Computational Science:
Introduction to Finite-Difference Time-Domain

Convolutional Perfectly Matched Layer (CPML)

Lecture Outline

- The PML Parameters
- Maxwell's Equations with Simplified CFS-PML
- Discrete Convolution
- 3D Update Equations for B_x and D_x with a CPML
- 3D Update Equations for E_x and H_x with a CPML
- 2D Update Equations for B_x and D_x with a CPML
- 2D Update Equations for E_x and H_x with a CPML

Normalized Maxwell's Equations

$$\vec{D} = \varepsilon_0 \tilde{\vec{D}}$$

$$\vec{B} = \frac{1}{c_0} \tilde{\vec{B}}$$

$$\vec{H} = \eta_0^{-1} \tilde{\vec{H}}$$

$$\nabla \times \vec{E} = -\frac{1}{c_0} \frac{\partial \tilde{\vec{B}}}{\partial t}$$

$$\nabla \times \tilde{\vec{H}} = \frac{1}{c_0} \frac{\partial \tilde{\vec{D}}}{\partial t}$$

$$\nabla \bullet \tilde{\vec{D}} = 0$$

$$\nabla \bullet \tilde{\vec{B}} = 0$$

$$\tilde{\vec{D}} = \varepsilon_\mathrm{r} \vec{E}$$

$$\tilde{\vec{B}} = \mu_\mathrm{r} \tilde{\vec{H}}$$

EMPossible

Slide 3

The PML Parameters

Slide 4

Maxwell's Equations with a Stretched-Coordinate Perfectly Matched Layer (SC-PML)

The stretched-coordinate perfectly matched layer (SC-PML) is incorporated into Maxwell's curl equations as follows:

$$[S]^{-1}\nabla\times\vec{E}(\omega)=-\frac{j\omega}{c_0}\vec{\tilde{B}}(\omega) \quad\Rightarrow\quad \begin{aligned} \frac{1}{s_y}\frac{\partial E_z(\omega)}{\partial y}-\frac{1}{s_z}\frac{\partial E_y(\omega)}{\partial z}&=-\frac{j\omega}{c_0}\tilde{B}_x(\omega)\\ \frac{1}{s_z}\frac{\partial E_x(\omega)}{\partial z}-\frac{1}{s_x}\frac{\partial E_z(\omega)}{\partial x}&=-\frac{j\omega}{c_0}\tilde{B}_y(\omega)\\ \frac{1}{s_x}\frac{\partial E_y(\omega)}{\partial x}-\frac{1}{s_y}\frac{\partial E_x(\omega)}{\partial y}&=-\frac{j\omega}{c_0}\tilde{B}_z(\omega) \end{aligned}$$

$$[S]^{-1}\nabla\times\vec{\tilde{H}}(\omega)=\frac{j\omega}{c_0}\vec{\tilde{D}}(\omega) \quad\Rightarrow\quad \begin{aligned} \frac{1}{s_y}\frac{\partial \tilde{H}_z(\omega)}{\partial y}-\frac{1}{s_z}\frac{\partial \tilde{H}_y(\omega)}{\partial z}&=\frac{j\omega}{c_0}\tilde{D}_x(\omega)\\ \frac{1}{s_z}\frac{\partial \tilde{H}_x(\omega)}{\partial z}-\frac{1}{s_x}\frac{\partial \tilde{H}_z(\omega)}{\partial x}&=\frac{j\omega}{c_0}\tilde{D}_y(\omega)\\ \frac{1}{s_x}\frac{\partial \tilde{H}_y(\omega)}{\partial x}-\frac{1}{s_y}\frac{\partial \tilde{H}_x(\omega)}{\partial y}&=\frac{j\omega}{c_0}\tilde{D}_z(\omega) \end{aligned}$$

EMPossible

Slide 5

The SC-PML Parameters

The SC-PML is represented as a tensor quantity $[S]$.

$$[S]=\begin{bmatrix} s_x & 0 & 0\\ 0 & s_y & 0\\ 0 & 0 & s_z \end{bmatrix}$$

The parameters s_x, s_y and s_z are complex stretching parameters that introduce loss in the PML regions to absorb outgoing waves while keeping the impedance matched to prevent reflection from the PML.

EMPossible

Slide 6

Berenger's PML Parameters

Berenger's PML is one of the simplest expressions for the PML parameters s_x, s_y and s_z.

PML Parameters

$$s_x(x) = 1 + \frac{\sigma_x(x)}{j\omega\varepsilon_0}$$

$$s_y(y) = 1 + \frac{\sigma_y(y)}{j\omega\varepsilon_0}$$

$$s_z(z) = 1 + \frac{\sigma_z(z)}{j\omega\varepsilon_0}$$

PML Profile

$$\sigma_x(x) = \frac{\varepsilon_0}{2\Delta t}\left(\frac{x}{L_x}\right)^3$$

$$\sigma_y(y) = \frac{\varepsilon_0}{2\Delta t}\left(\frac{y}{L_y}\right)^3$$

$$\sigma_z(z) = \frac{\varepsilon_0}{2\Delta t}\left(\frac{z}{L_z}\right)^3$$

EMPossible

Slide 7

Complex Frequency-Shifted PML Parameters (1 of 2)

The complex frequency-shifted PML (CFS-PML) defines unique PML parameters to the electric and magnetic field quantities.

Magnetic PML Parameters

$$s_{mx}(x) = \kappa_{mx} + \frac{\sigma_{mx}(x)}{a_{mx} + j\omega\mu_0}$$

$$s_{my}(y) = \kappa_{my} + \frac{\sigma_{my}(y)}{a_{my} + j\omega\mu_0}$$

$$s_{mz}(z) = \kappa_{mz} + \frac{\sigma_{mz}(z)}{a_{mz} + j\omega\mu_0}$$

Electric PML Parameters

$$s_{ex}(x) = \kappa_{ex} + \frac{\sigma_{ex}(x)}{a_{ex} + j\omega\varepsilon_0}$$

$$s_{ey}(y) = \kappa_{ey} + \frac{\sigma_{ey}(y)}{a_{ey} + j\omega\varepsilon_0}$$

$$s_{ez}(z) = \kappa_{ez} + \frac{\sigma_{ez}(z)}{a_{ez} + j\omega\varepsilon_0}$$

PML Profile

$$\sigma_{fx}(x) = \sigma_{f,\max}\left(\frac{x}{L_x}\right)^{n_{\text{PML}}}$$

$$\sigma_{fy}(y) = \sigma_{f,\max}\left(\frac{x}{L_x}\right)^{n_{\text{PML}}}$$

$$\sigma_{fz}(z) = \sigma_{f,\max}\left(\frac{x}{L_x}\right)^{n_{\text{PML}}}$$

$$f = \text{m or e}$$

EMPossible

Slide 8

Complex Frequency-Shifted PML Parameters (2 of 2)

For zero reflections at the PML-to-PML interface, it is necessary that

$$s_{\mathrm{mx}}(x) = s_{\mathrm{ex}}(x) \qquad s_{\mathrm{my}}(y) = s_{\mathrm{ey}}(y) \qquad s_{\mathrm{mz}}(z) = s_{\mathrm{ez}}(z)$$

This leads to the following conditions.

$$\kappa_{\mathrm{mx}} = \kappa_{\mathrm{ex}} \qquad \kappa_{\mathrm{my}} = \kappa_{\mathrm{ey}} \qquad \kappa_{\mathrm{mz}} = \kappa_{\mathrm{ez}}$$

$$\frac{\sigma_{\mathrm{mx}}(x)}{a_{\mathrm{mx}} + j\omega\varepsilon_0} = \frac{\sigma_{\mathrm{ex}}(x)}{a_{\mathrm{ex}} + j\omega\varepsilon_0} \qquad \frac{\sigma_{\mathrm{my}}(y)}{a_{\mathrm{my}} + j\omega\varepsilon_0} = \frac{\sigma_{\mathrm{ey}}(y)}{a_{\mathrm{ey}} + j\omega\varepsilon_0} \qquad \frac{\sigma_{\mathrm{mz}}(z)}{a_{\mathrm{mz}} + j\omega\varepsilon_0} = \frac{\sigma_{\mathrm{ez}}(z)}{a_{\mathrm{ez}} + j\omega\varepsilon_0}$$

