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Electromagnetics:

Microwave Engineering

Transmission Matrices and Cascading Multiple Networks

Lecture Outline

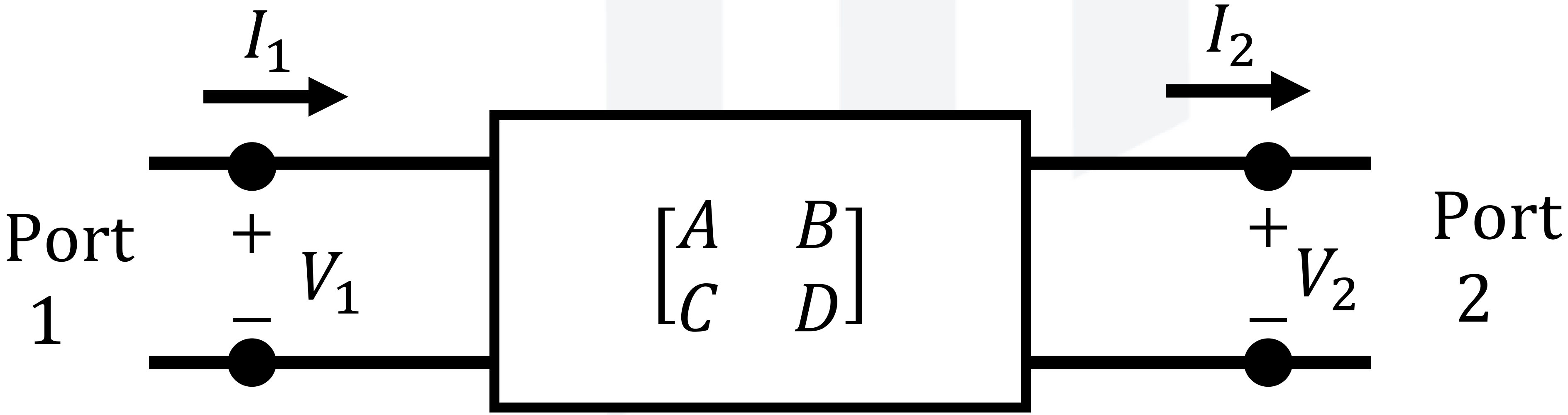
- The Transmission (ABCD) Matrix
- Cascading Multiple Networks
- Example: Analysis of a Large Circuit

The Transmission (ABCD) Matrix

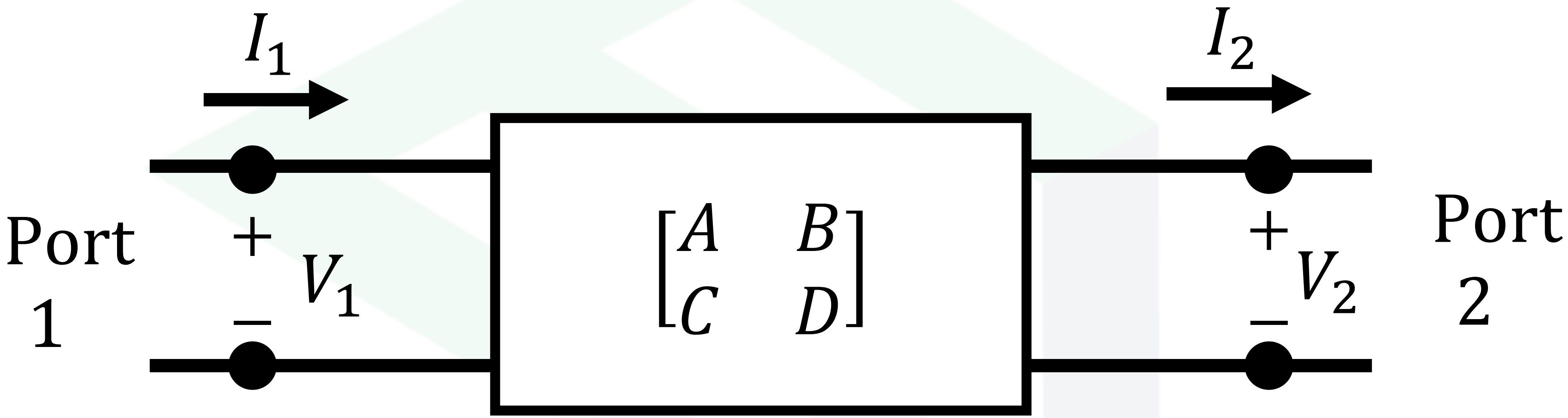
The ABCD Matrix

Relates the total voltage and current at the first port to the total voltage and current at the second port. This is purely for 2-port networks and makes cascading matrices very easy.

