Computational Science:
Computational Methods in Engineering

Common Linear Algebra Problems

$[A][x] = [b]$

This problem arises when a problem $[A]$ is given some excitation $[b]$ and produces a <u>single</u> solution $[x]$.

Examples: (1) waves scattering from an object, (2) heat through a device, (3) solving currents and voltages in a circuit.

It produces a single solution.

Step 1 – Differential equation

$$\frac{d^2 f}{dx^2} + \gamma \frac{df}{dx} + f = b$$

Step 2 – ODE is converted to system of equations using finite-differences, finite elements, etc.

$$\begin{gathered} a_{11} f_1 + a_{12} f_2 + \cdots + a_{1n} f_n = b_1 \\ a_{21} f_1 + a_{22} f_2 + \cdots + a_{2n} f_n = b_2 \\ \vdots \\ a_{n1} f_1 + a_{n2} f_2 + \cdots + a_{nn} f_n = b_n \end{gathered}$$

Step 3 – System of equations is put into matrix form.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \\ a_{n1} & a_{n2} & & a_{nn} \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Step 4 – Matrix problem is solved for [f]

$$[f] = [A]^{-1}[b]$$

Step 5 – [f] is post processed to learn something.

Eigen-Value Problems $[A][x] = \lambda[x]$

Eigen-value problems arise when multiple solutions exist. No excitation is needed.

Examples: (1) resonating modes on a string, (2) electromagnetic modes in a waveguide, (3) electronic bands in a semiconductor.

$$[A][x] = \lambda[x] \quad \text{Standard eigen-value problem}$$

$$[A][x] = \lambda[B][x] \quad \text{Generalized eigen-value problem}$$

$[A]$ is the linear operation

$[x]$ is the unknown (eigen-vector)

λ is the eigen-value and is just a scalar number

$[B]$ is potentially another part of the linear operation

EMPossible

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Determinants

The determinant is an important number associated with square matrices.

It is sort of a magnitude or volume.

Unique solutions to systems of equations do not exist when the determinant is zero.

<u>2×2 Matrices</u>

$$\det[A] = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

<u>3×3 Matrices</u>

$$\det[A] = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

This can be calculated by walking across any of the rows.

EMPossible

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Cramer's Rule

Cramer's rule provides a methodical approach for calculating the unknowns of a system of equations.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$x_1 = \frac{1}{D} \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix} \qquad x_2 = \frac{1}{D} \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix} \qquad x_3 = \frac{1}{D} \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}$$

$$D = \det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

EMPossible

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