Simplified CFS-PML Parameters

An easy way to satisfy the conditions for zero reflection at the PML-to-PML interface and to simplify the number of terms that have to be correctly determined is to use the same set of parameters for both magnetic and electric fields.

$$s_{\mathrm{mx}}(x) = s_{\mathrm{ex}}(x) = s_x(x) = \kappa_x + \frac{\sigma_x(x)}{a_x + j\omega\varepsilon_0}$$

$$s_{\mathrm{my}}(x) = s_{\mathrm{ey}}(x) = s_y(y) = \kappa_y + \frac{\sigma_y(y)}{a_y + j\omega\varepsilon_0}$$

$$s_{\mathrm{my}}(x) = s_{\mathrm{ey}}(x) = s_z(z) = \kappa_z + \frac{\sigma_z(z)}{a_z + j\omega\varepsilon_0}$$

$$\sigma_x(x) = \sigma_{\max}\left(\frac{x}{L_x}\right)^{n_{\mathrm{PML}}}$$

$$\sigma_y(y) = \sigma_{\max}\left(\frac{x}{L_x}\right)^{n_{\mathrm{PML}}}$$

$$\sigma_z(z) = \sigma_{\max}\left(\frac{x}{L_x}\right)^{n_{\mathrm{PML}}}$$

$$\kappa_x \geq 1 \qquad \kappa_y \geq 1 \qquad \kappa_z \geq 1$$

$$a_x > 0 \qquad a_y > 0 \qquad a_z > 0$$

Some Rough Numbers

The follow are some rough numbers for the PML parameters to get started with.

$$\kappa_x = \kappa_y = \kappa_z = 1 \qquad a_x = a_y = a_z = 10^{-6}$$

The parameter $\sigma_{\max}$ can be calculated using the following equation.

$$\sigma_{\max} = -\frac{(n_{\text{PML}} + 1)\log_{10}(R_0)}{2\eta_0 L}$$

$R_0 \equiv$ theorectical eflection at normal incidence
$L \equiv$ physical size of PML
$n_{\text{PML}} \equiv$ PML profile parameter
$\eta_0 \equiv$ free space impedance

$$R_0 \approx 10^{-8}$$

EMPossible

Slide 11

Maxwell's Equations with Simplified CFS-PML

Slide 12

Frequency-Domain Curl Equations

$$\frac{1}{s_y}\frac{\partial E_z(\omega)}{\partial y}-\frac{1}{s_z}\frac{\partial E_y(\omega)}{\partial z}=-\frac{j\omega}{c_0}\tilde{B}_x(\omega)$$
$$\frac{1}{s_z}\frac{\partial E_x(\omega)}{\partial z}-\frac{1}{s_x}\frac{\partial E_z(\omega)}{\partial x}=-\frac{j\omega}{c_0}\tilde{B}_y(\omega)$$
$$\frac{1}{s_x}\frac{\partial E_y(\omega)}{\partial x}-\frac{1}{s_y}\frac{\partial E_x(\omega)}{\partial y}=-\frac{j\omega}{c_0}\tilde{B}_z(\omega)$$

$\Rightarrow$

$$\left(\kappa_y+\frac{\sigma_y(y)}{a_y+j\omega\varepsilon_0}\right)^{-1}\frac{\partial E_z(\omega)}{\partial y}-\left(\kappa_z+\frac{\sigma_z(z)}{a_z+j\omega\varepsilon_0}\right)^{-1}\frac{\partial E_y(\omega)}{\partial z}=-\frac{j\omega}{c_0}\tilde{B}_x(\omega)$$
$$\left(\kappa_z+\frac{\sigma_z(z)}{a_z+j\omega\varepsilon_0}\right)^{-1}\frac{\partial E_x(\omega)}{\partial z}-\left(\kappa_x+\frac{\sigma_x(x)}{a_x+j\omega\varepsilon_0}\right)^{-1}\frac{\partial E_z(\omega)}{\partial x}=-\frac{j\omega}{c_0}\tilde{B}_y(\omega)$$
$$\left(\kappa_x+\frac{\sigma_x(x)}{a_x+j\omega\varepsilon_0}\right)^{-1}\frac{\partial E_y(\omega)}{\partial x}-\left(\kappa_y+\frac{\sigma_y(y)}{a_y+j\omega\varepsilon_0}\right)^{-1}\frac{\partial E_x(\omega)}{\partial y}=-\frac{j\omega}{c_0}\tilde{B}_z(\omega)$$

$$\frac{1}{s_y}\frac{\partial \tilde{H}_z(\omega)}{\partial y}-\frac{1}{s_z}\frac{\partial \tilde{H}_y(\omega)}{\partial z}=\frac{j\omega}{c_0}\tilde{D}_x(\omega)$$
$$\frac{1}{s_z}\frac{\partial \tilde{H}_x(\omega)}{\partial z}-\frac{1}{s_x}\frac{\partial \tilde{H}_z(\omega)}{\partial x}=\frac{j\omega}{c_0}\tilde{D}_y(\omega)$$
$$\frac{1}{s_x}\frac{\partial \tilde{H}_y(\omega)}{\partial x}-\frac{1}{s_y}\frac{\partial \tilde{H}_x(\omega)}{\partial y}=\frac{j\omega}{c_0}\tilde{D}_z(\omega)$$

$\Rightarrow$

$$\left(\kappa_y+\frac{\sigma_y(y)}{a_y+j\omega\varepsilon_0}\right)^{-1}\frac{\partial \tilde{H}_z(\omega)}{\partial y}-\left(\kappa_z+\frac{\sigma_z(z)}{a_z+j\omega\varepsilon_0}\right)^{-1}\frac{\partial \tilde{H}_y(\omega)}{\partial z}=\frac{j\omega}{c_0}\tilde{D}_x(\omega)$$
$$\left(\kappa_z+\frac{\sigma_z(z)}{a_z+j\omega\varepsilon_0}\right)^{-1}\frac{\partial \tilde{H}_x(\omega)}{\partial z}-\left(\kappa_x+\frac{\sigma_x(x)}{a_x+j\omega\varepsilon_0}\right)^{-1}\frac{\partial \tilde{H}_z(\omega)}{\partial x}=\frac{j\omega}{c_0}\tilde{D}_y(\omega)$$
$$\left(\kappa_x+\frac{\sigma_x(x)}{a_x+j\omega\varepsilon_0}\right)^{-1}\frac{\partial \tilde{H}_y(\omega)}{\partial x}-\left(\kappa_y+\frac{\sigma_y(y)}{a_y+j\omega\varepsilon_0}\right)^{-1}\frac{\partial \tilde{H}_x(\omega)}{\partial y}=\frac{j\omega}{c_0}\tilde{D}_z(\omega)$$

EMPossible

Slide 13

Conversion to Time-Domain

Start with the first equation and inverse Fourier transform each term.

$$\left(\kappa_y+\frac{\sigma_y(y)}{a_y+j\omega\varepsilon_0}\right)^{-1}\frac{\partial E_z(\omega)}{\partial y}-\left(\kappa_z+\frac{\sigma_z(z)}{a_z+j\omega\varepsilon_0}\right)^{-1}\frac{\partial E_y(\omega)}{\partial z}=-\frac{j\omega}{c_0}\tilde{B}_x(\omega)$$

?

$$\frac{\partial E_z(t)}{\partial y}$$

?

$$\frac{\partial E_y(t)}{\partial z}$$

$$-\frac{1}{c_0}\frac{\partial \tilde{B}_x(t)}{\partial z}$$

Altogether this is

$$(?)*\frac{\partial E_z(t)}{\partial y}-(?)*\frac{\partial E_y(t)}{\partial z}=-\frac{j\omega}{c_0}\frac{\partial \tilde{B}_x(t)}{\partial t}$$

EMPossible

Slide 14

Inverse Fourier Transform of $\left(\kappa + \frac{\sigma}{a + j\omega\varepsilon_0}\right)^{-1}$

After a lot of work, the inverse Fourier transform is

$$\Im^{-1}\left\{\left(\kappa + \frac{\sigma}{a + j\omega\varepsilon_0}\right)^{-1}\right\} = \frac{\delta(t)}{\kappa} - \frac{\sigma}{\varepsilon_0\kappa^2} e^{-\left(\frac{\sigma}{\varepsilon_0\kappa} + \frac{a}{\varepsilon_0}\right)t} u(t)$$