$$\begin{aligned} V_1 &= AV_2 + BI_2 \\ I_1 &= CV_2 + DI_2 \end{aligned} \quad \rightarrow \quad \begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix}$$



The ABCD Parameters



$$A = \left.\frac{V_1}{V_2}\right|_{I_2=0}$$ open-circuit voltage ratio

$$C = \left.\frac{I_1}{V_2}\right|_{I_2=0}$$ open-circuit transfer impedance

$$B = \left.\frac{V_1}{I_2}\right|_{V_2=0}$$ short-circuit transfer impedance

$$D = \left.\frac{I_1}{I_2}\right|_{V_2=0}$$ short-circuit current ratio

Properties of ABCD Matrices

Reciprocal Networks

$$\det\begin{bmatrix} A & B \\ C & D \end{bmatrix} = 1 \quad \rightarrow \quad AD - BC = 1$$

Lossless Networks

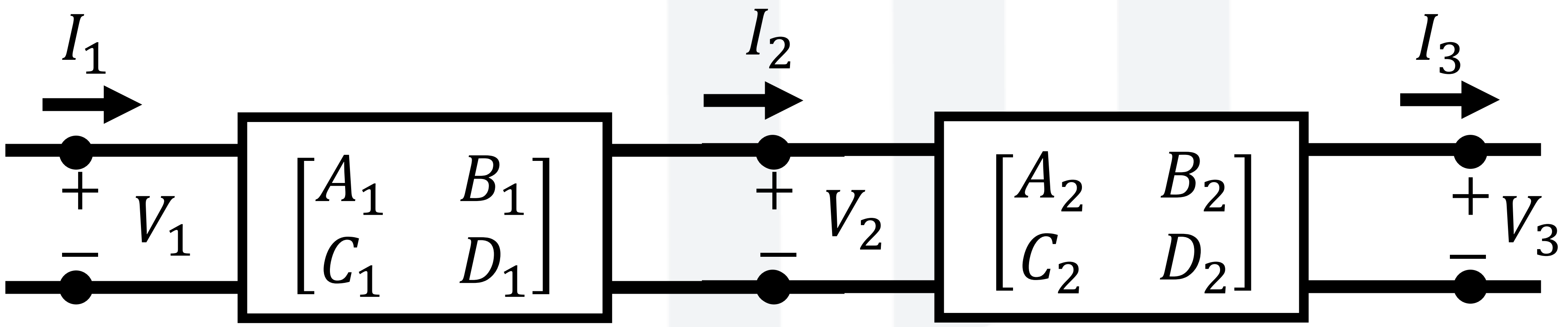
The network is lossless if (1) diagonal elements are purely real, and (2) off diagonal elements are purely imaginary

Cascading Multiple Networks

Combining Two ABCD Matrices

Given the ABCD matrices of two networks,

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix} \qquad \begin{bmatrix} V_2 \\ I_2 \end{bmatrix} = \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} \begin{bmatrix} V_3 \\ I_3 \end{bmatrix}$$



Combining Two ABCD Matrices

We can combine these by substituting and doing simple matrix multiplication.

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} \begin{bmatrix} V_3 \\ I_3 \end{bmatrix}$$

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A_{12} & B_{12} \\ C_{12} & D_{12} \end{bmatrix} \begin{bmatrix} V_3 \\ I_3 \end{bmatrix}$$

$$A_{12} = A_1 A_2 + B_1 C_2$$
$$B_{12} = A_1 B_2 + B_1 D_2$$
$$C_{12} = C_1 A_2 + D_1 C_2$$
$$D_{12} = C_1 B_2 + D_1 D_2$$

Note: $[A][B] \neq [B][A]$

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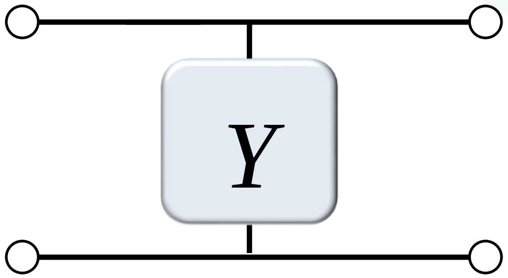
ABCD Parameters of Common Networks

