



Computational Science:
Computational Methods in Engineering

Derivative Matrices on a Collocated Grid



Outline

- Collocated Derivative Matrix $[D_x]$
- Collocated Derivative Matrix $[D_x^2]$
- Collocated Derivative Matrix $[D_y]$
- Collocated Derivative Matrix $[D_y^2]$
- Final Notes

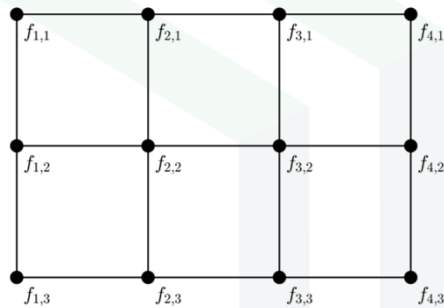


Collocated Derivative Matrix

$$[D_x]$$



$[D_x]$ Derivative Matrix (1 of 14)

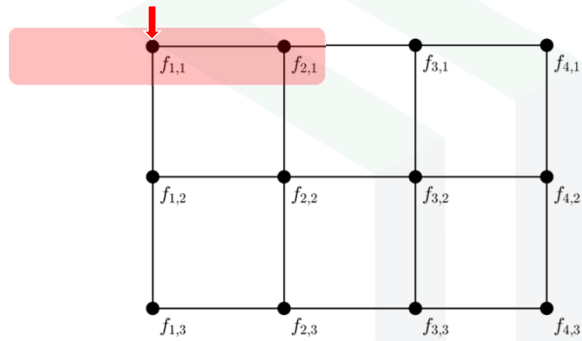


$$\frac{\partial f_{i,j}}{\partial x} \approx \frac{f_{i+1,j} - f_{i-1,j}}{2\Delta x}$$

$$\begin{aligned} \frac{\partial f_{1,1}}{\partial x} &\approx \frac{f_{2,1} - 0}{2\Delta x} \\ \frac{\partial f_{2,1}}{\partial x} &\approx \frac{f_{3,1} - f_{1,1}}{2\Delta x} \\ \frac{\partial f_{3,1}}{\partial x} &\approx \frac{f_{4,1} - f_{2,1}}{2\Delta x} \\ \frac{\partial f_{4,1}}{\partial x} &\approx \frac{0 - f_{3,1}}{2\Delta x} \\ \frac{\partial f_{1,2}}{\partial x} &\approx \frac{f_{2,2} - 0}{2\Delta x} \\ \frac{\partial f_{2,2}}{\partial x} &\approx \frac{f_{3,2} - f_{1,2}}{2\Delta x} \\ \frac{\partial f_{3,2}}{\partial x} &\approx \frac{f_{4,2} - f_{2,2}}{2\Delta x} \\ \frac{\partial f_{4,2}}{\partial x} &\approx \frac{0 - f_{3,2}}{2\Delta x} \\ \frac{\partial f_{1,3}}{\partial x} &\approx \frac{f_{2,3} - 0}{2\Delta x} \\ \frac{\partial f_{2,3}}{\partial x} &\approx \frac{f_{3,3} - f_{1,3}}{2\Delta x} \\ \frac{\partial f_{3,3}}{\partial x} &\approx \frac{f_{4,3} - f_{2,3}}{2\Delta x} \\ \frac{\partial f_{4,3}}{\partial x} &\approx \frac{0 - f_{3,3}}{2\Delta x} \end{aligned}$$



$[D_x]$ Derivative Matrix (2 of 14)



$$\frac{\partial f_{i,j}}{\partial x} \approx \frac{f_{i+1,j} - f_{i-1,j}}{2\Delta x}$$

$$\frac{\partial f_{1,1}}{\partial x} \approx \frac{f_{2,1} - 0}{2\Delta x}$$

$$\frac{\partial f_{2,1}}{\partial x} \approx \frac{f_{3,1} - f_{1,1}}{2\Delta x}$$

$$\frac{\partial f_{3,1}}{\partial x} \approx \frac{f_{4,1} - f_{2,1}}{2\Delta x}$$

$$\frac{\partial f_{4,1}}{\partial x} \approx \frac{0 - f_{3,1}}{2\Delta x}$$

$$\frac{\partial f_{1,2}}{\partial x} \approx \frac{f_{2,2} - 0}{2\Delta x}$$

$$\frac{\partial f_{2,2}}{\partial x} \approx \frac{f_{3,2} - f_{1,2}}{2\Delta x}$$

$$\frac{\partial f_{3,2}}{\partial x} \approx \frac{f_{4,2} - f_{2,2}}{2\Delta x}$$

$$\frac{\partial f_{4,2}}{\partial x} \approx \frac{0 - f_{3,2}}{2\Delta x}$$

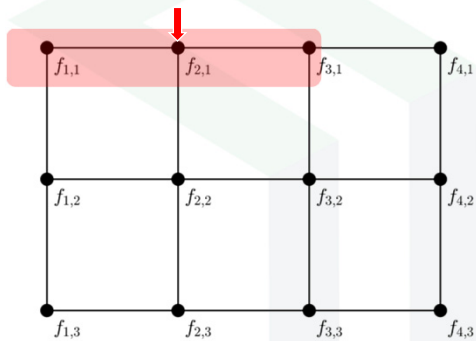
$$\frac{\partial f_{1,3}}{\partial x} \approx \frac{f_{2,3} - 0}{2\Delta x}$$

$$\frac{\partial f_{2,3}}{\partial x} \approx \frac{f_{3,3} - f_{1,3}}{2\Delta x}$$

$$\frac{\partial f_{3,3}}{\partial x} \approx \frac{f_{4,3} - f_{2,3}}{2\Delta x}$$

$$\frac{\partial f_{4,3}}{\partial x} \approx \frac{0 - f_{3,3}}{2\Delta x}$$

$[D_x]$ Derivative Matrix (3 of 14)



$$\frac{\partial f_{i,j}}{\partial x} \approx \frac{f_{i+1,j} - f_{i-1,j}}{2\Delta x}$$

$$\frac{\partial f_{1,1}}{\partial x} \approx \frac{f_{2,1} - 0}{2\Delta x}$$

$$\frac{\partial f_{2,1}}{\partial x} \approx \frac{f_{3,1} - f_{1,1}}{2\Delta x}$$

$$\frac{\partial f_{3,1}}{\partial x} \approx \frac{f_{4,1} - f_{2,1}}{2\Delta x}$$

$$\frac{\partial f_{4,1}}{\partial x} \approx \frac{0 - f_{3,1}}{2\Delta x}$$

$$\frac{\partial f_{1,2}}{\partial x} \approx \frac{f_{2,2} - 0}{2\Delta x}$$

$$\frac{\partial f_{2,2}}{\partial x} \approx \frac{f_{3,2} - f_{1,2}}{2\Delta x}$$

$$\frac{\partial f_{3,2}}{\partial x} \approx \frac{f_{4,2} - f_{2,2}}{2\Delta x}$$

$$\frac{\partial f_{4,2}}{\partial x} \approx \frac{0 - f_{3,2}}{2\Delta x}$$

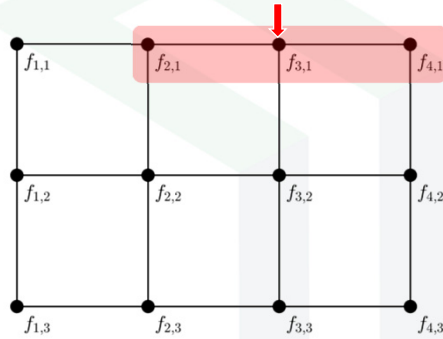
$$\frac{\partial f_{1,3}}{\partial x} \approx \frac{f_{2,3} - 0}{2\Delta x}$$

$$\frac{\partial f_{2,3}}{\partial x} \approx \frac{f_{3,3} - f_{1,3}}{2\Delta x}$$

$$\frac{\partial f_{3,3}}{\partial x} \approx \frac{f_{4,3} - f_{2,3}}{2\Delta x}$$

$$\frac{\partial f_{4,3}}{\partial x} \approx \frac{0 - f_{3,3}}{2\Delta x}$$

$[D_x]$ Derivative Matrix (4 of 14)

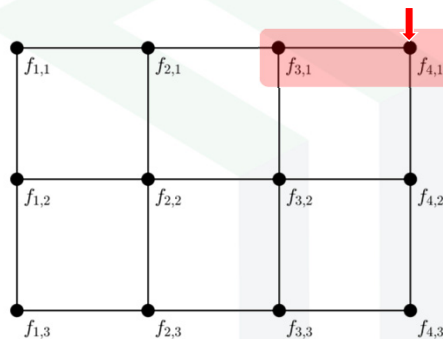


$$\frac{\partial f_{i,j}}{\partial x} \approx \frac{f_{i+1,j} - f_{i-1,j}}{2\Delta x}$$

$$\begin{aligned} \frac{\partial f_{1,1}}{\partial x} &\approx \frac{f_{2,1} - 0}{2\Delta x} \\ \frac{\partial f_{2,1}}{\partial x} &\approx \frac{f_{3,1} - f_{1,1}}{2\Delta x} \\ \frac{\partial f_{3,1}}{\partial x} &\approx \frac{f_{4,1} - f_{2,1}}{2\Delta x} \\ \frac{\partial f_{4,1}}{\partial x} &\approx \frac{0 - f_{3,1}}{2\Delta x} \\ \frac{\partial f_{1,2}}{\partial x} &\approx \frac{f_{2,2} - 0}{2\Delta x} \\ \frac{\partial f_{2,2}}{\partial x} &\approx \frac{f_{3,2} - f_{1,2}}{2\Delta x} \\ \frac{\partial f_{3,2}}{\partial x} &\approx \frac{f_{4,2} - f_{2,2}}{2\Delta x} \\ \frac{\partial f_{4,2}}{\partial x} &\approx \frac{0 - f_{3,2}}{2\Delta x} \\ \frac{\partial f_{1,3}}{\partial x} &\approx \frac{f_{2,3} - 0}{2\Delta x} \\ \frac{\partial f_{2,3}}{\partial x} &\approx \frac{f_{3,3} - f_{1,3}}{2\Delta x} \\ \frac{\partial f_{3,3}}{\partial x} &\approx \frac{f_{4,3} - f_{2,3}}{2\Delta x} \\ \frac{\partial f_{4,3}}{\partial x} &\approx \frac{0 - f_{3,3}}{2\Delta x} \end{aligned}$$



$[D_x]$ Derivative Matrix (5 of 14)