$\delta(t)$ is an impulse function.

$$\delta(t) * f(t) = f(t)$$

$u(t)$ is a step function

$$\int_{\tau=-\infty}^{\infty} u(\tau) f(\tau) d\tau = \int_{\tau=0}^{\infty} f(\tau) d\tau$$

EMPossible

Slide 15

Inverse Fourier Transform of $\left(\kappa + \frac{\sigma}{a + j\omega\varepsilon_0}\right)^{-1}$

After a lot of work, the inverse Fourier transform is

$$\Im^{-1}\left\{\left(\kappa + \frac{\sigma}{a + j\omega\varepsilon_0}\right)^{-1}\right\} = \frac{\delta(t)}{\kappa} + \xi(t)$$

$$\xi(t) = -\frac{\sigma}{\varepsilon_0\kappa^2} e^{-\left(\frac{\sigma}{\varepsilon_0\kappa} + \frac{a}{\varepsilon_0}\right)t} u(t)$$

EMPossible

Slide 16

Back to the Time-Domain Equation

Recall the time-domain equation was

$$(?)*\frac{\partial E_z(t)}{\partial y}-(?)*\frac{\partial E_y(t)}{\partial z}=-\frac{j\omega}{c_0}\frac{\partial \tilde{B}_x(t)}{\partial t}$$

Substituting in the expression for $(?)$ from the previous slide gives

$$\left[\frac{\delta(t)}{\kappa_y}+\xi_y(t)\right]*\frac{\partial E_z(t)}{\partial y}-\left[\frac{\delta(t)}{\kappa_z}+\xi_z(t)\right]*\frac{\partial E_y(t)}{\partial z}=-\frac{j\omega}{c_0}\frac{\partial \tilde{B}_x(t)}{\partial t}$$

Substitute in expression

$$\frac{\delta(t)}{\kappa_y}*\frac{\partial E_z(t)}{\partial y}+\xi_y(t)*\frac{\partial E_z(t)}{\partial y}-\frac{\delta(t)}{\kappa_z}*\frac{\partial E_y(t)}{\partial z}-\xi_z(t)*\frac{\partial E_y(t)}{\partial z}=-\frac{j\omega}{c_0}\frac{\partial \tilde{B}_x(t)}{\partial t}$$

Expand equation

$$\frac{1}{\kappa_y}\frac{\partial E_z(t)}{\partial y}+\xi_y(t)*\frac{\partial E_z(t)}{\partial y}-\frac{1}{\kappa_z}\frac{\partial E_y(t)}{\partial z}-\xi_z(t)*\frac{\partial E_y(t)}{\partial z}=-\frac{j\omega}{c_0}\frac{\partial \tilde{B}_x(t)}{\partial t}$$

$\delta(t)*f(t)=f(t)$

$$\frac{1}{\kappa_y}\frac{\partial E_z(t)}{\partial y}-\frac{1}{\kappa_z}\frac{\partial E_y(t)}{\partial z}+\xi_y(t)*\frac{\partial E_z(t)}{\partial y}-\xi_z(t)*\frac{\partial E_y(t)}{\partial z}=-\frac{j\omega}{c_0}\frac{\partial \tilde{B}_x(t)}{\partial t}$$

Rearrange equation

EMPossible

Time-Domain Maxwell's Equations with CFS-PML

$$\left(\kappa_y+\frac{\sigma_y(y)}{a_y+j\omega\varepsilon_0}\right)^{-1}\frac{\partial E_z(\omega)}{\partial y}-\left(\kappa_z+\frac{\sigma_z(z)}{a_z+j\omega\varepsilon_0}\right)^{-1}\frac{\partial E_y(\omega)}{\partial z}=-\frac{j\omega}{c_0}\tilde{B}_x(\omega)$$
$$\left(\kappa_z+\frac{\sigma_z(z)}{a_z+j\omega\varepsilon_0}\right)^{-1}\frac{\partial E_x(\omega)}{\partial z}-\left(\kappa_x+\frac{\sigma_x(x)}{a_x+j\omega\varepsilon_0}\right)^{-1}\frac{\partial E_z(\omega)}{\partial x}=-\frac{j\omega}{c_0}\tilde{B}_y(\omega)$$
$$\left(\kappa_x+\frac{\sigma_x(x)}{a_x+j\omega\varepsilon_0}\right)^{-1}\frac{\partial E_y(\omega)}{\partial x}-\left(\kappa_y+\frac{\sigma_y(y)}{a_y+j\omega\varepsilon_0}\right)^{-1}\frac{\partial E_x(\omega)}{\partial y}=-\frac{j\omega}{c_0}\tilde{B}_z(\omega)$$

→

$$\frac{1}{\kappa_y}\frac{\partial E_z(t)}{\partial y}-\frac{1}{\kappa_z}\frac{\partial E_y(t)}{\partial z}+\xi_y(t)*\frac{\partial E_z(t)}{\partial y}-\xi_z(t)*\frac{\partial E_y(t)}{\partial z}=-\frac{1}{c_0}\frac{\partial \tilde{B}_x(t)}{\partial t}$$
$$\frac{1}{\kappa_z}\frac{\partial E_x(t)}{\partial z}-\frac{1}{\kappa_x}\frac{\partial E_z(t)}{\partial x}+\xi_z(t)*\frac{\partial E_x(t)}{\partial z}-\xi_x(t)*\frac{\partial E_z(t)}{\partial x}=-\frac{1}{c_0}\frac{\partial \tilde{B}_y(t)}{\partial t}$$
$$\frac{1}{\kappa_x}\frac{\partial E_y(t)}{\partial x}-\frac{1}{\kappa_y}\frac{\partial E_x(t)}{\partial y}+\xi_x(t)*\frac{\partial E_y(t)}{\partial x}-\xi_y(t)*\frac{\partial E_x(t)}{\partial y}=-\frac{1}{c_0}\frac{\partial \tilde{B}_z(t)}{\partial t}$$

$$\left(\kappa_y+\frac{\sigma_y(y)}{a_y+j\omega\varepsilon_0}\right)^{-1}\frac{\partial \tilde{H}_z(\omega)}{\partial y}-\left(\kappa_z+\frac{\sigma_z(z)}{a_z+j\omega\varepsilon_0}\right)^{-1}\frac{\partial \tilde{H}_y(\omega)}{\partial z}=\frac{j\omega}{c_0}\tilde{D}_x(\omega)$$
$$\left(\kappa_z+\frac{\sigma_z(z)}{a_z+j\omega\varepsilon_0}\right)^{-1}\frac{\partial \tilde{H}_x(\omega)}{\partial z}-\left(\kappa_x+\frac{\sigma_x(x)}{a_x+j\omega\varepsilon_0}\right)^{-1}\frac{\partial \tilde{H}_z(\omega)}{\partial x}=\frac{j\omega}{c_0}\tilde{D}_y(\omega)$$
$$\left(\kappa_x+\frac{\sigma_x(x)}{a_x+j\omega\varepsilon_0}\right)^{-1}\frac{\partial \tilde{H}_y(\omega)}{\partial x}-\left(\kappa_y+\frac{\sigma_y(y)}{a_y+j\omega\varepsilon_0}\right)^{-1}\frac{\partial \tilde{H}_x(\omega)}{\partial y}=\frac{j\omega}{c_0}\tilde{D}_z(\omega)$$

→

$$\frac{1}{\kappa_y}\frac{\partial \tilde{H}_z(t)}{\partial y}-\frac{1}{\kappa_z}\frac{\partial \tilde{H}_y(t)}{\partial z}+\xi_y(t)*\frac{\partial \tilde{H}_z(t)}{\partial y}-\xi_z(t)*\frac{\partial \tilde{H}_y(t)}{\partial z}=\frac{1}{c_0}\frac{\partial \tilde{D}_x(t)}{\partial t}$$
$$\frac{1}{\kappa_z}\frac{\partial \tilde{H}_x(t)}{\partial z}-\frac{1}{\kappa_x}\frac{\partial \tilde{H}_z(t)}{\partial x}+\xi_z(t)*\frac{\partial \tilde{H}_x(t)}{\partial z}-\xi_x(t)*\frac{\partial \tilde{H}_z(t)}{\partial x}=\frac{1}{c_0}\frac{\partial \tilde{D}_y(t)}{\partial t}$$
$$\frac{1}{\kappa_x}\frac{\partial \tilde{H}_y(t)}{\partial x}-\frac{1}{\kappa_y}\frac{\partial \tilde{H}_x(t)}{\partial y}+\xi_x(t)*\frac{\partial \tilde{H}_y(t)}{\partial x}-\xi_y(t)*\frac{\partial \tilde{H}_x(t)}{\partial y}=\frac{1}{c_0}\frac{\partial \tilde{D}_z(t)}{\partial t}$$

