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Advanced Electromagnetics:

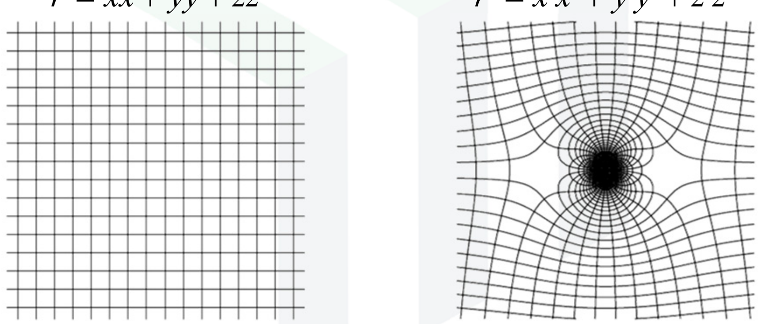
21st Century Electromagnetics

Coordinate Transforms

Coordinate Transformation

One coordinate space $\vec{r}$ can be mapped to another $\vec{r}\,'$.

$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z} \qquad \vec{r}\,' = x'\hat{x}' + y'\hat{y}' + z'\hat{z}'$$



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Jacobian Transformation Matrix $[\,J\,]$

The Jacobian transformation matrix is defined as

$$[J] = (\nabla \vec{r}')^T = \begin{bmatrix} \frac{\partial x'}{\partial x} & \frac{\partial x'}{\partial y} & \frac{\partial x'}{\partial z} \\ \frac{\partial y'}{\partial x} & \frac{\partial y'}{\partial y} & \frac{\partial y'}{\partial z} \\ \frac{\partial z'}{\partial x} & \frac{\partial z'}{\partial y} & \frac{\partial z'}{\partial z} \end{bmatrix}$$

Gradient of a vector is a tensor! Wow! ☺

Think of this as a matrix that characterizes the "stretching" of the transform. It describes how much the coordinate changes in the transformed system with respect to a change in the original system.

The Jacobian transformation matrix does not perform a coordinate transform. It transforms functions and operations between different coordinate systems.

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General Form of the Jacobian Matrix $[J]$

$$[J] = \begin{bmatrix} \frac{h_1'}{h_1}\frac{\partial x_1'}{\partial x_1} & \frac{h_1'}{h_2}\frac{\partial x_1'}{\partial x_2} & \frac{h_1'}{h_3}\frac{\partial x_1'}{\partial x_3} \\ \frac{h_2'}{h_1}\frac{\partial x_2'}{\partial x_1} & \frac{h_2'}{h_2}\frac{\partial x_2'}{\partial x_2} & \frac{h_2'}{h_3}\frac{\partial x_2'}{\partial x_3} \\ \frac{h_3'}{h_1}\frac{\partial x_3'}{\partial x_1} & \frac{h_3'}{h_2}\frac{\partial x_3'}{\partial x_2} & \frac{h_3'}{h_3}\frac{\partial x_3'}{\partial x_3} \end{bmatrix}$$

$$h_i = \sqrt{\sum_{k=1}^{n}\left(\frac{\partial x_k}{\partial x'_i}\right)^2}$$

Coordinate System	h_1	h_2	h_3
Cartesian (x,y,z)	1	1	1
Cylindrical (ρ,ϕ,z)	1	ρ	1
Spherical (r,ϕ,θ)	1	r	$r\sin\theta$

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Example #1: Cylindrical to Cartesian

The Cartesian and cylindrical coordinates are related through

$$x = \rho\cos\phi$$
$$y = \rho\sin\phi$$
$$z = z$$

The Jacobian matrix is then

$$[J] = \begin{bmatrix} \partial x/\partial\rho & \partial x/\partial\phi & \partial x/\partial z \\ \partial y/\partial\rho & \partial y/\partial\phi & \partial y/\partial z \\ \partial z/\partial\rho & \partial z/\partial\phi & \partial z/\partial z \end{bmatrix}$$

$$\frac{\partial x}{\partial\rho} = \cos\phi \quad \frac{\partial x}{\partial\phi} = -\rho\sin\phi \quad \frac{\partial x}{\partial z} = 0$$
$$\frac{\partial y}{\partial\rho} = \sin\phi \quad \frac{\partial y}{\partial\phi} = \rho\cos\phi \quad \frac{\partial y}{\partial z} = 0$$
$$\frac{\partial z}{\partial\rho} = 0 \quad \frac{\partial z}{\partial\phi} = 0 \quad \frac{\partial z}{\partial z} = 1$$

$$[J] = \begin{bmatrix} \cos\phi & -\rho\sin\phi & 0 \\ \sin\phi & \rho\cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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Example #2: Spherical to Cartesian