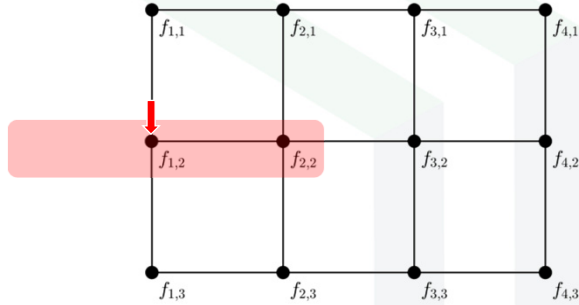


$$\frac{\partial f_{i,j}}{\partial x} \approx \frac{f_{i+1,j} - f_{i-1,j}}{2\Delta x}$$

$$\begin{aligned} \frac{\partial f_{1,1}}{\partial x} &\approx \frac{f_{2,1} - 0}{2\Delta x} \\ \frac{\partial f_{2,1}}{\partial x} &\approx \frac{f_{3,1} - f_{1,1}}{2\Delta x} \\ \frac{\partial f_{3,1}}{\partial x} &\approx \frac{f_{4,1} - f_{2,1}}{2\Delta x} \\ \frac{\partial f_{4,1}}{\partial x} &\approx \frac{0 - f_{3,1}}{2\Delta x} \\ \frac{\partial f_{1,2}}{\partial x} &\approx \frac{f_{2,2} - 0}{2\Delta x} \\ \frac{\partial f_{2,2}}{\partial x} &\approx \frac{f_{3,2} - f_{1,2}}{2\Delta x} \\ \frac{\partial f_{3,2}}{\partial x} &\approx \frac{f_{4,2} - f_{2,2}}{2\Delta x} \\ \frac{\partial f_{4,2}}{\partial x} &\approx \frac{0 - f_{3,2}}{2\Delta x} \\ \frac{\partial f_{1,3}}{\partial x} &\approx \frac{f_{2,3} - 0}{2\Delta x} \\ \frac{\partial f_{2,3}}{\partial x} &\approx \frac{f_{3,3} - f_{1,3}}{2\Delta x} \\ \frac{\partial f_{3,3}}{\partial x} &\approx \frac{f_{4,3} - f_{2,3}}{2\Delta x} \\ \frac{\partial f_{4,3}}{\partial x} &\approx \frac{0 - f_{3,3}}{2\Delta x} \end{aligned}$$



$[D_x]$ Derivative Matrix (6 of 14)

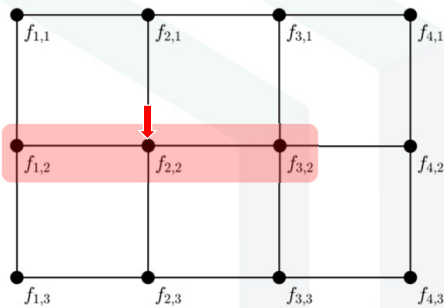


$$\frac{\partial f_{i,j}}{\partial x} \approx \frac{f_{i+1,j} - f_{i-1,j}}{2\Delta x}$$

$$\begin{aligned} \frac{\partial f_{1,1}}{\partial x} &\approx \frac{f_{2,1} - 0}{2\Delta x} \\ \frac{\partial f_{2,1}}{\partial x} &\approx \frac{f_{3,1} - f_{1,1}}{2\Delta x} \\ \frac{\partial f_{3,1}}{\partial x} &\approx \frac{f_{4,1} - f_{2,1}}{2\Delta x} \\ \frac{\partial f_{4,1}}{\partial x} &\approx \frac{0 - f_{3,1}}{2\Delta x} \\ \frac{\partial f_{1,2}}{\partial x} &\approx \frac{f_{2,2} - 0}{2\Delta x} \\ \frac{\partial f_{2,2}}{\partial x} &\approx \frac{f_{3,2} - f_{1,2}}{2\Delta x} \\ \frac{\partial f_{3,2}}{\partial x} &\approx \frac{f_{4,2} - f_{2,2}}{2\Delta x} \\ \frac{\partial f_{4,2}}{\partial x} &\approx \frac{0 - f_{3,2}}{2\Delta x} \\ \frac{\partial f_{1,3}}{\partial x} &\approx \frac{f_{2,3} - 0}{2\Delta x} \\ \frac{\partial f_{2,3}}{\partial x} &\approx \frac{f_{3,3} - f_{1,3}}{2\Delta x} \\ \frac{\partial f_{3,3}}{\partial x} &\approx \frac{f_{4,3} - f_{2,3}}{2\Delta x} \\ \frac{\partial f_{4,3}}{\partial x} &\approx \frac{0 - f_{3,3}}{2\Delta x} \end{aligned}$$



$[D_x]$ Derivative Matrix (7 of 14)

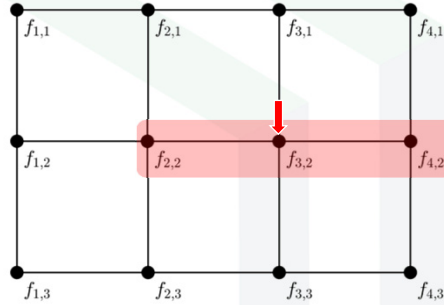


$$\frac{\partial f_{i,j}}{\partial x} \approx \frac{f_{i+1,j} - f_{i-1,j}}{2\Delta x}$$

$$\begin{aligned} \frac{\partial f_{1,1}}{\partial x} &\approx \frac{f_{2,1} - 0}{2\Delta x} \\ \frac{\partial f_{2,1}}{\partial x} &\approx \frac{f_{3,1} - f_{1,1}}{2\Delta x} \\ \frac{\partial f_{3,1}}{\partial x} &\approx \frac{f_{4,1} - f_{2,1}}{2\Delta x} \\ \frac{\partial f_{4,1}}{\partial x} &\approx \frac{0 - f_{3,1}}{2\Delta x} \\ \frac{\partial f_{1,2}}{\partial x} &\approx \frac{f_{2,2} - 0}{2\Delta x} \\ \frac{\partial f_{2,2}}{\partial x} &\approx \frac{f_{3,2} - f_{1,2}}{2\Delta x} \\ \frac{\partial f_{3,2}}{\partial x} &\approx \frac{f_{4,2} - f_{2,2}}{2\Delta x} \\ \frac{\partial f_{4,2}}{\partial x} &\approx \frac{0 - f_{3,2}}{2\Delta x} \\ \frac{\partial f_{1,3}}{\partial x} &\approx \frac{f_{2,3} - 0}{2\Delta x} \\ \frac{\partial f_{2,3}}{\partial x} &\approx \frac{f_{3,3} - f_{1,3}}{2\Delta x} \\ \frac{\partial f_{3,3}}{\partial x} &\approx \frac{f_{4,3} - f_{2,3}}{2\Delta x} \\ \frac{\partial f_{4,3}}{\partial x} &\approx \frac{0 - f_{3,3}}{2\Delta x} \end{aligned}$$



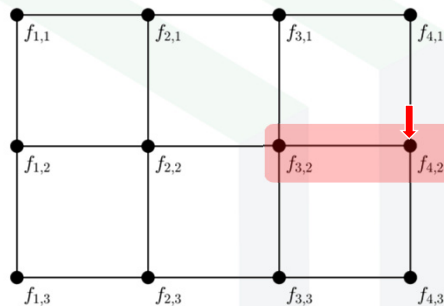
$[D_x]$ Derivative Matrix (8 of 14)



$$\frac{\partial f_{i,j}}{\partial x} \approx \frac{f_{i+1,j} - f_{i-1,j}}{2\Delta x}$$

$$\begin{aligned} \frac{\partial f_{1,1}}{\partial x} &\approx \frac{f_{2,1} - 0}{2\Delta x} \\ \frac{\partial f_{2,1}}{\partial x} &\approx \frac{f_{3,1} - f_{1,1}}{2\Delta x} \\ \frac{\partial f_{3,1}}{\partial x} &\approx \frac{f_{4,1} - f_{2,1}}{2\Delta x} \\ \frac{\partial f_{4,1}}{\partial x} &\approx \frac{0 - f_{3,1}}{2\Delta x} \\ \frac{\partial f_{1,2}}{\partial x} &\approx \frac{f_{2,2} - 0}{2\Delta x} \\ \frac{\partial f_{2,2}}{\partial x} &\approx \frac{f_{3,2} - f_{1,2}}{2\Delta x} \\ \frac{\partial f_{3,2}}{\partial x} &\approx \frac{f_{4,2} - f_{2,2}}{2\Delta x} \\ \frac{\partial f_{4,2}}{\partial x} &\approx \frac{0 - f_{3,2}}{2\Delta x} \\ \frac{\partial f_{1,3}}{\partial x} &\approx \frac{f_{2,3} - 0}{2\Delta x} \\ \frac{\partial f_{2,3}}{\partial x} &\approx \frac{f_{3,3} - f_{1,3}}{2\Delta x} \\ \frac{\partial f_{3,3}}{\partial x} &\approx \frac{f_{4,3} - f_{2,3}}{2\Delta x} \\ \frac{\partial f_{4,3}}{\partial x} &\approx \frac{0 - f_{3,3}}{2\Delta x} \end{aligned}$$

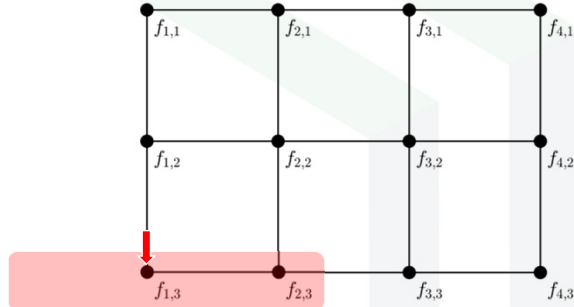
$[D_x]$ Derivative Matrix (9 of 14)



$$\frac{\partial f_{i,j}}{\partial x} \approx \frac{f_{i+1,j} - f_{i-1,j}}{2\Delta x}$$

$$\begin{aligned} \frac{\partial f_{1,1}}{\partial x} &\approx \frac{f_{2,1} - 0}{2\Delta x} \\ \frac{\partial f_{2,1}}{\partial x} &\approx \frac{f_{3,1} - f_{1,1}}{2\Delta x} \\ \frac{\partial f_{3,1}}{\partial x} &\approx \frac{f_{4,1} - f_{2,1}}{2\Delta x} \\ \frac{\partial f_{4,1}}{\partial x} &\approx \frac{0 - f_{3,1}}{2\Delta x} \\ \frac{\partial f_{1,2}}{\partial x} &\approx \frac{f_{2,2} - 0}{2\Delta x} \\ \frac{\partial f_{2,2}}{\partial x} &\approx \frac{f_{3,2} - f_{1,2}}{2\Delta x} \\ \frac{\partial f_{3,2}}{\partial x} &\approx \frac{f_{4,2} - f_{2,2}}{2\Delta x} \\ \frac{\partial f_{4,2}}{\partial x} &\approx \frac{0 - f_{3,2}}{2\Delta x} \\ \frac{\partial f_{1,3}}{\partial x} &\approx \frac{f_{2,3} - 0}{2\Delta x} \\ \frac{\partial f_{2,3}}{\partial x} &\approx \frac{f_{3,3} - f_{1,3}}{2\Delta x} \\ \frac{\partial f_{3,3}}{\partial x} &\approx \frac{f_{4,3} - f_{2,3}}{2\Delta x} \\ \frac{\partial f_{4,3}}{\partial x} &\approx \frac{0 - f_{3,3}}{2\Delta x} \end{aligned}$$