EMPossible

Discrete Convolution

Closer Look at the Convolution

Focus on the first convolution in the first time-domain equation.

$$\frac{1}{\kappa_y}\frac{\partial E_z(t)}{\partial y} - \frac{1}{\kappa_z}\frac{\partial E_y(t)}{\partial z} + \xi_y(t) * \frac{\partial E_z(t)}{\partial y} - \xi_z(t) * \frac{\partial E_y(t)}{\partial z} = -\frac{1}{c_0}\frac{\partial \tilde{B}_x(t)}{\partial t}$$

$$\xi_y(t) * \frac{\partial E_z(t)}{\partial y} = \int_{\tau=-\infty}^{\infty} \xi_y(\tau)\frac{\partial E_z(t-\tau)}{\partial y}d\tau$$

Apply the definition of a convolution.

$$= \int_{\tau=0}^{\infty} \xi_y(\tau)\frac{\partial E_z(t-\tau)}{\partial y}d\tau$$

Step function $u(t)$ allows integral to start at $\tau = 0$.

Numerical Calculation of the Convolution

The convolution is written in discrete form as

$$\xi_y(t) * \frac{\partial E_z(t)}{\partial y} = \int_{\tau=0}^{\infty} \xi_y(\tau) \frac{\partial E_z(t-\tau)}{\partial y} d\tau \approx \sum_{m=0}^{n-1} Z_{B_x,y}(m) \left(\frac{E_z\big|_{n-m}^{i,j+1,k} - E_z\big|_{n-m}^{i,j,k}}{\Delta y} \right)$$

$n \equiv$ FDTD time step 0,1,2,3...
$m \equiv$ intermediate time step for convolution

The $Z_{B_x,y}(m)$ term is calculated as

$$Z_{B_x,y}(m) = \int_{\tau=m\Delta t}^{(m+1)\Delta t} \xi_y(\tau) d\tau = c_{B_x,y} b_{B_x,y}^m$$

$$c_{B_x,y} = \frac{\sigma_y}{\sigma_y \kappa_y + a_y \kappa_y^2} \left(b_{B_x,y} - 1 \right)$$

$$b_{B_x,y} = e^{-\left(\frac{\sigma_y}{\varepsilon_0 \kappa_y} + \frac{a_y}{\varepsilon_0} \right) \Delta t}$$

WARNING: If $a_y = 0$ then $c_{B_x,y}$ will be undefined in the problem space where $\sigma_y = a_y = 0$. Either keep $a_y > 0$ or manually set $c_{B_x,y} = 0$ in the problem space when $a_y = 0$.

EMPossible

Recursive Convolution Technique (1 of 3)

The convolution is temporarily written in the following more compact form.

$$\psi(t) = \xi_y(t) * \frac{\partial E_z(t)}{\partial y} \quad \Leftrightarrow \quad \psi(n) = \sum_{m=0}^{n-1} A e^{mT} B(n-m)$$

$$A = c_{B_x,y} = \frac{\sigma_y}{\sigma_y \kappa_y + a_y \kappa_y^2} \left(b_{B_x,y} - 1 \right) \qquad B = \frac{E_z\big|_{n-m}^{i,j+1,k} - E_z\big|_{n-m}^{i,j,k}}{\Delta y} \qquad b_{B_x,y} = e^{-\left(\frac{\sigma_y}{\varepsilon_0 \kappa_y} + \frac{a_y}{\varepsilon_0} \right) \Delta t} \qquad T = -\left(\frac{\sigma_y}{\varepsilon_0 \kappa_y} + \frac{a_y}{\varepsilon_0} \right) \Delta t$$

Expand the summation for $\psi(n)$ and $\psi(n-1)$.

$$\psi(n) = AB(n) + Ae^T B(n-1) + Ae^{2T} B(n-2) + Ae^{3T} B(n-3) + Ae^{4T} B(n-4) + \cdots$$
$$\psi(n-1) = AB(n-1) + Ae^T B(n-2) + Ae^{2T} B(n-3) + Ae^{3T} B(n-4) + Ae^{4T} B(n-5) + \cdots$$

EMPossible

Recursive Convolution Technique (2 of 3)

$$\overbrace{e^{T}\left[AB(n-1)+Ae^{T}B(n-2)+Ae^{2T}B(n-3)+Ae^{3T}B(n-4)+\cdots\right]}^{\psi(n-1)}$$

$$Ae^{T}B(n-1)+Ae^{2T}B(n-2)+Ae^{3T}B(n-3)+Ae^{4T}B(n-4)+\cdots$$

$$\psi(n)=AB(n)+\underbrace{Ae^{T}B(n-1)+Ae^{2T}B(n-2)+Ae^{3T}B(n-3)+Ae^{4T}B(n-4)+\cdots}$$

$$\psi(n-1)=AB(n-1)+Ae^{T}B(n-2)+Ae^{2T}B(n-3)+Ae^{3T}B(n-4)+Ae^{4T}B(n-5)+\cdots$$

$$\psi(n)=AB(n)+e^{T}\psi(n-1)$$

EMPossible

Slide 23

Recursive Convolution Technique (3 of 3)

Recall the convolution: $\psi(t)=\xi_y(t)*\frac{\partial E_z(t)}{\partial y} \quad \Leftrightarrow \quad \psi(n)=\sum_{m=0}^{n-1}Ae^{mT}B(n-m)$

$$A=c_{B_x,y}=\frac{\sigma_y}{\sigma_y\kappa_y+a_y\kappa_y^2}\left(b_{B_x,y}-1\right) \qquad B=\frac{E_z\big|_{n-m}^{i,j+1,k}-E_z\big|_{n-m}^{i,j,k}}{\Delta y} \qquad b_{B_x,y}=e^{-\left(\frac{\sigma_y}{\varepsilon_0\kappa_y}+\frac{a_y}{\varepsilon_0}\right)\Delta t} \qquad T=-\left(\frac{\sigma_y}{\varepsilon_0\kappa_y}+\frac{a_y}{\varepsilon_0}\right)\Delta t$$

The recursive relation was found to be

$$\psi(n)=AB(n)+e^{T}\psi(n-1)$$

Substituting in the expressions for A, B and T gives

$$\psi_{B_x,y}\Big|_t^{i,j,k}=b_{B_x,y}\psi_{B_x,y}\Big|_{t-\Delta t}^{i,j,k}+c_{B_x,y}\left(\frac{E_z\big|_t^{i,j+1,k}-E_z\big|_t^{i,j,k}}{\Delta y}\right)$$

EMPossible

Slide 24

3D Update Equations for B_x and D_x with a CPML

Derivation of the Update Equation for B_x

Start with the time-domain equation.

$$\frac{1}{\kappa_y}\frac{\partial E_z(t)}{\partial y}-\frac{1}{\kappa_z}\frac{\partial E_y(t)}{\partial z}+\xi_y(t)*\frac{\partial E_z(t)}{\partial y}-\xi_z(t)*\frac{\partial E_y(t)}{\partial z}=-\frac{1}{c_0}\frac{\partial \tilde{B}_x(t)}{\partial t}$$

Approximate derivatives with finite-differences and the convolutions with recursion rules.

$$\frac{1}{\kappa_y}\frac{E_z\big|_t^{i,j+1,k}-E_z\big|_t^{i,j,k}}{\Delta y}-\frac{1}{\kappa_z}\frac{E_y\big|_t^{i,j,k+1}-E_y\big|_t^{i,j,k}}{\Delta z}+\psi_{B_x,y}\big|_t^{i,j,k}-\psi_{B_x,z}\big|_t^{i,j,k}=-\frac{1}{c_0}\frac{\tilde{B}_x\big|_{t+\Delta t/2}^{i,j,k}-\tilde{B}_x\big|_{t-\Delta t/2}^{i,j,k}}{\Delta t}$$