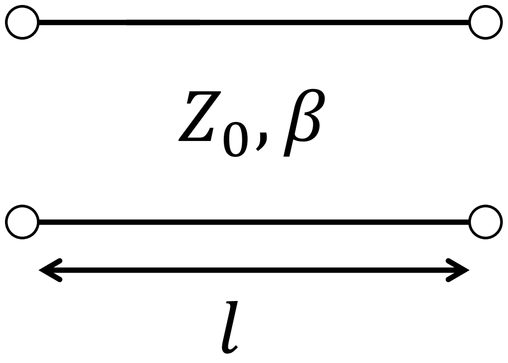
$A = 1 \qquad B = Z$

$C = 0 \qquad D = 1$



$A = 1 \qquad B = 0$

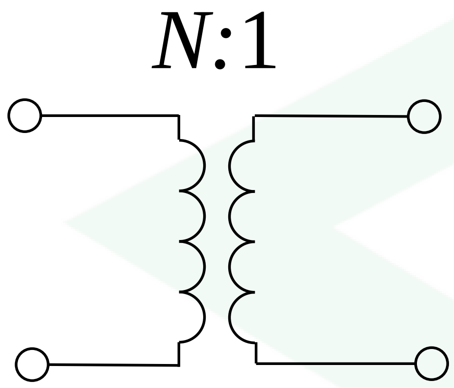
$C = Y \qquad D = 1$



$A = \cos \beta l \qquad B = jZ_0 \sin \beta l$

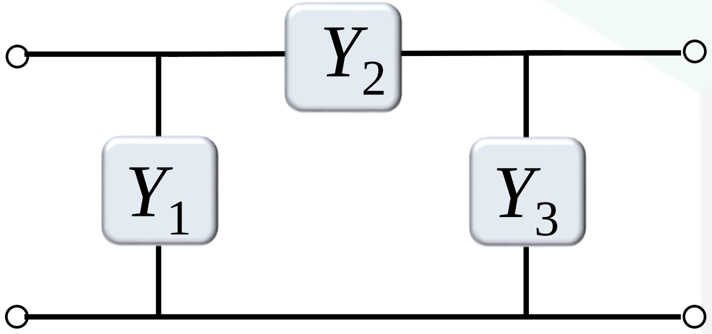
$C = jZ_0 \sin \beta l \qquad D = \cos \beta l$

