



Electromagnetics:  
Microwave Engineering

# Transmission Matrices and Cascading Multiple Networks



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## Lecture Outline

- The Transmission (ABCD) Matrix
- Cascading Multiple Networks
- Example: Analysis of a Large Circuit



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# The Transmission (ABCD) Matrix



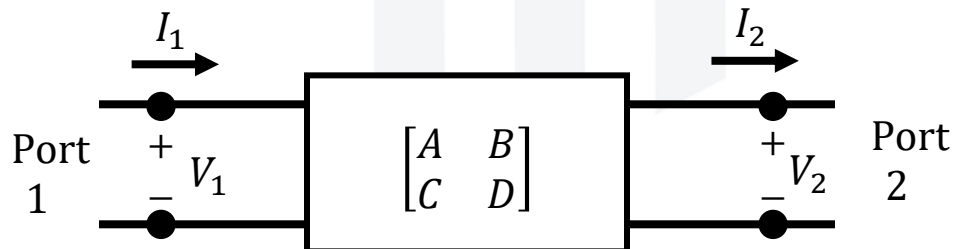
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## The ABCD Matrix

Relates the total voltage and current at the first port to the total voltage and current at the second port. This is purely for 2-port networks and makes cascading matrices very easy.

$$\begin{aligned} V_1 &= AV_2 + BI_2 \\ I_1 &= CV_2 + DI_2 \end{aligned} \quad \rightarrow \quad \begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix}$$



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## The ABCD Parameters

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

$A = \left. \frac{V_1}{V_2} \right|_{I_2=0}$     open-circuit voltage ratio   
  $C = \left. \frac{I_1}{V_2} \right|_{I_2=0}$     open-circuit transfer impedance

$B = \left. \frac{V_1}{I_2} \right|_{V_2=0}$     short-circuit transfer impedance   
 $D = \left. \frac{I_1}{I_2} \right|_{V_2=0}$     short-circuit current ratio

EMPossible

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## Properties of ABCD Matrices

Reciprocal Networks

$$\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = 1 \quad \rightarrow \quad AD - BC = 1$$

Lossless Networks

The network is lossless if (1) diagonal elements are purely real, and (2) off diagonal elements are purely imaginary

EMPossible

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# Cascading Multiple Networks



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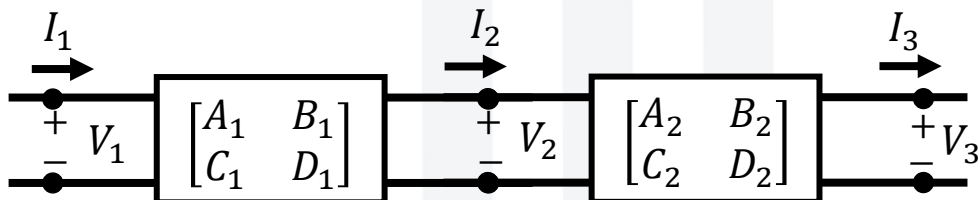
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## Combining Two ABCD Matrices

Given the ABCD matrices of two networks,

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix}$$

$$\begin{bmatrix} V_2 \\ I_2 \end{bmatrix} = \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} \begin{bmatrix} V_3 \\ I_3 \end{bmatrix}$$



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## Combining Two ABCD Matrices

We can combine these by substituting and doing simple matrix multiplication.

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} \begin{bmatrix} V_3 \\ I_3 \end{bmatrix}$$

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A_{12} & B_{12} \\ C_{12} & D_{12} \end{bmatrix} \begin{bmatrix} V_3 \\ I_3 \end{bmatrix}$$