Solve for B_x at the future time step.

$$\tilde{B}_x\big|_{t+\Delta t/2}^{i,j,k}=\tilde{B}_x\big|_{t-\Delta t/2}^{i,j,k}-c_0\Delta t\left(\frac{E_z\big|_t^{i,j+1,k}-E_z\big|_t^{i,j,k}}{\kappa_y\Delta y}-\frac{E_y\big|_t^{i,j,k+1}-E_y\big|_t^{i,j,k}}{\kappa_z\Delta z}+\psi_{B_x,y}\big|_t^{i,j,k}-\psi_{B_x,z}\big|_t^{i,j,k}\right)$$

EMPossible

Summary of the Update Equation for B_x

$$\tilde{B}_x\Big|_{t+\Delta t/2}^{i,j,k} = \tilde{B}_x\Big|_{t-\Delta t/2}^{i,j,k} - M\left(\frac{d_{E_z,y}\big|_t^{i,j,k}}{\kappa_y} - \frac{d_{E_y,z}\big|_t^{i,j,k}}{\kappa_z} + \psi_{B_x,y}\Big|_t^{i,j,k} - \psi_{B_x,z}\Big|_t^{i,j,k}\right)$$

$$d_{E_z,y}\Big|_t^{i,j,k} = \frac{E_z\big|_t^{i,j+1,k} - E_z\big|_t^{i,j,k}}{\Delta y}$$

$$d_{E_y,z}\Big|_t^{i,j,k} = \frac{E_y\big|_t^{i,j,k+1} - E_y\big|_t^{i,j,k}}{\Delta z}$$

$$M = c_0 \Delta t$$

$$\psi_{B_x,y}\Big|_t^{i,j,k} = b_{B_x,y}\,\psi_{B_x,y}\Big|_{t-\Delta t}^{i,j,k} + c_{B_x,y}\,d_{E_z,y}\Big|_t^{i,j,k}$$

$$b_{B_x,y} = e^{-\left(\frac{\sigma_y}{\varepsilon_0 \kappa_y} + \frac{a_y}{\varepsilon_0}\right)\Delta t} \qquad c_{B_x,y} = \frac{\sigma_y}{\sigma_y \kappa_y + a_y \kappa_y^2}\left(b_{B_x,y} - 1\right)$$

$$\psi_{B_x,z}\Big|_t^{i,j,k} = b_{B_x,z}\,\psi_{B_x,z}\Big|_{t-\Delta t}^{i,j,k} + c_{B_x,z}\,d_{E_y,z}\Big|_t^{i,j,k}$$

$$b_{B_x,z} = e^{-\left(\frac{\sigma_z}{\varepsilon_0 \kappa_z} + \frac{a_z}{\varepsilon_0}\right)\Delta t} \qquad c_{B_x,z} = \frac{\sigma_z}{\sigma_z \kappa_z + a_z \kappa_z^2}\left(b_{B_x,z} - 1\right)$$

EMPossible

Slide 27

Summary of the Update Equation for B_y

$$\tilde{B}_y\Big|_{t+\Delta t/2}^{i,j,k} = \tilde{B}_y\Big|_{t-\Delta t/2}^{i,j,k} - M\left(\frac{d_{E_x,z}\big|_t^{i,j,k}}{\kappa_z} - \frac{d_{E_z,x}\big|_t^{i,j,k}}{\kappa_x} + \psi_{B_y,z}\Big|_t^{i,j,k} - \psi_{B_y,x}\Big|_t^{i,j,k}\right)$$

$$d_{E_x,z}\Big|_t^{i,j,k} = \frac{E_x\big|_t^{i,j,k+1} - E_x\big|_t^{i,j,k}}{\Delta z}$$

$$d_{E_z,x}\Big|_t^{i,j,k} = \frac{E_z\big|_t^{i+1,j,k} - E_z\big|_t^{i,j,k}}{\Delta x}$$

$$M = c_0 \Delta t$$

$$\psi_{B_y,z}\Big|_t^{i,j,k} = b_{B_y,z}\,\psi_{B_y,z}\Big|_{t-\Delta t}^{i,j,k} + c_{B_y,z}\,d_{E_x,z}\Big|_t^{i,j,k}$$

$$b_{B_y,z} = e^{-\left(\frac{\sigma_z}{\varepsilon_0 \kappa_z} + \frac{a_z}{\varepsilon_0}\right)\Delta t} \qquad c_{B_y,z} = \frac{\sigma_z}{\sigma_z \kappa_z + a_z \kappa_z^2}\left(b_{B_y,z} - 1\right)$$

$$\psi_{B_y,x}\Big|_t^{i,j,k} = b_{B_y,x}\,\psi_{B_y,x}\Big|_{t-\Delta t}^{i,j,k} + c_{B_y,x}\,d_{E_z,x}\Big|_t^{i,j,k}$$

$$b_{B_y,x} = e^{-\left(\frac{\sigma_x}{\varepsilon_0 \kappa_x} + \frac{a_x}{\varepsilon_0}\right)\Delta t} \qquad c_{B_y,x} = \frac{\sigma_x}{\sigma_x \kappa_x + a_x \kappa_x^2}\left(b_{B_y,x} - 1\right)$$

EMPossible

Slide 28

Summary of the Update Equation for B_z

$$\tilde{B}_z\Big|_{t+\Delta t/2}^{i,j,k} = \tilde{B}_z\Big|_{t-\Delta t/2}^{i,j,k} - M\left(\frac{d_{E_y,x}\Big|_t^{i,j,k}}{\kappa_x} - \frac{d_{E_x,y}\Big|_t^{i,j,k}}{\kappa_y} + \psi_{B_z,x}\Big|_t^{i,j,k} - \psi_{B_z,y}\Big|_t^{i,j,k} \right)$$

$$d_{E_y,x}\Big|_t^{i,j,k} = \frac{E_y\Big|_t^{i+1,j,k} - E_y\Big|_t^{i,j,k}}{\Delta x}$$

$$d_{E_x,y}\Big|_t^{i,j,k} = \frac{E_x\Big|_t^{i,j+1,k} - E_x\Big|_t^{i,j,k}}{\Delta y}$$

$$M = c_0 \Delta t$$

$$\psi_{B_z,x}\Big|_t^{i,j,k} = b_{B_z,x}\,\psi_{B_z,x}\Big|_{t-\Delta t}^{i,j,k} + c_{B_z,x}\, d_{E_y,x}\Big|_t^{i,j,k}$$

$$b_{B_z,x} = e^{-\left(\frac{\sigma_x}{\varepsilon_0 \kappa_x} + \frac{a_x}{\varepsilon_0}\right)\Delta t} \qquad c_{B_z,x} = \frac{\sigma_x}{\sigma_x \kappa_x + a_x \kappa_x^2}\left(b_{B_z,x} - 1\right)$$

$$\psi_{B_z,y}\Big|_t^{i,j,k} = b_{B_z,y}\,\psi_{B_z,y}\Big|_{t-\Delta t}^{i,j,k} + c_{B_z,y}\, d_{E_x,y}\Big|_t^{i,j,k}$$

$$b_{B_z,y} = e^{-\left(\frac{\sigma_y}{\varepsilon_0 \kappa_y} + \frac{a_y}{\varepsilon_0}\right)\Delta t} \qquad c_{B_z,y} = \frac{\sigma_y}{\sigma_y \kappa_y + a_y \kappa_y^2}\left(b_{B_z,y} - 1\right)$$