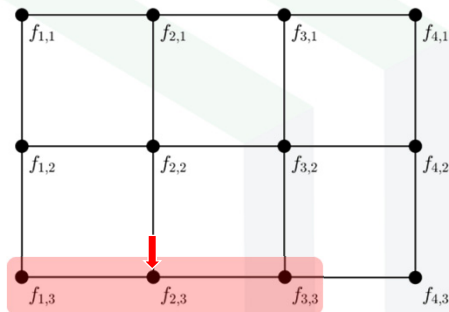
$[D_x]$ Derivative Matrix (10 of 14)



$$\frac{\partial f_{i,j}}{\partial x} \approx \frac{f_{i+1,j} - f_{i-1,j}}{2\Delta x}$$

$$\begin{aligned} \frac{\partial f_{1,1}}{\partial x} &\approx \frac{f_{2,1} - 0}{2\Delta x} \\ \frac{\partial f_{2,1}}{\partial x} &\approx \frac{f_{3,1} - f_{1,1}}{2\Delta x} \\ \frac{\partial f_{3,1}}{\partial x} &\approx \frac{f_{4,1} - f_{2,1}}{2\Delta x} \\ \frac{\partial f_{4,1}}{\partial x} &\approx \frac{0 - f_{3,1}}{2\Delta x} \\ \frac{\partial f_{1,2}}{\partial x} &\approx \frac{f_{2,2} - 0}{2\Delta x} \\ \frac{\partial f_{2,2}}{\partial x} &\approx \frac{f_{3,2} - f_{1,2}}{2\Delta x} \\ \frac{\partial f_{3,2}}{\partial x} &\approx \frac{f_{4,2} - f_{2,2}}{2\Delta x} \\ \frac{\partial f_{4,2}}{\partial x} &\approx \frac{0 - f_{3,2}}{2\Delta x} \\ \frac{\partial f_{1,3}}{\partial x} &\approx \frac{f_{2,3} - 0}{2\Delta x} \\ \frac{\partial f_{2,3}}{\partial x} &\approx \frac{f_{3,3} - f_{1,3}}{2\Delta x} \\ \frac{\partial f_{3,3}}{\partial x} &\approx \frac{f_{4,3} - f_{2,3}}{2\Delta x} \\ \frac{\partial f_{4,3}}{\partial x} &\approx \frac{0 - f_{3,3}}{2\Delta x} \end{aligned}$$

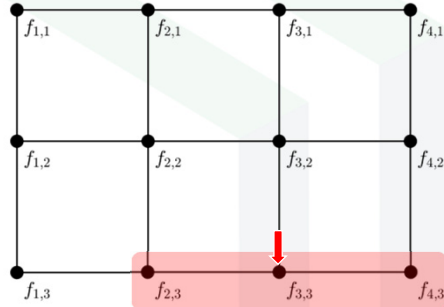
$[D_x]$ Derivative Matrix (11 of 14)



$$\frac{\partial f_{i,j}}{\partial x} \approx \frac{f_{i+1,j} - f_{i-1,j}}{2\Delta x}$$

$$\begin{aligned} \frac{\partial f_{1,1}}{\partial x} &\approx \frac{f_{2,1} - 0}{2\Delta x} \\ \frac{\partial f_{2,1}}{\partial x} &\approx \frac{f_{3,1} - f_{1,1}}{2\Delta x} \\ \frac{\partial f_{3,1}}{\partial x} &\approx \frac{f_{4,1} - f_{2,1}}{2\Delta x} \\ \frac{\partial f_{4,1}}{\partial x} &\approx \frac{0 - f_{3,1}}{2\Delta x} \\ \frac{\partial f_{1,2}}{\partial x} &\approx \frac{f_{2,2} - 0}{2\Delta x} \\ \frac{\partial f_{2,2}}{\partial x} &\approx \frac{f_{3,2} - f_{1,2}}{2\Delta x} \\ \frac{\partial f_{3,2}}{\partial x} &\approx \frac{f_{4,2} - f_{2,2}}{2\Delta x} \\ \frac{\partial f_{4,2}}{\partial x} &\approx \frac{0 - f_{3,2}}{2\Delta x} \\ \frac{\partial f_{1,3}}{\partial x} &\approx \frac{f_{2,3} - 0}{2\Delta x} \\ \frac{\partial f_{2,3}}{\partial x} &\approx \frac{f_{3,3} - f_{1,3}}{2\Delta x} \\ \frac{\partial f_{3,3}}{\partial x} &\approx \frac{f_{4,3} - f_{2,3}}{2\Delta x} \\ \frac{\partial f_{4,3}}{\partial x} &\approx \frac{0 - f_{3,3}}{2\Delta x} \end{aligned}$$

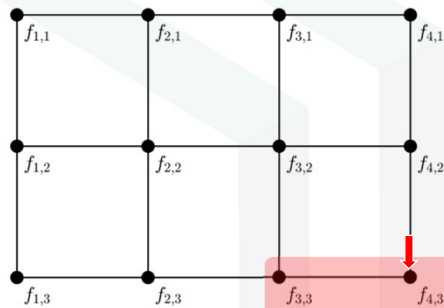
$[D_x]$ Derivative Matrix (12 of 14)



$$\frac{\partial f_{i,j}}{\partial x} \approx \frac{f_{i+1,j} - f_{i-1,j}}{2\Delta x}$$

$$\begin{aligned} \frac{\partial f_{1,1}}{\partial x} &\approx \frac{f_{2,1} - 0}{2\Delta x} \\ \frac{\partial f_{2,1}}{\partial x} &\approx \frac{f_{3,1} - f_{1,1}}{2\Delta x} \\ \frac{\partial f_{3,1}}{\partial x} &\approx \frac{f_{4,1} - f_{2,1}}{2\Delta x} \\ \frac{\partial f_{4,1}}{\partial x} &\approx \frac{0 - f_{3,1}}{2\Delta x} \\ \frac{\partial f_{1,2}}{\partial x} &\approx \frac{f_{2,2} - 0}{2\Delta x} \\ \frac{\partial f_{2,2}}{\partial x} &\approx \frac{f_{3,2} - f_{1,2}}{2\Delta x} \\ \frac{\partial f_{3,2}}{\partial x} &\approx \frac{f_{4,2} - f_{2,2}}{2\Delta x} \\ \frac{\partial f_{4,2}}{\partial x} &\approx \frac{0 - f_{3,2}}{2\Delta x} \\ \frac{\partial f_{1,3}}{\partial x} &\approx \frac{f_{2,3} - 0}{2\Delta x} \\ \frac{\partial f_{2,3}}{\partial x} &\approx \frac{f_{3,3} - f_{1,3}}{2\Delta x} \\ \frac{\partial f_{3,3}}{\partial x} &\approx \frac{f_{4,3} - f_{2,3}}{2\Delta x} \\ \frac{\partial f_{4,3}}{\partial x} &\approx \frac{0 - f_{3,3}}{2\Delta x} \end{aligned}$$

$[D_x]$ Derivative Matrix (13 of 14)



$$\frac{\partial f_{i,j}}{\partial x} \approx \frac{f_{i+1,j} - f_{i-1,j}}{2\Delta x}$$

$$\begin{aligned} \frac{\partial f_{1,1}}{\partial x} &\approx \frac{f_{2,1} - 0}{2\Delta x} \\ \frac{\partial f_{2,1}}{\partial x} &\approx \frac{f_{3,1} - f_{1,1}}{2\Delta x} \\ \frac{\partial f_{3,1}}{\partial x} &\approx \frac{f_{4,1} - f_{2,1}}{2\Delta x} \\ \frac{\partial f_{4,1}}{\partial x} &\approx \frac{0 - f_{3,1}}{2\Delta x} \\ \frac{\partial f_{1,2}}{\partial x} &\approx \frac{f_{2,2} - 0}{2\Delta x} \\ \frac{\partial f_{2,2}}{\partial x} &\approx \frac{f_{3,2} - f_{1,2}}{2\Delta x} \\ \frac{\partial f_{3,2}}{\partial x} &\approx \frac{f_{4,2} - f_{2,2}}{2\Delta x} \\ \frac{\partial f_{4,2}}{\partial x} &\approx \frac{0 - f_{3,2}}{2\Delta x} \\ \frac{\partial f_{1,3}}{\partial x} &\approx \frac{f_{2,3} - 0}{2\Delta x} \\ \frac{\partial f_{2,3}}{\partial x} &\approx \frac{f_{3,3} - f_{1,3}}{2\Delta x} \\ \frac{\partial f_{3,3}}{\partial x} &\approx \frac{f_{4,3} - f_{2,3}}{2\Delta x} \\ \frac{\partial f_{4,3}}{\partial x} &\approx \frac{0 - f_{3,3}}{2\Delta x} \end{aligned}$$

$[D_x]$ Derivative Matrix (14 of 14)

$$\frac{1}{2\Delta x} \begin{bmatrix} f_{1,1} \\ f_{2,1} \\ f_{3,1} \\ f_{4,1} \\ f_{1,2} \\ f_{2,2} \\ f_{3,2} \\ f_{4,2} \\ f_{1,3} \\ f_{2,3} \\ f_{3,3} \\ f_{4,3} \end{bmatrix} = \frac{1}{2\Delta x} \begin{bmatrix} f_{2,1} - 0 \\ f_{3,1} - f_{1,1} \\ f_{4,1} - f_{2,1} \\ 0 - f_{3,1} \\ f_{2,2} - 0 \\ f_{3,2} - f_{1,2} \\ f_{4,2} - f_{2,2} \\ 0 - f_{3,2} \\ f_{2,3} - 0 \\ f_{3,3} - f_{1,3} \\ f_{4,3} - f_{2,3} \\ 0 - f_{3,3} \end{bmatrix}$$

$$\begin{aligned} \frac{\partial f_{1,1}}{\partial x} &\equiv \frac{f_{2,1} - 0}{2\Delta x} \\ \frac{\partial f_{2,1}}{\partial x} &\equiv \frac{f_{3,1} - f_{1,1}}{2\Delta x} \\ \frac{\partial f_{3,1}}{\partial x} &\equiv \frac{f_{4,1} - f_{2,1}}{2\Delta x} \\ \frac{\partial f_{4,1}}{\partial x} &\equiv \frac{0 - f_{3,1}}{2\Delta x} \\ \frac{\partial f_{1,2}}{\partial x} &\equiv \frac{f_{2,2} - 0}{2\Delta x} \\ \frac{\partial f_{2,2}}{\partial x} &\equiv \frac{f_{3,2} - f_{1,2}}{2\Delta x} \\ \frac{\partial f_{3,2}}{\partial x} &\equiv \frac{f_{4,2} - f_{2,2}}{2\Delta x} \\ \frac{\partial f_{4,2}}{\partial x} &\equiv \frac{0 - f_{3,2}}{2\Delta x} \\ \frac{\partial f_{1,3}}{\partial x} &\equiv \frac{f_{2,3} - 0}{2\Delta x} \\ \frac{\partial f_{2,3}}{\partial x} &\equiv \frac{f_{3,3} - f_{1,3}}{2\Delta x} \\ \frac{\partial f_{3,3}}{\partial x} &\equiv \frac{f_{4,3} - f_{2,3}}{2\Delta x} \\ \frac{\partial f_{4,3}}{\partial x} &\equiv \frac{0 - f_{3,3}}{2\Delta x} \end{aligned}$$