ABCD Parameters of Common Networks



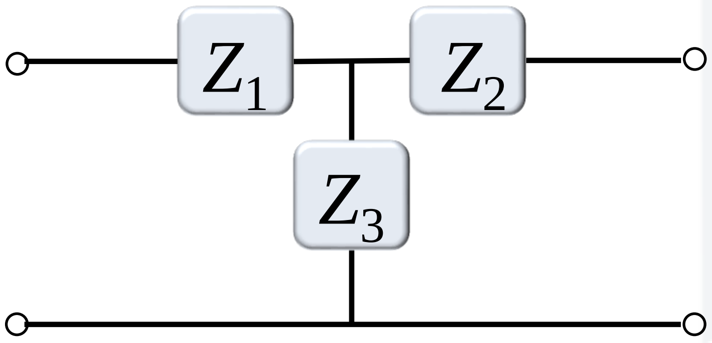
$$A = N \qquad B = 0$$

$$C = 0 \qquad D = \frac{1}{N}$$



$$A = 1 + \frac{Y_2}{Y_3} \qquad B = \frac{1}{Y_3}$$

$$C = Y_1 + Y_2 + \frac{Y_1 Y_2}{Y_3} \qquad D = 1 + \frac{Y_1}{Y_3}$$



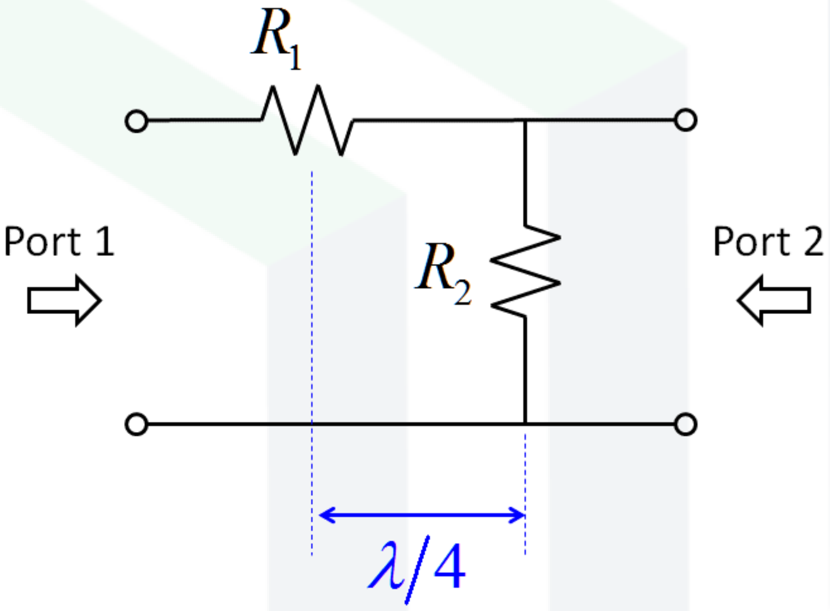
$$A = 1 + \frac{Z_1}{Z_3} \qquad B = Z_1 + Z_2 + \frac{Z_1 Z_2}{Z_3}$$

$$C = \frac{1}{Z_3} \qquad D = 1 + \frac{Z_2}{Z_3}$$

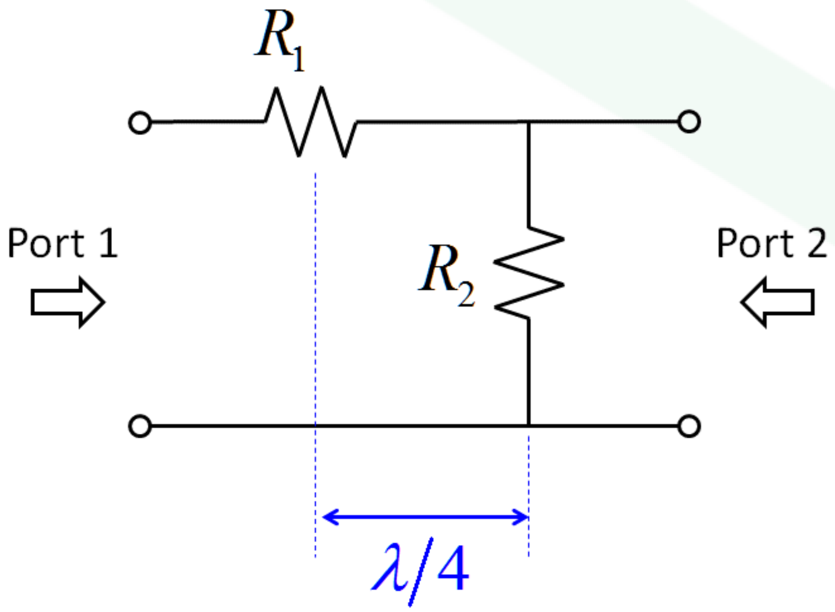
Example: Analysis of a Large Circuit

Example: Analysis of a Large Circuit

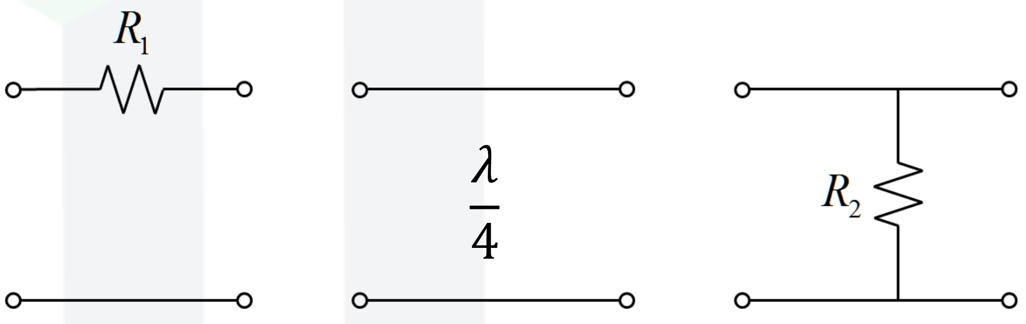
Suppose we have a voltage divider, but it is large relative to the wavelength. Assuming a $50\ \Omega$ transmission line impedance, what is the ABCD matrix?