EMPossible Slide 29

Summary of the Update Equation for $\tilde{D}_x$

$$\tilde{D}_x\Big|_{t+\Delta t}^{i,j,k} = \tilde{D}_x\Big|_t^{i,j,k} + M\left(\frac{d_{H_z,y}\Big|_{t+\Delta t/2}^{i,j,k}}{\kappa_y} - \frac{d_{H_y,z}\Big|_{t+\Delta t/2}^{i,j,k}}{\kappa_z} + \psi_{D_x,y}\Big|_{t+\Delta t/2}^{i,j,k} - \psi_{D_x,z}\Big|_{t+\Delta t/2}^{i,j,k} \right)$$

$$d_{H_z,y}\Big|_{t+\Delta t/2}^{i,j,k} = \frac{\tilde{H}_z\Big|_{t+\Delta t/2}^{i,j,k} - \tilde{H}_z\Big|_{t+\Delta t/2}^{i,j-1,k}}{\Delta y}$$

$$d_{H_y,z}\Big|_{t+\Delta t/2}^{i,j,k} = \frac{\tilde{H}_y\Big|_{t+\Delta t/2}^{i,j,k} - \tilde{H}_y\Big|_{t+\Delta t/2}^{i,j,k-1}}{\Delta z}$$

$$M = c_0 \Delta t$$

$$\psi_{D_x,y}\Big|_{t+\Delta t/2}^{i,j,k} = b_{D_x,y}\,\psi_{D_x,y}\Big|_{t-\Delta t/2}^{i,j,k} + c_{D_x,y}\, d_{H_z,y}\Big|_{t+\Delta t/2}^{i,j,k}$$

$$b_{D_x,y} = e^{-\left(\frac{\sigma_y}{\varepsilon_0 \kappa_y} + \frac{a_y}{\varepsilon_0}\right)\Delta t} \qquad c_{D_x,y} = \frac{\sigma_y}{\sigma_y \kappa_y + a_y \kappa_y^2}\left(b_{D_x,y} - 1\right)$$

$$\psi_{D_x,z}\Big|_{t+\Delta t/2}^{i,j,k} = b_{D_x,z}\,\psi_{D_x,z}\Big|_{t-\Delta t/2}^{i,j,k} + c_{D_x,z}\, d_{H_y,z}\Big|_{t+\Delta t/2}^{i,j,k}$$

$$b_{D_x,z} = e^{-\left(\frac{\sigma_z}{\varepsilon_0 \kappa_z} + \frac{a_z}{\varepsilon_0}\right)\Delta t} \qquad c_{D_x,z} = \frac{\sigma_z}{\sigma_z \kappa_z + a_z \kappa_z^2}\left(b_{D_x,z} - 1\right)$$

EMPossible Slide 30

Summary of the Update Equation for $\tilde{D}_y$

$$\tilde{D}_y\Big|_{t+\Delta t}^{i,j,k} = \tilde{D}_y\Big|_{t}^{i,j,k} + M\left(\frac{d_{H_x,z}\big|_{t+\Delta t/2}^{i,j,k}}{\kappa_z} - \frac{d_{H_z,x}\big|_{t+\Delta t/2}^{i,j,k}}{\kappa_x} + \psi_{D_y,z}\Big|_{t+\Delta t/2}^{i,j,k} - \psi_{D_y,x}\Big|_{t+\Delta t/2}^{i,j,k} \right)$$

$$d_{H_x,z}\Big|_{t+\Delta t/2}^{i,j,k} = \frac{\tilde{H}_x\big|_{t+\Delta t/2}^{i,j,k} - \tilde{H}_x\big|_{t+\Delta t/2}^{i,j,k-1}}{\Delta z}$$

$$d_{H_z,x}\Big|_{t+\Delta t/2}^{i,j,k} = \frac{\tilde{H}_z\big|_{t+\Delta t/2}^{i,j,k} - \tilde{H}_z\big|_{t+\Delta t/2}^{i-1,j,k}}{\Delta x}$$

$$M = c_0 \Delta t$$

$$\psi_{D_y,z}\Big|_{t+\Delta t/2}^{i,j,k} = b_{D_y,z}\,\psi_{D_y,z}\Big|_{t-\Delta t/2}^{i,j,k} + c_{D_y,z}\,d_{H_x,z}\Big|_{t+\Delta t/2}^{i,j,k}$$

$$b_{D_y,z} = e^{-\left(\frac{\sigma_z}{\varepsilon_0 \kappa_z} + \frac{a_z}{\varepsilon_0}\right)\Delta t} \qquad c_{D_y,z} = \frac{\sigma_z}{\sigma_z \kappa_z + a_z \kappa_z^2}\left(b_{D_y,z} - 1\right)$$

$$\psi_{D_y,x}\Big|_{t+\Delta t/2}^{i,j,k} = b_{D_y,x}\,\psi_{D_y,x}\Big|_{t-\Delta t/2}^{i,j,k} + c_{D_y,x}\,d_{H_z,x}\Big|_{t+\Delta t/2}^{i,j,k}$$

$$b_{D_y,x} = e^{-\left(\frac{\sigma_x}{\varepsilon_0 \kappa_x} + \frac{a_x}{\varepsilon_0}\right)\Delta t} \qquad c_{D_y,x} = \frac{\sigma_x}{\sigma_x \kappa_x + a_x \kappa_x^2}\left(b_{D_y,x} - 1\right)$$

EMPossible

Slide 31

Summary of the Update Equation for $\tilde{D}_z$

$$\tilde{D}_z\Big|_{t+\Delta t}^{i,j,k} = \tilde{D}_z\Big|_{t}^{i,j,k} + M\left(\frac{d_{H_y,x}\big|_{t+\Delta t/2}^{i,j,k}}{\kappa_x} - \frac{d_{H_x,y}\big|_{t+\Delta t/2}^{i,j,k}}{\kappa_y} + \psi_{D_z,x}\Big|_{t+\Delta t/2}^{i,j,k} - \psi_{D_z,y}\Big|_{t+\Delta t/2}^{i,j,k} \right)$$

$$d_{H_y,x}\Big|_{t+\Delta t/2}^{i,j,k} = \frac{\tilde{H}_y\big|_{t+\Delta t/2}^{i,j,k} - \tilde{H}_y\big|_{t+\Delta t/2}^{i-1,j,k}}{\Delta x}$$

$$d_{H_x,y}\Big|_{t+\Delta t/2}^{i,j,k} = \frac{\tilde{H}_x\big|_{t+\Delta t/2}^{i,j,k} - \tilde{H}_x\big|_{t+\Delta t/2}^{i,j-1,k}}{\Delta y}$$

$$M = c_0 \Delta t$$

$$\psi_{D_z,x}\Big|_{t+\Delta t/2}^{i,j,k} = b_{D_z,x}\,\psi_{D_z,x}\Big|_{t-\Delta t/2}^{i,j,k} + c_{D_z,x}\,d_{H_y,x}\Big|_{t+\Delta t/2}^{i,j,k}$$

$$b_{D_z,x} = e^{-\left(\frac{\sigma_x}{\varepsilon_0 \kappa_x} + \frac{a_x}{\varepsilon_0}\right)\Delta t} \qquad c_{D_z,x} = \frac{\sigma_x}{\sigma_x \kappa_x + a_x \kappa_x^2}\left(b_{D_z,x} - 1\right)$$

$$\psi_{D_z,y}\Big|_{t+\Delta t/2}^{i,j,k} = b_{D_z,y}\,\psi_{D_z,y}\Big|_{t-\Delta t/2}^{i,j,k} + c_{D_z,y}\,d_{H_x,y}\Big|_{t+\Delta t/2}^{i,j,k}$$

$$b_{D_z,y} = e^{-\left(\frac{\sigma_y}{\varepsilon_0 \kappa_y} + \frac{a_y}{\varepsilon_0}\right)\Delta t} \qquad c_{D_z,y} = \frac{\sigma_y}{\sigma_y \kappa_y + a_y \kappa_y^2}\left(b_{D_z,y} - 1\right)$$

EMPossible

Slide 32

3D Update Equations for E_x and H_x

Impermittivity & Impermeability Tensors

The update equations are derived from the constitutive relations.

$$\vec{\tilde{D}} = [\varepsilon_r]\vec{E} \qquad \vec{\tilde{B}} = [\mu_r]\vec{\tilde{H}}$$

It is best to express these equations in terms of the impermittivity tensor $[\varepsilon_r^{-1}]$ and impermeability tensor $[\mu_r^{-1}]$.