$[D_x]$ Derivative Matrix (14 of 14)

$$\frac{1}{2\Delta x} \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} f_{1,1} \\ f_{2,1} \\ f_{3,1} \\ f_{4,1} \\ f_{1,2} \\ f_{2,2} \\ f_{3,2} \\ f_{4,2} \\ f_{1,3} \\ f_{2,3} \\ f_{3,3} \\ f_{4,3} \end{bmatrix} = \frac{1}{2\Delta x} \begin{bmatrix} f_{2,1} - 0 \\ f_{3,1} - f_{1,1} \\ f_{4,1} - f_{2,1} \\ 0 - f_{3,1} \\ f_{2,2} - 0 \\ f_{3,2} - f_{1,2} \\ f_{4,2} - f_{2,2} \\ 0 - f_{3,2} \\ f_{2,3} - 0 \\ f_{3,3} - f_{1,3} \\ f_{4,3} - f_{2,3} \\ 0 - f_{3,3} \end{bmatrix}$$

$[D_x]$

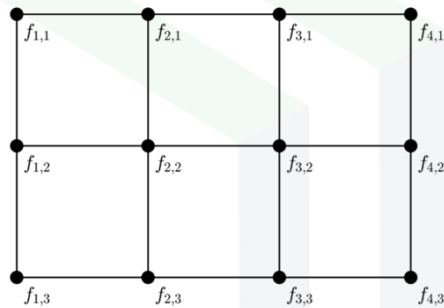
$$\begin{aligned} \frac{\partial f_{1,1}}{\partial x} &\equiv \frac{f_{2,1} - 0}{2\Delta x} \\ \frac{\partial f_{2,1}}{\partial x} &\equiv \frac{f_{3,1} - f_{1,1}}{2\Delta x} \\ \frac{\partial f_{3,1}}{\partial x} &\equiv \frac{f_{4,1} - f_{2,1}}{2\Delta x} \\ \frac{\partial f_{4,1}}{\partial x} &\equiv \frac{0 - f_{3,1}}{2\Delta x} \\ \frac{\partial f_{1,2}}{\partial x} &\equiv \frac{f_{2,2} - 0}{2\Delta x} \\ \frac{\partial f_{2,2}}{\partial x} &\equiv \frac{f_{3,2} - f_{1,2}}{2\Delta x} \\ \frac{\partial f_{3,2}}{\partial x} &\equiv \frac{f_{4,2} - f_{2,2}}{2\Delta x} \\ \frac{\partial f_{4,2}}{\partial x} &\equiv \frac{0 - f_{3,2}}{2\Delta x} \\ \frac{\partial f_{1,3}}{\partial x} &\equiv \frac{f_{2,3} - 0}{2\Delta x} \\ \frac{\partial f_{2,3}}{\partial x} &\equiv \frac{f_{3,3} - f_{1,3}}{2\Delta x} \\ \frac{\partial f_{3,3}}{\partial x} &\equiv \frac{f_{4,3} - f_{2,3}}{2\Delta x} \\ \frac{\partial f_{4,3}}{\partial x} &\equiv \frac{0 - f_{3,3}}{2\Delta x} \end{aligned}$$

Collocated Derivative Matrix

$$\left[D_x^2 \right]$$



$\left[D_x^2 \right]$ Derivative Matrix (1 of 14)



$$\frac{\partial^2 f_{i,j}}{\partial x^2} \cong \frac{f_{i+1,j} - 2f_{i,j} + f_{i-1,j}}{(\Delta x)^2}$$

$$\frac{\partial^2 f_{1,1}}{\partial x^2} \cong \frac{0 - 2f_{1,1} + f_{2,1}}{(\Delta x)^2}$$

$$\frac{\partial^2 f_{2,1}}{\partial x^2} \cong \frac{f_{1,1} - 2f_{2,1} + f_{3,1}}{(\Delta x)^2}$$

$$\frac{\partial^2 f_{3,1}}{\partial x^2} \cong \frac{f_{2,1} - 2f_{3,1} + f_{4,1}}{(\Delta x)^2}$$

$$\frac{\partial^2 f_{4,1}}{\partial x^2} \cong \frac{f_{3,1} - 2f_{4,1} + 0}{(\Delta x)^2}$$

$$\frac{\partial^2 f_{1,2}}{\partial x^2} \cong \frac{0 - 2f_{1,2} + f_{2,2}}{(\Delta x)^2}$$

$$\frac{\partial^2 f_{2,2}}{\partial x^2} \cong \frac{f_{1,2} - 2f_{2,2} + f_{3,2}}{(\Delta x)^2}$$

$$\frac{\partial^2 f_{3,2}}{\partial x^2} \cong \frac{f_{2,2} - 2f_{3,2} + f_{4,2}}{(\Delta x)^2}$$

$$\frac{\partial^2 f_{4,2}}{\partial x^2} \cong \frac{f_{3,2} - 2f_{4,2} + 0}{(\Delta x)^2}$$

$$\frac{\partial^2 f_{1,3}}{\partial x^2} \cong \frac{0 - 2f_{1,3} + f_{2,3}}{(\Delta x)^2}$$

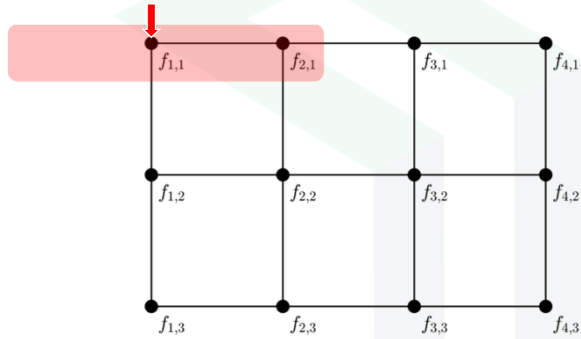
$$\frac{\partial^2 f_{2,3}}{\partial x^2} \cong \frac{f_{1,3} - 2f_{2,3} + f_{3,3}}{(\Delta x)^2}$$

$$\frac{\partial^2 f_{3,3}}{\partial x^2} \cong \frac{f_{2,3} - 2f_{3,3} + f_{4,3}}{(\Delta x)^2}$$

$$\frac{\partial^2 f_{4,3}}{\partial x^2} \cong \frac{f_{3,3} - 2f_{4,3} + 0}{(\Delta x)^2}$$



$[D_x^2]$ Derivative Matrix (2 of 14)



$$\frac{\partial^2 f_{i,j}}{\partial x^2} \cong \frac{f_{i+1,j} - 2f_{i,j} + f_{i-1,j}}{(\Delta x)^2}$$

$$\frac{\partial^2 f_{1,1}}{\partial x^2} \cong \frac{0 - 2f_{1,1} + f_{2,1}}{(\Delta x)^2}$$

$$\frac{\partial^2 f_{2,1}}{\partial x^2} \cong \frac{f_{1,1} - 2f_{2,1} + f_{3,1}}{(\Delta x)^2}$$

$$\frac{\partial^2 f_{3,1}}{\partial x^2} \cong \frac{f_{2,1} - 2f_{3,1} + f_{4,1}}{(\Delta x)^2}$$

$$\frac{\partial^2 f_{4,1}}{\partial x^2} \cong \frac{f_{3,1} - 2f_{4,1} + 0}{(\Delta x)^2}$$

$$\frac{\partial^2 f_{1,2}}{\partial x^2} \cong \frac{0 - 2f_{1,2} + f_{2,2}}{(\Delta x)^2}$$

$$\frac{\partial^2 f_{2,2}}{\partial x^2} \cong \frac{f_{1,2} - 2f_{2,2} + f_{3,2}}{(\Delta x)^2}$$

$$\frac{\partial^2 f_{3,2}}{\partial x^2} \cong \frac{f_{2,2} - 2f_{3,2} + f_{4,2}}{(\Delta x)^2}$$

$$\frac{\partial^2 f_{4,2}}{\partial x^2} \cong \frac{f_{3,2} - 2f_{4,2} + 0}{(\Delta x)^2}$$

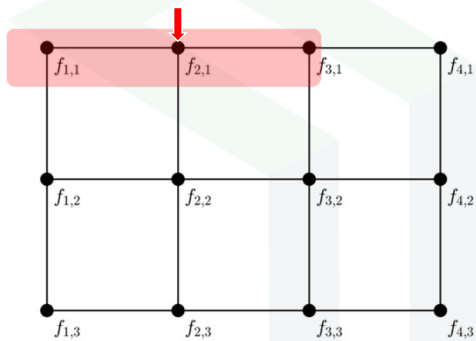
$$\frac{\partial^2 f_{1,3}}{\partial x^2} \cong \frac{0 - 2f_{1,3} + f_{2,3}}{(\Delta x)^2}$$

$$\frac{\partial^2 f_{2,3}}{\partial x^2} \cong \frac{f_{1,3} - 2f_{2,3} + f_{3,3}}{(\Delta x)^2}$$

$$\frac{\partial^2 f_{3,3}}{\partial x^2} \cong \frac{f_{2,3} - 2f_{3,3} + f_{4,3}}{(\Delta x)^2}$$

$$\frac{\partial^2 f_{4,3}}{\partial x^2} \cong \frac{f_{3,3} - 2f_{4,3} + 0}{(\Delta x)^2}$$

$[D_x^2]$ Derivative Matrix (3 of 14)



$$\frac{\partial^2 f_{i,j}}{\partial x^2} \cong \frac{f_{i+1,j} - 2f_{i,j} + f_{i-1,j}}{(\Delta x)^2}$$

$$\frac{\partial^2 f_{1,1}}{\partial x^2} \cong \frac{0 - 2f_{1,1} + f_{2,1}}{(\Delta x)^2}$$

$$\frac{\partial^2 f_{2,1}}{\partial x^2} \cong \frac{f_{1,1} - 2f_{2,1} + f_{3,1}}{(\Delta x)^2}$$

$$\frac{\partial^2 f_{3,1}}{\partial x^2} \cong \frac{f_{2,1} - 2f_{3,1} + f_{4,1}}{(\Delta x)^2}$$

$$\frac{\partial^2 f_{4,1}}{\partial x^2} \cong \frac{f_{3,1} - 2f_{4,1} + 0}{(\Delta x)^2}$$

$$\frac{\partial^2 f_{1,2}}{\partial x^2} \cong \frac{0 - 2f_{1,2} + f_{2,2}}{(\Delta x)^2}$$

$$\frac{\partial^2 f_{2,2}}{\partial x^2} \cong \frac{f_{1,2} - 2f_{2,2} + f_{3,2}}{(\Delta x)^2}$$

$$\frac{\partial^2 f_{3,2}}{\partial x^2} \cong \frac{f_{2,2} - 2f_{3,2} + f_{4,2}}{(\Delta x)^2}$$

$$\frac{\partial^2 f_{4,2}}{\partial x^2} \cong \frac{f_{3,2} - 2f_{4,2} + 0}{(\Delta x)^2}$$

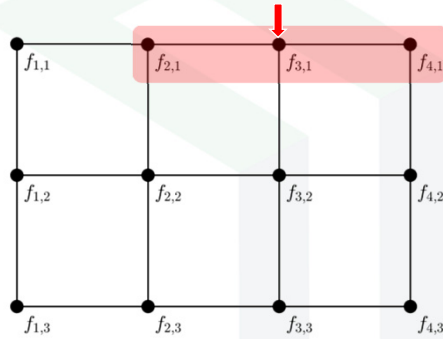
$$\frac{\partial^2 f_{1,3}}{\partial x^2} \cong \frac{0 - 2f_{1,3} + f_{2,3}}{(\Delta x)^2}$$

$$\frac{\partial^2 f_{2,3}}{\partial x^2} \cong \frac{f_{1,3} - 2f_{2,3} + f_{3,3}}{(\Delta x)^2}$$

$$\frac{\partial^2 f_{3,3}}{\partial x^2} \cong \frac{f_{2,3} - 2f_{3,3} + f_{4,3}}{(\Delta x)^2}$$

$$\frac{\partial^2 f_{4,3}}{\partial x^2} \cong \frac{f_{3,3} - 2f_{4,3} + 0}{(\Delta x)^2}$$