$$A_{12} = A_1 A_2 + B_1 C_2$$

$$B_{12} = A_1 B_2 + B_1 D_2$$

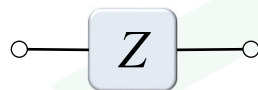
$$C_{12} = C_1 A_2 + D_1 C_2$$

$$D_{12} = C_1 B_2 + D_1 D_2$$

Note:  $[A][B] \neq [B][A]$



## ABCD Parameters of Common Networks

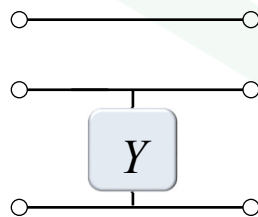


$$A = 1$$

$$C = 0$$

$$B = Z$$

$$D = 1$$

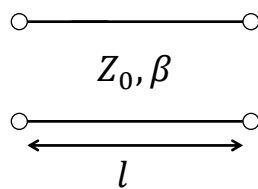


$$A = 1$$

$$C = Y$$

$$B = 0$$

$$D = 1$$



$$A = \cos \beta l$$

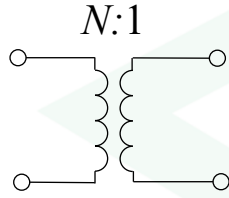
$$C = jZ_0^{-1} \sin \beta l$$

$$B = jZ_0 \sin \beta l$$

$$D = \cos \beta l$$



## ABCD Parameters of Common Networks

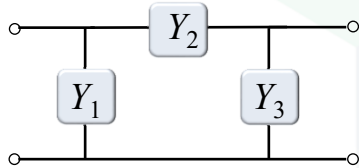


$$A = N$$

$$B = 0$$

$$C = 0$$

$$D = \frac{1}{N}$$

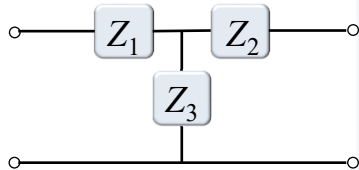


$$A = 1 + \frac{Y_2}{Y_3}$$

$$B = \frac{1}{Y_3}$$

$$C = Y_1 + Y_2 + \frac{Y_1 Y_2}{Y_3}$$

$$D = 1 + \frac{Y_1}{Y_3}$$



$$A = 1 + \frac{Z_1}{Z_3}$$

$$B = Z_1 + Z_2 + \frac{Z_1 Z_2}{Z_3}$$

$$C = \frac{1}{Z_3}$$

$$D = 1 + \frac{Z_2}{Z_3}$$

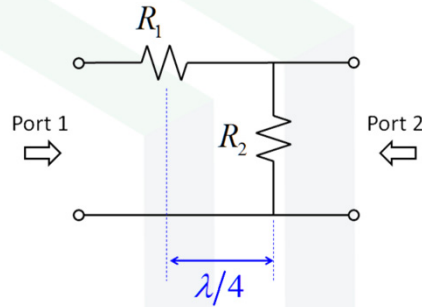


## Example: Analysis of a Large Circuit



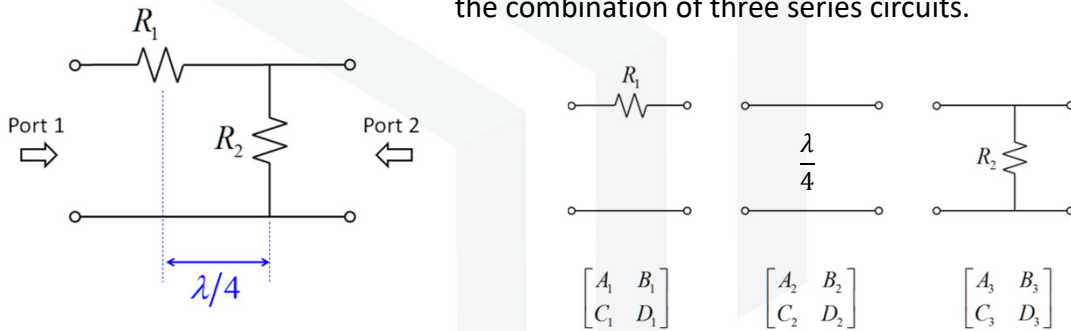
## Example: Analysis of a Large Circuit

Suppose we have a voltage divider, but it is large relative to the wavelength. Assuming a  $50\ \Omega$  transmission line impedance, what is the ABCD matrix?

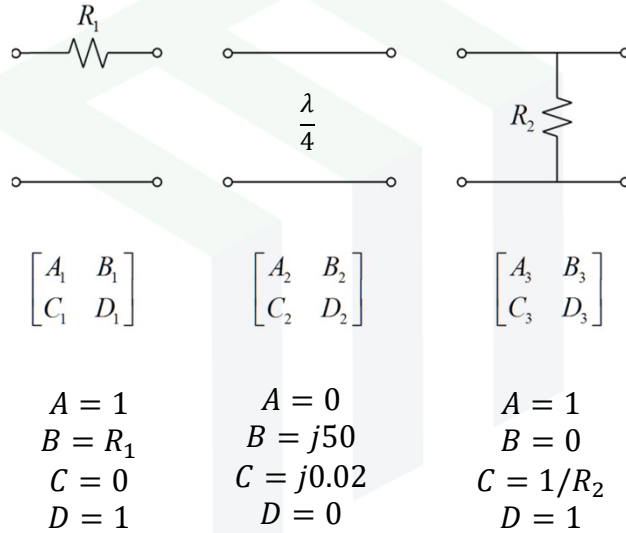


## Example: Analysis of a Large Circuit

To obtain the parameters, we can think of this circuit as the combination of three series circuits.



## Example: Analysis of a Large Circuit



## Example: Analysis of a Large Circuit

Now we combine the three matrices

$$\begin{bmatrix} 1 & R_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & j50 \\ j0.02 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1/R_2 & 1 \end{bmatrix} = \begin{bmatrix} j\left(\frac{R_1}{50} + \frac{50}{R_2}\right) & j50 \\ j0.02 & 0 \end{bmatrix}$$

Compare to an ideal voltage divider

$$\begin{bmatrix} 1 + R_1/R_2 & R_1 \\ 1/R_2 & 1 \end{bmatrix}$$

The size of the circuit makes a difference!



## Example: Analysis of a Large Circuit

Let's say we have 10 V amplitude input signal,  $R_1 = R_2 = 1\text{k}\Omega$ . What is the output voltage if the source, load, and transmission lines are all  $50\ \Omega$

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} j20.05 & j50 \\ j0.02 & 0 \end{bmatrix}$$

Let's convert this to a scattering matrix

$$S_{11} = \frac{AZ_L + B - CZ_s^*Z_L - DZ_s^*}{AZ_L + B + CZ_sZ_L + DZ_s}$$

$$S_{21} = \frac{2\sqrt{\text{Re}[Z_s]\text{Re}[Z_L]}}{AZ_L + B + CZ_sZ_L + DZ_s}$$

$$S_{12} = \frac{2(AD - BC)\sqrt{\text{Re}[Z_s]\text{Re}[Z_L]}}{AZ_L + B + CZ_sZ_L + DZ_s}$$

$$S_{22} = \frac{-AZ_L^* + B - CZ_sZ_L^* - DZ_s}{AZ_L + B + CZ_sZ_L + DZ_s}$$



## Example: Analysis of a Large Circuit

$$[S] = \begin{bmatrix} 0.9093 & -j0.0907 \\ j0.0907 & -0.9093 \end{bmatrix}$$

$$|S_{21}|^2 = |-j0.0907|^2 = 0.0082$$

$$V_2 = S_{21}V_1 = (-j0.0907)V_1 = (0.0907 \angle -90^\circ)V_1$$

This should be 0.5 for an ordinary voltage divider!