$$\vec{E} = [\varepsilon_r^{-1}]\vec{\tilde{D}} \qquad \vec{\tilde{H}} = [\mu_r^{-1}]\vec{\tilde{B}}$$

$$\begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} = \begin{bmatrix} \varepsilon_{xx}^{-1} & \varepsilon_{xy}^{-1} & \varepsilon_{xz}^{-1} \\ \varepsilon_{yx}^{-1} & \varepsilon_{yy}^{-1} & \varepsilon_{yz}^{-1} \\ \varepsilon_{zx}^{-1} & \varepsilon_{zy}^{-1} & \varepsilon_{zz}^{-1} \end{bmatrix} \begin{bmatrix} \tilde{D}_x \\ \tilde{D}_y \\ \tilde{D}_z \end{bmatrix} \qquad \begin{bmatrix} \tilde{H}_x \\ \tilde{H}_y \\ \tilde{H}_z \end{bmatrix} = \begin{bmatrix} \mu_{xx}^{-1} & \mu_{xy}^{-1} & \mu_{xz}^{-1} \\ \mu_{yx}^{-1} & \mu_{yy}^{-1} & \mu_{yz}^{-1} \\ \mu_{zx}^{-1} & \mu_{zy}^{-1} & \mu_{zz}^{-1} \end{bmatrix} \begin{bmatrix} \tilde{B}_x \\ \tilde{B}_y \\ \tilde{B}_z \end{bmatrix}$$

Do not be confused by the notation.

$$\varepsilon_{mn}^{-1} \neq \frac{1}{\varepsilon_{mn}}$$

$$\mu_{mn}^{-1} \neq \frac{1}{\mu_{mn}}$$

EMPossible

Drop Off-Diagonal Terms

To simplify the update equations, the off-diagonal terms are dropped from the tensors.

$$\begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} = \begin{bmatrix} \varepsilon_{xx}^{-1} & 0 & 0 \\ 0 & \varepsilon_{yy}^{-1} & 0 \\ 0 & 0 & \varepsilon_{zz}^{-1} \end{bmatrix} \begin{bmatrix} \tilde{D}_x \\ \tilde{D}_y \\ \tilde{D}_z \end{bmatrix} \qquad \begin{bmatrix} \tilde{H}_x \\ \tilde{H}_y \\ \tilde{H}_z \end{bmatrix} = \begin{bmatrix} \mu_{xx}^{-1} & 0 & 0 \\ 0 & \mu_{yy}^{-1} & 0 \\ 0 & 0 & \mu_{zz}^{-1} \end{bmatrix} \begin{bmatrix} \tilde{B}_x \\ \tilde{B}_y \\ \tilde{B}_z \end{bmatrix}$$

The update equations are readily written as

$$E_x\Big|_t^{i,j,k} = \varepsilon_{xx}^{-1}\Big|^{i,j,k} \tilde{D}_x\Big|_t^{i,j,k} \qquad \tilde{H}_x\Big|_t^{i,j,k} = \mu_{xx}^{-1}\Big|^{i,j,k} \tilde{B}_x\Big|_t^{i,j,k}$$

$$E_y\Big|_t^{i,j,k} = \varepsilon_{yy}^{-1}\Big|^{i,j,k} \tilde{D}_y\Big|_t^{i,j,k} \qquad \tilde{H}_y\Big|_t^{i,j,k} = \mu_{yy}^{-1}\Big|^{i,j,k} \tilde{B}_y\Big|_t^{i,j,k}$$

$$E_z\Big|_t^{i,j,k} = \varepsilon_{zz}^{-1}\Big|^{i,j,k} \tilde{D}_z\Big|_t^{i,j,k} \qquad \tilde{H}_z\Big|_t^{i,j,k} = \mu_{zz}^{-1}\Big|^{i,j,k} \tilde{B}_z\Big|_t^{i,j,k}$$

2D Update Equations for B_x and D_x with a CPML

Reducing the 3D Update Equations to 2D

Confine the simulation to the xy plane. Let the z direction be uniform and infinite.

$$\frac{\partial}{\partial z} = 0$$

No change in the z direction.

For the B_x update equation,

$$d_{E_y,z}\Big|_t^{i,j,k} = \frac{E_y\big|_t^{i,j,k+1} - E_y\big|_t^{i,j,k}}{\Delta z} = 0$$

$$\psi_{B_x,z}\Big|_t^{i,j,k} = b_{B_x,z}\,\psi_{B_x,z}\Big|_{t-\Delta t}^{i,j,k} + c_{B_x,z}\,d_{E_y,z}\Big|_t^{i,j,k} = 0$$

$$\tilde{B}_x\Big|_{t+\Delta t/2}^{i,j,k} = \tilde{B}_x\Big|_{t-\Delta t/2}^{i,j,k} - M\left(\frac{d_{E_z,y}\big|_t^{i,j,k}}{\kappa_y} - \frac{d_{E_y,z}\big|_t^{i,j,k}}{\kappa_z} + \psi_{B_x,y}\Big|_t^{i,j,k} - \psi_{B_x,z}\Big|_t^{i,j,k}\right)$$

EMPossible

Slide 37

Summary of the Update Equation for B_x

$$\tilde{B}_x\Big|_{t+\Delta t/2}^{i,j,k} = \tilde{B}_x\Big|_{t-\Delta t/2}^{i,j,k} - M\left(\frac{d_{E_z,y}\big|_t^{i,j,k}}{\kappa_y} + \psi_{B_x,y}\Big|_t^{i,j,k}\right)$$

$$d_{E_z,y}\Big|_t^{i,j,k} = \frac{E_z\big|_t^{i,j+1,k} - E_z\big|_t^{i,j,k}}{\Delta y}$$

$$M = c_0 \Delta t$$

$$\psi_{B_x,y}\Big|_t^{i,j,k} = b_{B_x,y}\,\psi_{B_x,y}\Big|_{t-\Delta t}^{i,j,k} + c_{B_x,y}\,d_{E_z,y}\Big|_t^{i,j,k}$$

$$b_{B_x,y} = e^{-\left(\frac{\sigma_y}{\varepsilon_0 \kappa_y} + \frac{a_y}{\varepsilon_0}\right)\Delta t} \qquad c_{B_x,y} = \frac{\sigma_y}{\sigma_y \kappa_y + a_y \kappa_y^2}\left(b_{B_x,y} - 1\right)$$

EMPossible

Slide 38

Summary of the Update Equation for B_y

$$\tilde{B}_y\Big|_{t+\Delta t/2}^{i,j,k} = \tilde{B}_y\Big|_{t-\Delta t/2}^{i,j,k} - M\left(-\frac{d_{E_z,x}\Big|_t^{i,j,k}}{\kappa_x} - \psi_{B_y,x}\Big|_t^{i,j,k} \right)$$

$$d_{E_z,x}\Big|_t^{i,j,k} = \frac{E_z\Big|_t^{i+1,j,k} - E_z\Big|_t^{i,j,k}}{\Delta x}$$

$$M = c_0 \Delta t$$

$$\psi_{B_y,x}\Big|_t^{i,j,k} = b_{B_y,x}\,\psi_{B_y,x}\Big|_{t-\Delta t}^{i,j,k} + c_{B_y,x}\,d_{E_z,x}\Big|_t^{i,j,k}$$

$$b_{B_y,x} = e^{-\left(\frac{\sigma_x}{\varepsilon_0 \kappa_x} + \frac{a_x}{\varepsilon_0}\right)\Delta t} \qquad c_{B_y,x} = \frac{\sigma_x}{\sigma_x \kappa_x + a_x \kappa_x^2}\left(b_{B_y,x} - 1\right)$$