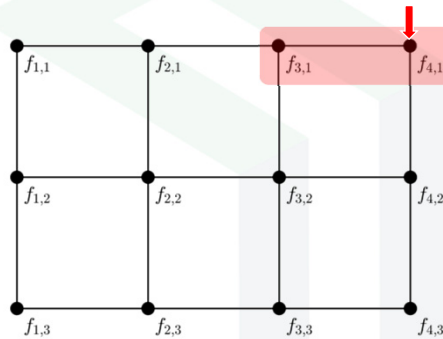
$[D_x^2]$ Derivative Matrix (4 of 14)



$$\frac{\partial^2 f_{i,j}}{\partial x^2} \cong \frac{f_{i+1,j} - 2f_{i,j} + f_{i-1,j}}{(\Delta x)^2}$$

$$\begin{aligned} \frac{\partial^2 f_{1,1}}{\partial x^2} &\cong \frac{0 - 2f_{1,1} + f_{2,1}}{(\Delta x)^2} \\ \frac{\partial^2 f_{2,1}}{\partial x^2} &\cong \frac{f_{1,1} - 2f_{2,1} + f_{3,1}}{(\Delta x)^2} \\ \frac{\partial^2 f_{3,1}}{\partial x^2} &\cong \frac{f_{2,1} - 2f_{3,1} + f_{4,1}}{(\Delta x)^2} \\ \frac{\partial^2 f_{4,1}}{\partial x^2} &\cong \frac{f_{3,1} - 2f_{4,1} + 0}{(\Delta x)^2} \\ \frac{\partial^2 f_{1,2}}{\partial x^2} &\cong \frac{0 - 2f_{1,2} + f_{2,2}}{(\Delta x)^2} \\ \frac{\partial^2 f_{2,2}}{\partial x^2} &\cong \frac{f_{1,2} - 2f_{2,2} + f_{3,2}}{(\Delta x)^2} \\ \frac{\partial^2 f_{3,2}}{\partial x^2} &\cong \frac{f_{2,2} - 2f_{3,2} + f_{4,2}}{(\Delta x)^2} \\ \frac{\partial^2 f_{4,2}}{\partial x^2} &\cong \frac{f_{3,2} - 2f_{4,2} + 0}{(\Delta x)^2} \\ \frac{\partial^2 f_{1,3}}{\partial x^2} &\cong \frac{0 - 2f_{1,3} + f_{2,3}}{(\Delta x)^2} \\ \frac{\partial^2 f_{2,3}}{\partial x^2} &\cong \frac{f_{1,3} - 2f_{2,3} + f_{3,3}}{(\Delta x)^2} \\ \frac{\partial^2 f_{3,3}}{\partial x^2} &\cong \frac{f_{2,3} - 2f_{3,3} + f_{4,3}}{(\Delta x)^2} \\ \frac{\partial^2 f_{4,3}}{\partial x^2} &\cong \frac{f_{3,3} - 2f_{4,3} + 0}{(\Delta x)^2} \end{aligned}$$

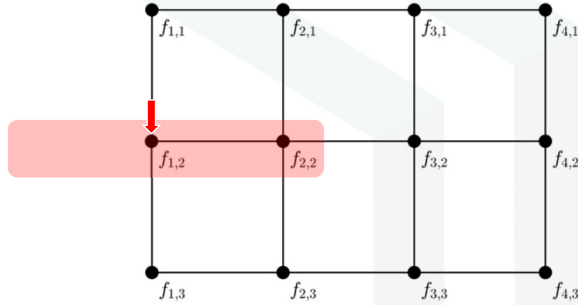
$[D_x^2]$ Derivative Matrix (5 of 14)



$$\frac{\partial^2 f_{i,j}}{\partial x^2} \cong \frac{f_{i+1,j} - 2f_{i,j} + f_{i-1,j}}{(\Delta x)^2}$$

$$\begin{aligned} \frac{\partial^2 f_{1,1}}{\partial x^2} &\cong \frac{0 - 2f_{1,1} + f_{2,1}}{(\Delta x)^2} \\ \frac{\partial^2 f_{2,1}}{\partial x^2} &\cong \frac{f_{1,1} - 2f_{2,1} + f_{3,1}}{(\Delta x)^2} \\ \frac{\partial^2 f_{3,1}}{\partial x^2} &\cong \frac{f_{2,1} - 2f_{3,1} + f_{4,1}}{(\Delta x)^2} \\ \frac{\partial^2 f_{4,1}}{\partial x^2} &\cong \frac{f_{3,1} - 2f_{4,1} + 0}{(\Delta x)^2} \\ \frac{\partial^2 f_{1,2}}{\partial x^2} &\cong \frac{0 - 2f_{1,2} + f_{2,2}}{(\Delta x)^2} \\ \frac{\partial^2 f_{2,2}}{\partial x^2} &\cong \frac{f_{1,2} - 2f_{2,2} + f_{3,2}}{(\Delta x)^2} \\ \frac{\partial^2 f_{3,2}}{\partial x^2} &\cong \frac{f_{2,2} - 2f_{3,2} + f_{4,2}}{(\Delta x)^2} \\ \frac{\partial^2 f_{4,2}}{\partial x^2} &\cong \frac{f_{3,2} - 2f_{4,2} + 0}{(\Delta x)^2} \\ \frac{\partial^2 f_{1,3}}{\partial x^2} &\cong \frac{0 - 2f_{1,3} + f_{2,3}}{(\Delta x)^2} \\ \frac{\partial^2 f_{2,3}}{\partial x^2} &\cong \frac{f_{1,3} - 2f_{2,3} + f_{3,3}}{(\Delta x)^2} \\ \frac{\partial^2 f_{3,3}}{\partial x^2} &\cong \frac{f_{2,3} - 2f_{3,3} + f_{4,3}}{(\Delta x)^2} \\ \frac{\partial^2 f_{4,3}}{\partial x^2} &\cong \frac{f_{3,3} - 2f_{4,3} + 0}{(\Delta x)^2} \end{aligned}$$

$[D_x^2]$ Derivative Matrix (6 of 14)



$$\frac{\partial^2 f_{i,j}}{\partial x^2} \cong \frac{f_{i+1,j} - 2f_{i,j} + f_{i-1,j}}{(\Delta x)^2}$$

$$\frac{\partial^2 f_{1,1}}{\partial x^2} \cong \frac{0 - 2f_{1,1} + f_{2,1}}{(\Delta x)^2}$$

$$\frac{\partial^2 f_{2,1}}{\partial x^2} \cong \frac{f_{1,1} - 2f_{2,1} + f_{3,1}}{(\Delta x)^2}$$

$$\frac{\partial^2 f_{3,1}}{\partial x^2} \cong \frac{f_{2,1} - 2f_{3,1} + f_{4,1}}{(\Delta x)^2}$$

$$\frac{\partial^2 f_{4,1}}{\partial x^2} \cong \frac{f_{3,1} - 2f_{4,1} + 0}{(\Delta x)^2}$$

$$\frac{\partial^2 f_{1,2}}{\partial x^2} \cong \frac{0 - 2f_{1,2} + f_{2,2}}{(\Delta x)^2}$$

$$\frac{\partial^2 f_{2,2}}{\partial x^2} \cong \frac{f_{1,2} - 2f_{2,2} + f_{3,2}}{(\Delta x)^2}$$

$$\frac{\partial^2 f_{3,2}}{\partial x^2} \cong \frac{f_{2,2} - 2f_{3,2} + f_{4,2}}{(\Delta x)^2}$$

$$\frac{\partial^2 f_{4,2}}{\partial x^2} \cong \frac{f_{3,2} - 2f_{4,2} + 0}{(\Delta x)^2}$$

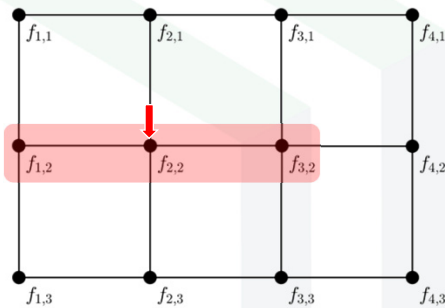
$$\frac{\partial^2 f_{1,3}}{\partial x^2} \cong \frac{0 - 2f_{1,3} + f_{2,3}}{(\Delta x)^2}$$

$$\frac{\partial^2 f_{2,3}}{\partial x^2} \cong \frac{f_{1,3} - 2f_{2,3} + f_{3,3}}{(\Delta x)^2}$$

$$\frac{\partial^2 f_{3,3}}{\partial x^2} \cong \frac{f_{2,3} - 2f_{3,3} + f_{4,3}}{(\Delta x)^2}$$

$$\frac{\partial^2 f_{4,3}}{\partial x^2} \cong \frac{f_{3,3} - 2f_{4,3} + 0}{(\Delta x)^2}$$

$[D_x^2]$ Derivative Matrix (7 of 14)



$$\frac{\partial^2 f_{i,j}}{\partial x^2} \cong \frac{f_{i+1,j} - 2f_{i,j} + f_{i-1,j}}{(\Delta x)^2}$$

$$\frac{\partial^2 f_{1,1}}{\partial x^2} \cong \frac{0 - 2f_{1,1} + f_{2,1}}{(\Delta x)^2}$$

$$\frac{\partial^2 f_{2,1}}{\partial x^2} \cong \frac{f_{1,1} - 2f_{2,1} + f_{3,1}}{(\Delta x)^2}$$

$$\frac{\partial^2 f_{3,1}}{\partial x^2} \cong \frac{f_{2,1} - 2f_{3,1} + f_{4,1}}{(\Delta x)^2}$$

$$\frac{\partial^2 f_{4,1}}{\partial x^2} \cong \frac{f_{3,1} - 2f_{4,1} + 0}{(\Delta x)^2}$$

$$\frac{\partial^2 f_{1,2}}{\partial x^2} \cong \frac{0 - 2f_{1,2} + f_{2,2}}{(\Delta x)^2}$$

$$\frac{\partial^2 f_{2,2}}{\partial x^2} \cong \frac{f_{1,2} - 2f_{2,2} + f_{3,2}}{(\Delta x)^2}$$

$$\frac{\partial^2 f_{3,2}}{\partial x^2} \cong \frac{f_{2,2} - 2f_{3,2} + f_{4,2}}{(\Delta x)^2}$$

$$\frac{\partial^2 f_{4,2}}{\partial x^2} \cong \frac{f_{3,2} - 2f_{4,2} + 0}{(\Delta x)^2}$$