Example: Analysis of a Large Circuit

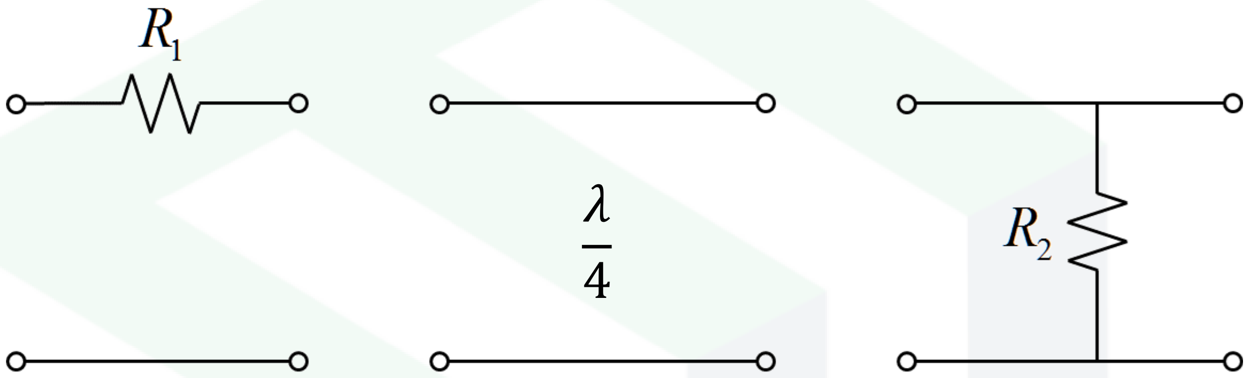


To obtain the parameters, we can think of this circuit as the combination of three series circuits.



$$\begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \quad \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} \quad \begin{bmatrix} A_3 & B_3 \\ C_3 & D_3 \end{bmatrix}$$

Example: Analysis of a Large Circuit



$$\begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \quad \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} \quad \begin{bmatrix} A_3 & B_3 \\ C_3 & D_3 \end{bmatrix}$$

$A = 1$	$A = 0$	$A = 1$
$B = R_1$	$B = j50$	$B = 0$
$C = 0$	$C = j0.02$	$C = 1/R_2$
$D = 1$	$D = 0$	$D = 1$

Example: Analysis of a Large Circuit

Now we combine the three matrices

$$\begin{bmatrix} 1 & R_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & j50 \\ j0.02 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1/R_2 & 1 \end{bmatrix} = \begin{bmatrix} j\left(\frac{R_1}{50} + \frac{50}{R_2}\right) & j50 \\ j0.02 & 0 \end{bmatrix}$$

Compare to an ideal voltage divider

$$\begin{bmatrix} 1 + R_1/R_2 & R_1 \\ 1/R_2 & 1 \end{bmatrix}$$

The size of the circuit makes a difference!

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Example: Analysis of a Large Circuit

Let's say we have 10 V amplitude input signal, $R_1 = R_2 = 1\text{k}\Omega$. What is the output voltage if the source, load, and transmission lines are all $50\ \Omega$

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} j20.05 & j50 \\ j0.02 & 0 \end{bmatrix}$$

Let's convert this to a scattering matrix

$$S_{11} = \frac{AZ_L + B - CZ_s^* Z_L - DZ_S^*}{AZ_L + B + CZ_s Z_L + DZ_s}$$

$$S_{21} = \frac{2\sqrt{\text{Re}[Z_s]\text{Re}\,[Z_L]}}{AZ_L + B + CZ_s Z_L + DZ_s}$$

$$S_{12} = \frac{2(AD - BC)\sqrt{\text{Re}[Z_s]\text{Re}\,[Z_L]}}{AZ_L + B + CZ_s Z_L + DZ_s}$$

$$S_{22} = \frac{-AZ_L^* + B - CZ_s Z_L^* - DZ_s}{AZ_L + B + CZ_s Z_L + DZ_s}$$

Example: Analysis of a Large Circuit

$$[S] = \begin{bmatrix} 0.9093 & -j0.0907 \\ j0.0907 & -0.9093 \end{bmatrix}$$

$$|S_{21}|^2 = |-j0.0907|^2 = 0.0082$$

$$V_2 = S_{21}V_1 = (-j0.0907)V_1 = \left(0.0907\angle-90^\circ\right)V_1$$

This should be 0.5 for an ordinary voltage divider!