EMPossible

Slide 39

Summary of the Update Equation for B_z

$$\tilde{B}_z\Big|_{t+\Delta t/2}^{i,j,k} = \tilde{B}_z\Big|_{t-\Delta t/2}^{i,j,k} - M\left(\frac{d_{E_y,x}\Big|_t^{i,j,k}}{\kappa_x} - \frac{d_{E_x,y}\Big|_t^{i,j,k}}{\kappa_y} + \psi_{B_z,x}\Big|_t^{i,j,k} - \psi_{B_z,y}\Big|_t^{i,j,k} \right)$$

$$d_{E_y,x}\Big|_t^{i,j,k} = \frac{E_y\Big|_t^{i+1,j,k} - E_y\Big|_t^{i,j,k}}{\Delta x}$$

$$d_{E_x,y}\Big|_t^{i,j,k} = \frac{E_x\Big|_t^{i,j+1,k} - E_x\Big|_t^{i,j,k}}{\Delta y}$$

$$M = c_0 \Delta t$$

$$\psi_{B_z,x}\Big|_t^{i,j,k} = b_{B_z,x}\,\psi_{B_z,x}\Big|_{t-\Delta t}^{i,j,k} + c_{B_z,x}\,d_{E_y,x}\Big|_t^{i,j,k}$$

$$b_{B_z,x} = e^{-\left(\frac{\sigma_x}{\varepsilon_0 \kappa_x} + \frac{a_x}{\varepsilon_0}\right)\Delta t} \qquad c_{B_z,x} = \frac{\sigma_x}{\sigma_x \kappa_x + a_x \kappa_x^2}\left(b_{B_z,x} - 1\right)$$

$$\psi_{B_z,y}\Big|_t^{i,j,k} = b_{B_z,y}\,\psi_{B_z,y}\Big|_{t-\Delta t}^{i,j,k} + c_{B_z,y}\,d_{E_x,y}\Big|_t^{i,j,k}$$

$$b_{B_z,y} = e^{-\left(\frac{\sigma_y}{\varepsilon_0 \kappa_y} + \frac{a_y}{\varepsilon_0}\right)\Delta t} \qquad c_{B_z,y} = \frac{\sigma_y}{\sigma_y \kappa_y + a_y \kappa_y^2}\left(b_{B_z,y} - 1\right)$$

EMPossible

Slide 40

Summary of the Update Equation for $\tilde{D}_x$

$$\left.\tilde{D}_x\right|_{t+\Delta t}^{i,j,k} = \left.\tilde{D}_x\right|_{t}^{i,j,k} + M\left(\frac{\left.d_{H_z,y}\right|_{t+\Delta t/2}^{i,j,k}}{\kappa_y} + \left.\psi_{D_x,y}\right|_{t+\Delta t/2}^{i,j,k}\right)$$

$$\left.d_{H_z,y}\right|_{t+\Delta t/2}^{i,j,k} = \frac{\left.\tilde{H}_z\right|_{t+\Delta t/2}^{i,j,k} - \left.\tilde{H}_z\right|_{t+\Delta t/2}^{i,j-1,k}}{\Delta y}$$

$$M = c_0 \Delta t$$

$$\left.\psi_{D_x,y}\right|_{t+\Delta t/2}^{i,j,k} = b_{D_x,y}\left.\psi_{D_x,y}\right|_{t-\Delta t/2}^{i,j,k} + c_{D_x,y}\left.d_{H_z,y}\right|_{t+\Delta t/2}^{i,j,k}$$

$$b_{D_x,y} = e^{-\left(\frac{\sigma_y}{\varepsilon_0 \kappa_y} + \frac{a_y}{\varepsilon_0}\right)\Delta t} \qquad c_{D_x,y} = \frac{\sigma_y}{\sigma_y \kappa_y + a_y \kappa_y^2}\left(b_{D_x,y} - 1\right)$$

EMPossible

Summary of the Update Equation for $\tilde{D}_y$

$$\left.\tilde{D}_y\right|_{t+\Delta t}^{i,j,k} = \left.\tilde{D}_y\right|_{t}^{i,j,k} + M\left(-\frac{\left.d_{H_z,x}\right|_{t+\Delta t/2}^{i,j,k}}{\kappa_x} - \left.\psi_{D_y,x}\right|_{t+\Delta t/2}^{i,j,k}\right)$$

$$\left.d_{H_z,x}\right|_{t+\Delta t/2}^{i,j,k} = \frac{\left.\tilde{H}_z\right|_{t+\Delta t/2}^{i,j,k} - \left.\tilde{H}_z\right|_{t+\Delta t/2}^{i-1,j,k}}{\Delta x}$$

$$M = c_0 \Delta t$$

$$\left.\psi_{D_y,x}\right|_{t+\Delta t/2}^{i,j,k} = b_{D_y,x}\left.\psi_{D_y,x}\right|_{t-\Delta t/2}^{i,j,k} + c_{D_y,x}\left.d_{H_z,x}\right|_{t+\Delta t/2}^{i,j,k}$$

$$b_{D_y,x} = e^{-\left(\frac{\sigma_x}{\varepsilon_0 \kappa_x} + \frac{a_x}{\varepsilon_0}\right)\Delta t} \qquad c_{D_y,x} = \frac{\sigma_x}{\sigma_x \kappa_x + a_x \kappa_x^2}\left(b_{D_y,x} - 1\right)$$

EMPossible

Summary of the Update Equation for $\tilde{D}_z$

$$\tilde{D}_z\Big|_{t+\Delta t}^{i,j,k} = \tilde{D}_z\Big|_{t}^{i,j,k} + M\left(\frac{d_{H_y,x}\Big|_{t+\Delta t/2}^{i,j,k}}{\kappa_x} - \frac{d_{H_x,y}\Big|_{t+\Delta t/2}^{i,j,k}}{\kappa_y} + \psi_{D_z,x}\Big|_{t+\Delta t/2}^{i,j,k} - \psi_{D_z,y}\Big|_{t+\Delta t/2}^{i,j,k} \right)$$

$$d_{H_y,x}\Big|_{t+\Delta t/2}^{i,j,k} = \frac{\tilde{H}_y\Big|_{t+\Delta t/2}^{i,j,k} - \tilde{H}_y\Big|_{t+\Delta t/2}^{i-1,j,k}}{\Delta x}$$

$$d_{H_x,y}\Big|_{t+\Delta t/2}^{i,j,k} = \frac{\tilde{H}_x\Big|_{t+\Delta t/2}^{i,j,k} - \tilde{H}_x\Big|_{t+\Delta t/2}^{i,j-1,k}}{\Delta y}$$

$$M = c_0 \Delta t$$

$$\psi_{D_z,x}\Big|_{t+\Delta t/2}^{i,j,k} = b_{D_z,x}\psi_{D_z,x}\Big|_{t-\Delta t/2}^{i,j,k} + c_{D_z,x} d_{H_y,x}\Big|_{t+\Delta t/2}^{i,j,k}$$

$$b_{D_z,x} = e^{-\left(\frac{\sigma_x}{\varepsilon_0\kappa_x} + \frac{a_x}{\varepsilon_0}\right)\Delta t} \qquad c_{D_z,x} = \frac{\sigma_x}{\sigma_x\kappa_x + a_x\kappa_x^2}\left(b_{D_z,x} - 1\right)$$

$$\psi_{D_z,y}\Big|_{t+\Delta t/2}^{i,j,k} = b_{D_z,y}\psi_{D_z,y}\Big|_{t-\Delta t/2}^{i,j,k} + c_{D_z,y} d_{H_x,y}\Big|_{t+\Delta t/2}^{i,j,k}$$

$$b_{D_z,y} = e^{-\left(\frac{\sigma_y}{\varepsilon_0\kappa_y} + \frac{a_y}{\varepsilon_0}\right)\Delta t} \qquad c_{D_z,y} = \frac{\sigma_y}{\sigma_y\kappa_y + a_y\kappa_y^2}\left(b_{D_z,y} - 1\right)$$

EMPossible

Slide 43

2D Update Equations for E_x and H_x

Slide 44

2D Update Equations for E_x and H_x

The update equations for E_x and H_x in 2D remain the same as for 3D.

$$E_x\Big|_t^{i,j,k} = \varepsilon_{xx}^{-1}\Big|^{i,j,k} \tilde{D}_x\Big|_t^{i,j,k} \qquad \tilde{H}_x\Big|_t^{i,j,k} = \mu_{xx}^{-1}\Big|^{i,j,k} \tilde{B}_x\Big|_t^{i,j,k}$$

$$E_y\Big|_t^{i,j,k} = \varepsilon_{yy}^{-1}\Big|^{i,j,k} \tilde{D}_y\Big|_t^{i,j,k} \qquad \tilde{H}_y\Big|_t^{i,j,k} = \mu_{yy}^{-1}\Big|^{i,j,k} \tilde{B}_y\Big|_t^{i,j,k}$$

$$E_z\Big|_t^{i,j,k} = \varepsilon_{zz}^{-1}\Big|^{i,j,k} \tilde{D}_z\Big|_t^{i,j,k} \qquad \tilde{H}_z\Big|_t^{i,j,k} = \mu_{zz}^{-1}\Big|^{i,j,k} \tilde{B}_z\Big|_t^{i,j,k}$$