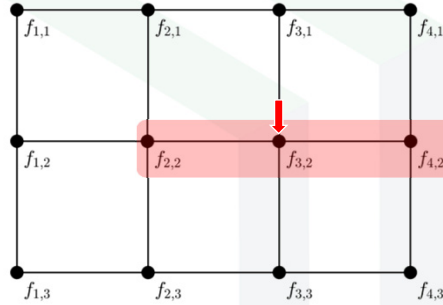
$$\frac{\partial^2 f_{1,3}}{\partial x^2} \cong \frac{0 - 2f_{1,3} + f_{2,3}}{(\Delta x)^2}$$

$$\frac{\partial^2 f_{2,3}}{\partial x^2} \cong \frac{f_{1,3} - 2f_{2,3} + f_{3,3}}{(\Delta x)^2}$$

$$\frac{\partial^2 f_{3,3}}{\partial x^2} \cong \frac{f_{2,3} - 2f_{3,3} + f_{4,3}}{(\Delta x)^2}$$

$$\frac{\partial^2 f_{4,3}}{\partial x^2} \cong \frac{f_{3,3} - 2f_{4,3} + 0}{(\Delta x)^2}$$

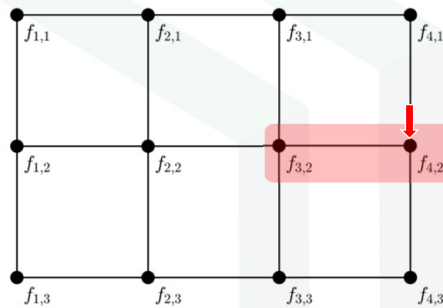
$[D_x^2]$ Derivative Matrix (8 of 14)



$$\frac{\partial^2 f_{i,j}}{\partial x^2} \cong \frac{f_{i+1,j} - 2f_{i,j} + f_{i-1,j}}{(\Delta x)^2}$$

$$\begin{aligned} \frac{\partial^2 f_{1,1}}{\partial x^2} &\cong \frac{0 - 2f_{1,1} + f_{2,1}}{(\Delta x)^2} \\ \frac{\partial^2 f_{2,1}}{\partial x^2} &\cong \frac{f_{1,1} - 2f_{2,1} + f_{3,1}}{(\Delta x)^2} \\ \frac{\partial^2 f_{3,1}}{\partial x^2} &\cong \frac{f_{2,1} - 2f_{3,1} + f_{4,1}}{(\Delta x)^2} \\ \frac{\partial^2 f_{4,1}}{\partial x^2} &\cong \frac{f_{3,1} - 2f_{4,1} + 0}{(\Delta x)^2} \\ \frac{\partial^2 f_{1,2}}{\partial x^2} &\cong \frac{0 - 2f_{1,2} + f_{2,2}}{(\Delta x)^2} \\ \frac{\partial^2 f_{2,2}}{\partial x^2} &\cong \frac{f_{1,2} - 2f_{2,2} + f_{3,2}}{(\Delta x)^2} \\ \frac{\partial^2 f_{3,2}}{\partial x^2} &\cong \frac{f_{2,2} - 2f_{3,2} + f_{4,2}}{(\Delta x)^2} \\ \frac{\partial^2 f_{4,2}}{\partial x^2} &\cong \frac{f_{3,2} - 2f_{4,2} + 0}{(\Delta x)^2} \\ \frac{\partial^2 f_{1,3}}{\partial x^2} &\cong \frac{0 - 2f_{1,3} + f_{2,3}}{(\Delta x)^2} \\ \frac{\partial^2 f_{2,3}}{\partial x^2} &\cong \frac{f_{1,3} - 2f_{2,3} + f_{3,3}}{(\Delta x)^2} \\ \frac{\partial^2 f_{3,3}}{\partial x^2} &\cong \frac{f_{2,3} - 2f_{3,3} + f_{4,3}}{(\Delta x)^2} \\ \frac{\partial^2 f_{4,3}}{\partial x^2} &\cong \frac{f_{3,3} - 2f_{4,3} + 0}{(\Delta x)^2} \end{aligned}$$

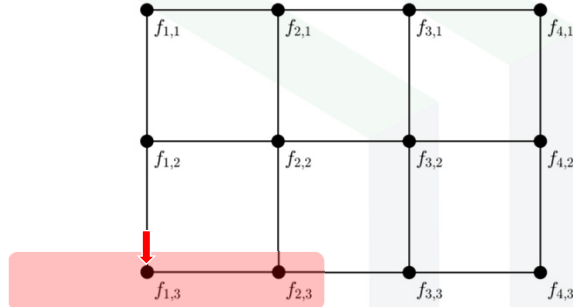
$[D_x^2]$ Derivative Matrix (9 of 14)



$$\frac{\partial^2 f_{i,j}}{\partial x^2} \cong \frac{f_{i+1,j} - 2f_{i,j} + f_{i-1,j}}{(\Delta x)^2}$$

$$\begin{aligned} \frac{\partial^2 f_{1,1}}{\partial x^2} &\cong \frac{0 - 2f_{1,1} + f_{2,1}}{(\Delta x)^2} \\ \frac{\partial^2 f_{2,1}}{\partial x^2} &\cong \frac{f_{1,1} - 2f_{2,1} + f_{3,1}}{(\Delta x)^2} \\ \frac{\partial^2 f_{3,1}}{\partial x^2} &\cong \frac{f_{2,1} - 2f_{3,1} + f_{4,1}}{(\Delta x)^2} \\ \frac{\partial^2 f_{4,1}}{\partial x^2} &\cong \frac{f_{3,1} - 2f_{4,1} + 0}{(\Delta x)^2} \\ \frac{\partial^2 f_{1,2}}{\partial x^2} &\cong \frac{0 - 2f_{1,2} + f_{2,2}}{(\Delta x)^2} \\ \frac{\partial^2 f_{2,2}}{\partial x^2} &\cong \frac{f_{1,2} - 2f_{2,2} + f_{3,2}}{(\Delta x)^2} \\ \frac{\partial^2 f_{3,2}}{\partial x^2} &\cong \frac{f_{2,2} - 2f_{3,2} + f_{4,2}}{(\Delta x)^2} \\ \frac{\partial^2 f_{4,2}}{\partial x^2} &\cong \frac{f_{3,2} - 2f_{4,2} + 0}{(\Delta x)^2} \\ \frac{\partial^2 f_{1,3}}{\partial x^2} &\cong \frac{0 - 2f_{1,3} + f_{2,3}}{(\Delta x)^2} \\ \frac{\partial^2 f_{2,3}}{\partial x^2} &\cong \frac{f_{1,3} - 2f_{2,3} + f_{3,3}}{(\Delta x)^2} \\ \frac{\partial^2 f_{3,3}}{\partial x^2} &\cong \frac{f_{2,3} - 2f_{3,3} + f_{4,3}}{(\Delta x)^2} \\ \frac{\partial^2 f_{4,3}}{\partial x^2} &\cong \frac{f_{3,3} - 2f_{4,3} + 0}{(\Delta x)^2} \end{aligned}$$

$[D_x^2]$ Derivative Matrix (10 of 14)



$$\frac{\partial^2 f_{i,j}}{\partial x^2} \cong \frac{f_{i+1,j} - 2f_{i,j} + f_{i-1,j}}{(\Delta x)^2}$$

$$\frac{\partial^2 f_{1,1}}{\partial x^2} \cong \frac{0 - 2f_{1,1} + f_{2,1}}{(\Delta x)^2}$$

$$\frac{\partial^2 f_{2,1}}{\partial x^2} \cong \frac{f_{1,1} - 2f_{2,1} + f_{3,1}}{(\Delta x)^2}$$

$$\frac{\partial^2 f_{3,1}}{\partial x^2} \cong \frac{f_{2,1} - 2f_{3,1} + f_{4,1}}{(\Delta x)^2}$$

$$\frac{\partial^2 f_{4,1}}{\partial x^2} \cong \frac{f_{3,1} - 2f_{4,1} + 0}{(\Delta x)^2}$$

$$\frac{\partial^2 f_{1,2}}{\partial x^2} \cong \frac{0 - 2f_{1,2} + f_{2,2}}{(\Delta x)^2}$$

$$\frac{\partial^2 f_{2,2}}{\partial x^2} \cong \frac{f_{1,2} - 2f_{2,2} + f_{3,2}}{(\Delta x)^2}$$

$$\frac{\partial^2 f_{3,2}}{\partial x^2} \cong \frac{f_{2,2} - 2f_{3,2} + f_{4,2}}{(\Delta x)^2}$$

$$\frac{\partial^2 f_{4,2}}{\partial x^2} \cong \frac{f_{3,2} - 2f_{4,2} + 0}{(\Delta x)^2}$$

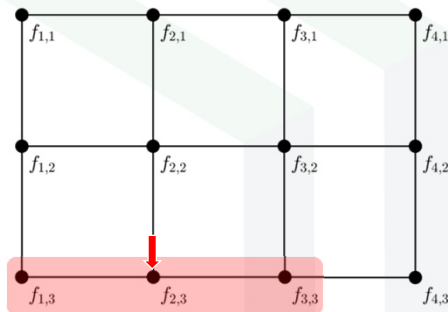
$$\frac{\partial^2 f_{1,3}}{\partial x^2} \cong \frac{0 - 2f_{1,3} + f_{2,3}}{(\Delta x)^2}$$

$$\frac{\partial^2 f_{2,3}}{\partial x^2} \cong \frac{f_{1,3} - 2f_{2,3} + f_{3,3}}{(\Delta x)^2}$$

$$\frac{\partial^2 f_{3,3}}{\partial x^2} \cong \frac{f_{2,3} - 2f_{3,3} + f_{4,3}}{(\Delta x)^2}$$

$$\frac{\partial^2 f_{4,3}}{\partial x^2} \cong \frac{f_{3,3} - 2f_{4,3} + 0}{(\Delta x)^2}$$

$[D_x^2]$ Derivative Matrix (11 of 14)



$$\frac{\partial^2 f_{i,j}}{\partial x^2} \cong \frac{f_{i+1,j} - 2f_{i,j} + f_{i-1,j}}{(\Delta x)^2}$$

$$\frac{\partial^2 f_{1,1}}{\partial x^2} \cong \frac{0 - 2f_{1,1} + f_{2,1}}{(\Delta x)^2}$$

$$\frac{\partial^2 f_{2,1}}{\partial x^2} \cong \frac{f_{1,1} - 2f_{2,1} + f_{3,1}}{(\Delta x)^2}$$

$$\frac{\partial^2 f_{3,1}}{\partial x^2} \cong \frac{f_{2,1} - 2f_{3,1} + f_{4,1}}{(\Delta x)^2}$$

$$\frac{\partial^2 f_{4,1}}{\partial x^2} \cong \frac{f_{3,1} - 2f_{4,1} + 0}{(\Delta x)^2}$$

$$\frac{\partial^2 f_{1,2}}{\partial x^2} \cong \frac{0 - 2f_{1,2} + f_{2,2}}{(\Delta x)^2}$$

$$\frac{\partial^2 f_{2,2}}{\partial x^2} \cong \frac{f_{1,2} - 2f_{2,2} + f_{3,2}}{(\Delta x)^2}$$

$$\frac{\partial^2 f_{3,2}}{\partial x^2} \cong \frac{f_{2,2} - 2f_{3,2} + f_{4,2}}{(\Delta x)^2}$$

$$\frac{\partial^2 f_{4,2}}{\partial x^2} \cong \frac{f_{3,2} - 2f_{4,2} + 0}{(\Delta x)^2}$$