The Cartesian and spherical coordinates are related through

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$

The Jacobian matrix is then

$$[J] = \begin{bmatrix} \partial x/\partial r & \partial x/\partial\theta & \partial x/\partial\phi \\ \partial y/\partial r & \partial y/\partial\theta & \partial y/\partial\phi \\ \partial z/\partial r & \partial z/\partial\theta & \partial z/\partial\phi \end{bmatrix}$$

$$\frac{\partial x}{\partial r} = \sin\theta\cos\phi \quad \frac{\partial x}{\partial\theta} = r\cos\theta\cos\phi \quad \frac{\partial x}{\partial\phi} = -r\sin\theta\sin\phi$$
$$\frac{\partial y}{\partial r} = \sin\theta\sin\phi \quad \frac{\partial y}{\partial\theta} = r\cos\theta\sin\phi \quad \frac{\partial y}{\partial\phi} = r\sin\theta\cos\phi$$
$$\frac{\partial z}{\partial r} = \cos\theta \quad \frac{\partial z}{\partial\theta} = -r\sin\theta \quad \frac{\partial z}{\partial\phi} = 0$$

$$[J] = \begin{bmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{bmatrix}$$

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Transforming Vector Functions

The same vector function (vector that changes as a function of position) expressed in two different coordinate systems is related through the Jacobian matrix $[J]$ as follows.

$$\vec{E}'(\vec{r}') = \left([J]^T\right)^{-1} \vec{E}(\vec{r})$$

$$\vec{E}(\vec{r}) = [J]^T \vec{E}'(\vec{r}')$$

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Transforming Operations

An operation (think derivatives, integrals, tensors, etc.) can also be transformed between two coordinate systems using the Jacobian matrix $[J]$.

$$\left[F'(\vec{r}')\right] = \frac{[J]\left[F(\vec{r})\right][J]^T}{\det[J]}$$

$$\left[F(\vec{r})\right] = \det[J] \cdot [J]^{-1} \left[F'(\vec{r}')\right]\left([J]^T\right)^{-1}$$

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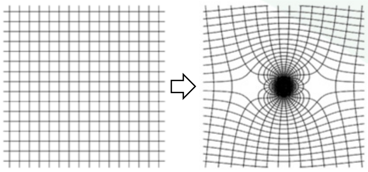
Summary

Coordinate Transform



$$x'(x,y,z)$$
$$y'(x,y,z)$$
$$z'(x,y,z)$$

Jacobian $[J]$

$$[J]=\begin{bmatrix} \frac{h_1'}{h_1}\frac{\partial x_1'}{\partial x_1} & \frac{h_1'}{h_2}\frac{\partial x_1'}{\partial x_2} & \frac{h_1'}{h_3}\frac{\partial x_1'}{\partial x_3} \\ \frac{h_2'}{h_1}\frac{\partial x_2'}{\partial x_1} & \frac{h_2'}{h_2}\frac{\partial x_2'}{\partial x_2} & \frac{h_2'}{h_3}\frac{\partial x_2'}{\partial x_3} \\ \frac{h_3'}{h_1}\frac{\partial x_3'}{\partial x_1} & \frac{h_3'}{h_2}\frac{\partial x_3'}{\partial x_2} & \frac{h_3'}{h_3}\frac{\partial x_3'}{\partial x_3} \end{bmatrix}$$

Transforming Vector Functions

$$\vec{E}'(\vec{r}')=\left([J]^T\right)^{-1}\vec{E}(\vec{r})$$
$$\vec{E}(\vec{r})=[J]^T\,\vec{E}'(\vec{r}')$$

Transforming Operations

$$\left[F'(\vec{r}')\right]=\frac{[J]\left[F(\vec{r})\right][J]^T}{\det[J]}$$
$$\left[F(\vec{r})\right]=\det[J]\cdot[J]^{-1}\left[F'(\vec{r}')\right]\left([J]^T\right)^{-1}$$