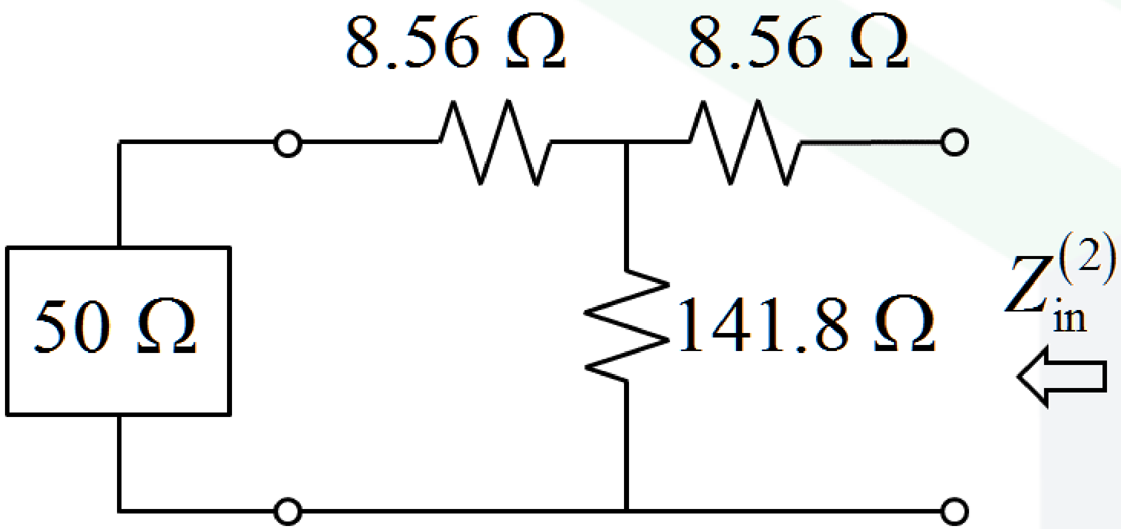
$$S_{22} = \Gamma^{(2)}\Big|_{V_1^+=0}$$



We can write this in terms of the backward reflection coefficient. In order to isolate S_{22} to the reflection at just Port 2, we match the impedance at Port 1.

$$S_{22} = \Gamma^{(2)}\Big|_{V_1^+=0} = \left.\frac{Z_{in}^{(2)} - Z_0}{Z_{in}^{(2)} + Z_0}\right|_{Z_0 \text{ on Port } 1}$$

Scattering Matrix Example



$$Z_{in}^{(2)} = 8.56 + 141.8||(8.56 + 50) = 50$$

Therefore, we obtain

$$S_{22} = \left.\frac{Z_{in}^{(1)} - Z_0}{Z_{in}^{(1)} - Z_0}\right|_{Z_0 \text{ on Port 1}} = \frac{50 - 50}{50 + 50} = \boxed{0}$$

Scattering Matrix Example

What does this network actually do?

Looking at the $[S]$ parameters, we first see that $S_{11} = S_{22} = 0$.

We also see that $S_{12} = S_{21} = 0.7077$.

$$|S_{12}|^2 = |S_{21}|^2 = |0.7077|^2 = 0.5$$

The network transmits half the power. The rest is dissipated by the network.
We call this a 3dB attenuator.

Reciprocal? Yes.

Lossless? No.

$$[S] = \begin{bmatrix} 0 & 0.7077 \\ 0.7077 & 0 \end{bmatrix}$$