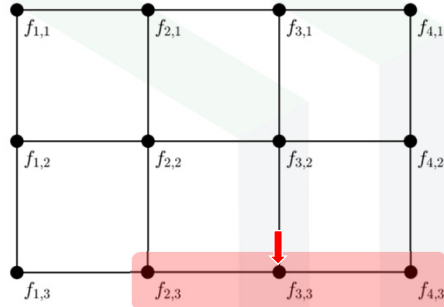
$$\frac{\partial^2 f_{1,3}}{\partial x^2} \cong \frac{0 - 2f_{1,3} + f_{2,3}}{(\Delta x)^2}$$

$$\frac{\partial^2 f_{2,3}}{\partial x^2} \cong \frac{f_{1,3} - 2f_{2,3} + f_{3,3}}{(\Delta x)^2}$$

$$\frac{\partial^2 f_{3,3}}{\partial x^2} \cong \frac{f_{2,3} - 2f_{3,3} + f_{4,3}}{(\Delta x)^2}$$

$$\frac{\partial^2 f_{4,3}}{\partial x^2} \cong \frac{f_{3,3} - 2f_{4,3} + 0}{(\Delta x)^2}$$

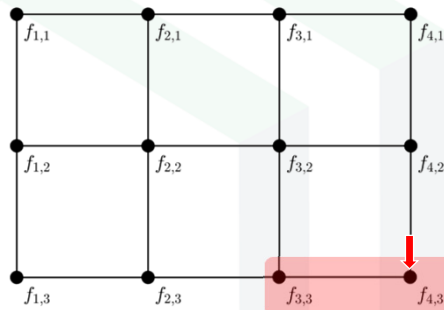
$[D_x^2]$ Derivative Matrix (12 of 14)



$$\frac{\partial^2 f_{i,j}}{\partial x^2} \cong \frac{f_{i+1,j} - 2f_{i,j} + f_{i-1,j}}{(\Delta x)^2}$$

$$\begin{aligned} \frac{\partial^2 f_{1,1}}{\partial x^2} &\cong \frac{0 - 2f_{1,1} + f_{2,1}}{(\Delta x)^2} \\ \frac{\partial^2 f_{2,1}}{\partial x^2} &\cong \frac{f_{1,1} - 2f_{2,1} + f_{3,1}}{(\Delta x)^2} \\ \frac{\partial^2 f_{3,1}}{\partial x^2} &\cong \frac{f_{2,1} - 2f_{3,1} + f_{4,1}}{(\Delta x)^2} \\ \frac{\partial^2 f_{4,1}}{\partial x^2} &\cong \frac{f_{3,1} - 2f_{4,1} + 0}{(\Delta x)^2} \\ \frac{\partial^2 f_{1,2}}{\partial x^2} &\cong \frac{0 - 2f_{1,2} + f_{2,2}}{(\Delta x)^2} \\ \frac{\partial^2 f_{2,2}}{\partial x^2} &\cong \frac{f_{1,2} - 2f_{2,2} + f_{3,2}}{(\Delta x)^2} \\ \frac{\partial^2 f_{3,2}}{\partial x^2} &\cong \frac{f_{2,2} - 2f_{3,2} + f_{4,2}}{(\Delta x)^2} \\ \frac{\partial^2 f_{4,2}}{\partial x^2} &\cong \frac{f_{3,2} - 2f_{4,2} + 0}{(\Delta x)^2} \\ \frac{\partial^2 f_{1,3}}{\partial x^2} &\cong \frac{0 - 2f_{1,3} + f_{2,3}}{(\Delta x)^2} \\ \frac{\partial^2 f_{2,3}}{\partial x^2} &\cong \frac{f_{1,3} - 2f_{2,3} + f_{3,3}}{(\Delta x)^2} \\ \frac{\partial^2 f_{3,3}}{\partial x^2} &\cong \frac{f_{2,3} - 2f_{3,3} + f_{4,3}}{(\Delta x)^2} \\ \frac{\partial^2 f_{4,3}}{\partial x^2} &\cong \frac{f_{3,3} - 2f_{4,3} + 0}{(\Delta x)^2} \end{aligned}$$

$[D_x^2]$ Derivative Matrix (13 of 14)



$$\frac{\partial^2 f_{i,j}}{\partial x^2} \cong \frac{f_{i+1,j} - 2f_{i,j} + f_{i-1,j}}{(\Delta x)^2}$$

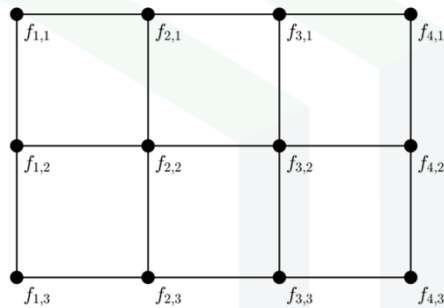
$$\begin{aligned} \frac{\partial^2 f_{1,1}}{\partial x^2} &\cong \frac{0 - 2f_{1,1} + f_{2,1}}{(\Delta x)^2} \\ \frac{\partial^2 f_{2,1}}{\partial x^2} &\cong \frac{f_{1,1} - 2f_{2,1} + f_{3,1}}{(\Delta x)^2} \\ \frac{\partial^2 f_{3,1}}{\partial x^2} &\cong \frac{f_{2,1} - 2f_{3,1} + f_{4,1}}{(\Delta x)^2} \\ \frac{\partial^2 f_{4,1}}{\partial x^2} &\cong \frac{f_{3,1} - 2f_{4,1} + 0}{(\Delta x)^2} \\ \frac{\partial^2 f_{1,2}}{\partial x^2} &\cong \frac{0 - 2f_{1,2} + f_{2,2}}{(\Delta x)^2} \\ \frac{\partial^2 f_{2,2}}{\partial x^2} &\cong \frac{f_{1,2} - 2f_{2,2} + f_{3,2}}{(\Delta x)^2} \\ \frac{\partial^2 f_{3,2}}{\partial x^2} &\cong \frac{f_{2,2} - 2f_{3,2} + f_{4,2}}{(\Delta x)^2} \\ \frac{\partial^2 f_{4,2}}{\partial x^2} &\cong \frac{f_{3,2} - 2f_{4,2} + 0}{(\Delta x)^2} \\ \frac{\partial^2 f_{1,3}}{\partial x^2} &\cong \frac{0 - 2f_{1,3} + f_{2,3}}{(\Delta x)^2} \\ \frac{\partial^2 f_{2,3}}{\partial x^2} &\cong \frac{f_{1,3} - 2f_{2,3} + f_{3,3}}{(\Delta x)^2} \\ \frac{\partial^2 f_{3,3}}{\partial x^2} &\cong \frac{f_{2,3} - 2f_{3,3} + f_{4,3}}{(\Delta x)^2} \\ \frac{\partial^2 f_{4,3}}{\partial x^2} &\cong \frac{f_{3,3} - 2f_{4,3} + 0}{(\Delta x)^2} \end{aligned}$$

Collocated Derivative Matrix

$$\left[D_y \right]$$



$[D_y]$ Derivative Matrix (1 of 14)



$$\frac{\partial f_{i,j}}{\partial y} \approx \frac{f_{i,j+1} - f_{i,j-1}}{2\Delta y}$$

$$\frac{\partial f_{1,1}}{\partial y} \approx \frac{f_{1,2} - 0}{2\Delta y}$$

$$\frac{\partial f_{2,1}}{\partial y} \approx \frac{f_{2,2} - 0}{2\Delta y}$$

$$\frac{\partial f_{3,1}}{\partial y} \approx \frac{f_{3,2} - 0}{2\Delta y}$$

$$\frac{\partial f_{4,1}}{\partial y} \approx \frac{f_{4,2} - 0}{2\Delta y}$$

$$\frac{\partial f_{1,2}}{\partial y} \approx \frac{f_{1,3} - f_{1,1}}{2\Delta y}$$

$$\frac{\partial f_{2,2}}{\partial y} \approx \frac{f_{2,3} - f_{2,1}}{2\Delta y}$$

$$\frac{\partial f_{3,2}}{\partial y} \approx \frac{f_{3,3} - f_{3,1}}{2\Delta y}$$

$$\frac{\partial f_{4,2}}{\partial y} \approx \frac{f_{4,3} - f_{4,1}}{2\Delta y}$$

$$\frac{\partial f_{1,3}}{\partial y} \approx \frac{0 - f_{1,2}}{2\Delta y}$$

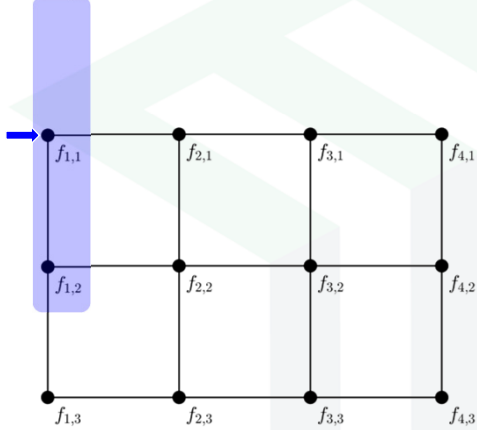
$$\frac{\partial f_{2,3}}{\partial y} \approx \frac{0 - f_{2,2}}{2\Delta y}$$

$$\frac{\partial f_{3,3}}{\partial y} \approx \frac{0 - f_{3,2}}{2\Delta y}$$

$$\frac{\partial f_{4,3}}{\partial y} \approx \frac{0 - f_{4,2}}{2\Delta y}$$



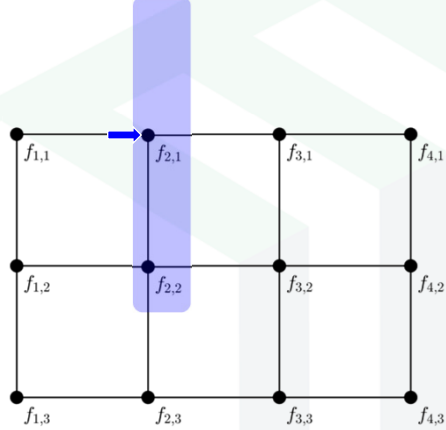
$[D_y]$ Derivative Matrix (2 of 14)



$$\frac{\partial f_{i,j}}{\partial y} \approx \frac{f_{i,j+1} - f_{i,j-1}}{2\Delta y}$$

$$\begin{aligned} \frac{\partial f_{1,1}}{\partial y} &\approx \frac{f_{1,2} - 0}{2\Delta y} \\ \frac{\partial f_{2,1}}{\partial y} &\approx \frac{f_{2,2} - 0}{2\Delta y} \\ \frac{\partial f_{3,1}}{\partial y} &\approx \frac{f_{3,2} - 0}{2\Delta y} \\ \frac{\partial f_{1,2}}{\partial y} &\approx \frac{f_{1,3} - f_{1,1}}{2\Delta y} \\ \frac{\partial f_{2,2}}{\partial y} &\approx \frac{f_{2,3} - f_{2,1}}{2\Delta y} \\ \frac{\partial f_{3,2}}{\partial y} &\approx \frac{f_{3,3} - f_{3,1}}{2\Delta y} \\ \frac{\partial f_{1,3}}{\partial y} &\approx \frac{0 - f_{1,2}}{2\Delta y} \\ \frac{\partial f_{2,3}}{\partial y} &\approx \frac{0 - f_{2,2}}{2\Delta y} \\ \frac{\partial f_{3,3}}{\partial y} &\approx \frac{0 - f_{3,2}}{2\Delta y} \\ \frac{\partial f_{4,2}}{\partial y} &\approx \frac{f_{4,3} - f_{4,1}}{2\Delta y} \end{aligned}$$

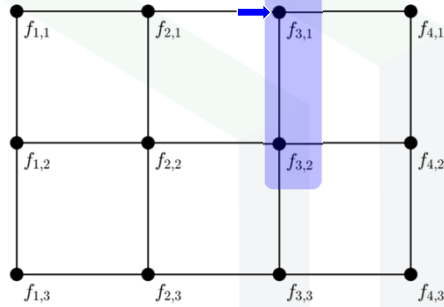
$[D_y]$ Derivative Matrix (3 of 14)



$$\frac{\partial f_{i,j}}{\partial y} \approx \frac{f_{i,j+1} - f_{i,j-1}}{2\Delta y}$$

$$\begin{aligned} \frac{\partial f_{1,1}}{\partial y} &\approx \frac{f_{1,2} - 0}{2\Delta y} \\ \frac{\partial f_{2,1}}{\partial y} &\approx \frac{f_{2,2} - 0}{2\Delta y} \\ \frac{\partial f_{3,1}}{\partial y} &\approx \frac{f_{3,2} - 0}{2\Delta y} \\ \frac{\partial f_{1,2}}{\partial y} &\approx \frac{f_{1,3} - f_{1,1}}{2\Delta y} \\ \frac{\partial f_{2,2}}{\partial y} &\approx \frac{f_{2,3} - f_{2,1}}{2\Delta y} \\ \frac{\partial f_{3,2}}{\partial y} &\approx \frac{f_{3,3} - f_{3,1}}{2\Delta y} \\ \frac{\partial f_{4,2}}{\partial y} &\approx \frac{f_{4,3} - f_{4,1}}{2\Delta y} \\ \frac{\partial f_{1,3}}{\partial y} &\approx \frac{0 - f_{1,2}}{2\Delta y} \\ \frac{\partial f_{2,3}}{\partial y} &\approx \frac{0 - f_{2,2}}{2\Delta y} \\ \frac{\partial f_{3,3}}{\partial y} &\approx \frac{0 - f_{3,2}}{2\Delta y} \\ \frac{\partial f_{4,3}}{\partial y} &\approx \frac{0 - f_{4,2}}{2\Delta y} \end{aligned}$$

$[D_y]$ Derivative Matrix (4 of 14)

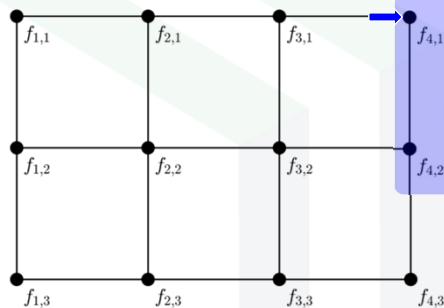


$$\frac{\partial f_{i,j}}{\partial y} \approx \frac{f_{i,j+1} - f_{i,j-1}}{2\Delta y}$$

$$\begin{aligned} \frac{\partial f_{1,1}}{\partial y} &\approx \frac{f_{1,2} - 0}{2\Delta y} \\ \frac{\partial f_{2,1}}{\partial y} &\approx \frac{f_{2,2} - 0}{2\Delta y} \\ \frac{\partial f_{3,1}}{\partial y} &\approx \frac{f_{3,2} - 0}{2\Delta y} \\ \frac{\partial f_{4,1}}{\partial y} &\approx \frac{f_{4,2} - 0}{2\Delta y} \\ \frac{\partial f_{1,2}}{\partial y} &\approx \frac{f_{1,3} - f_{1,1}}{2\Delta y} \\ \frac{\partial f_{2,2}}{\partial y} &\approx \frac{f_{2,3} - f_{2,1}}{2\Delta y} \\ \frac{\partial f_{3,2}}{\partial y} &\approx \frac{f_{3,3} - f_{3,1}}{2\Delta y} \\ \frac{\partial f_{4,2}}{\partial y} &\approx \frac{f_{4,3} - f_{4,1}}{2\Delta y} \\ \frac{\partial f_{1,3}}{\partial y} &\approx \frac{0 - f_{1,2}}{2\Delta y} \\ \frac{\partial f_{2,3}}{\partial y} &\approx \frac{0 - f_{2,2}}{2\Delta y} \\ \frac{\partial f_{3,3}}{\partial y} &\approx \frac{0 - f_{3,2}}{2\Delta y} \\ \frac{\partial f_{4,3}}{\partial y} &\approx \frac{0 - f_{4,2}}{2\Delta y} \end{aligned}$$



$[D_y]$ Derivative Matrix (5 of 14)

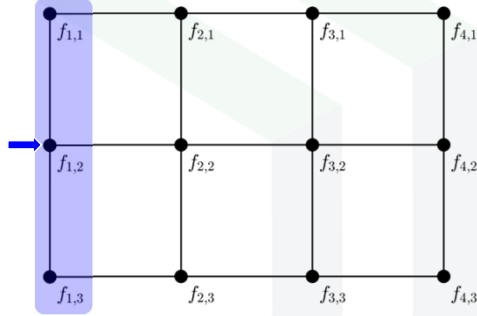


$$\frac{\partial f_{i,j}}{\partial y} \approx \frac{f_{i,j+1} - f_{i,j-1}}{2\Delta y}$$

$$\begin{aligned} \frac{\partial f_{1,1}}{\partial y} &\approx \frac{f_{1,2} - 0}{2\Delta y} \\ \frac{\partial f_{2,1}}{\partial y} &\approx \frac{f_{2,2} - 0}{2\Delta y} \\ \frac{\partial f_{3,1}}{\partial y} &\approx \frac{f_{3,2} - 0}{2\Delta y} \\ \frac{\partial f_{4,1}}{\partial y} &\approx \frac{f_{4,2} - 0}{2\Delta y} \\ \frac{\partial f_{1,2}}{\partial y} &\approx \frac{f_{1,3} - f_{1,1}}{2\Delta y} \\ \frac{\partial f_{2,2}}{\partial y} &\approx \frac{f_{2,3} - f_{2,1}}{2\Delta y} \\ \frac{\partial f_{3,2}}{\partial y} &\approx \frac{f_{3,3} - f_{3,1}}{2\Delta y} \\ \frac{\partial f_{4,2}}{\partial y} &\approx \frac{f_{4,3} - f_{4,1}}{2\Delta y} \\ \frac{\partial f_{1,3}}{\partial y} &\approx \frac{0 - f_{1,2}}{2\Delta y} \\ \frac{\partial f_{2,3}}{\partial y} &\approx \frac{0 - f_{2,2}}{2\Delta y} \\ \frac{\partial f_{3,3}}{\partial y} &\approx \frac{0 - f_{3,2}}{2\Delta y} \\ \frac{\partial f_{4,3}}{\partial y} &\approx \frac{0 - f_{4,2}}{2\Delta y} \end{aligned}$$



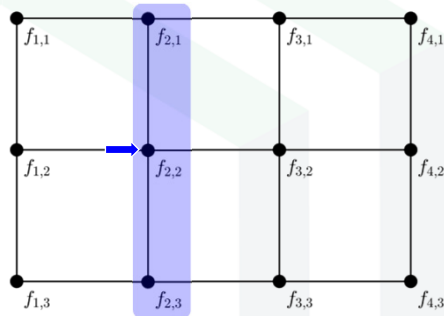
$[D_y]$ Derivative Matrix (6 of 14)



$$\frac{\partial f_{i,j}}{\partial y} \approx \frac{f_{i,j+1} - f_{i,j-1}}{2\Delta y}$$

$$\begin{aligned} \frac{\partial f_{1,1}}{\partial y} &\approx \frac{f_{1,2} - 0}{2\Delta y} \\ \frac{\partial f_{2,1}}{\partial y} &\approx \frac{f_{2,2} - 0}{2\Delta y} \\ \frac{\partial f_{3,1}}{\partial y} &\approx \frac{f_{3,2} - 0}{2\Delta y} \\ \frac{\partial f_{4,1}}{\partial y} &\approx \frac{f_{4,2} - 0}{2\Delta y} \\ \frac{\partial f_{1,2}}{\partial y} &\approx \frac{f_{1,3} - f_{1,1}}{2\Delta y} \\ \frac{\partial f_{2,2}}{\partial y} &\approx \frac{f_{2,3} - f_{2,1}}{2\Delta y} \\ \frac{\partial f_{3,2}}{\partial y} &\approx \frac{f_{3,3} - f_{3,1}}{2\Delta y} \\ \frac{\partial f_{4,2}}{\partial y} &\approx \frac{f_{4,3} - f_{4,1}}{2\Delta y} \\ \frac{\partial f_{1,3}}{\partial y} &\approx \frac{0 - f_{1,2}}{2\Delta y} \\ \frac{\partial f_{2,3}}{\partial y} &\approx \frac{0 - f_{2,2}}{2\Delta y} \\ \frac{\partial f_{3,3}}{\partial y} &\approx \frac{0 - f_{3,2}}{2\Delta y} \\ \frac{\partial f_{4,3}}{\partial y} &\approx \frac{0 - f_{4,2}}{2\Delta y} \end{aligned}$$

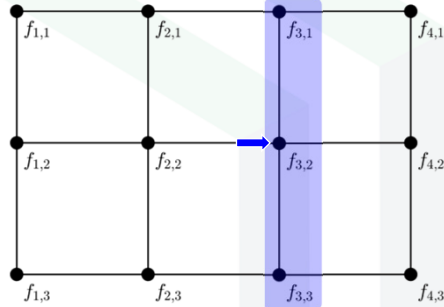
$[D_y]$ Derivative Matrix (7 of 14)



$$\frac{\partial f_{i,j}}{\partial y} \approx \frac{f_{i,j+1} - f_{i,j-1}}{2\Delta y}$$

$$\begin{aligned} \frac{\partial f_{1,1}}{\partial y} &\approx \frac{f_{1,2} - 0}{2\Delta y} \\ \frac{\partial f_{2,1}}{\partial y} &\approx \frac{f_{2,2} - 0}{2\Delta y} \\ \frac{\partial f_{3,1}}{\partial y} &\approx \frac{f_{3,2} - 0}{2\Delta y} \\ \frac{\partial f_{4,1}}{\partial y} &\approx \frac{f_{4,2} - 0}{2\Delta y} \\ \frac{\partial f_{1,2}}{\partial y} &\approx \frac{f_{1,3} - f_{1,1}}{2\Delta y} \\ \frac{\partial f_{2,2}}{\partial y} &\approx \frac{f_{2,3} - f_{2,1}}{2\Delta y} \\ \frac{\partial f_{3,2}}{\partial y} &\approx \frac{f_{3,3} - f_{3,1}}{2\Delta y} \\ \frac{\partial f_{4,2}}{\partial y} &\approx \frac{f_{4,3} - f_{4,1}}{2\Delta y} \\ \frac{\partial f_{1,3}}{\partial y} &\approx \frac{0 - f_{1,2}}{2\Delta y} \\ \frac{\partial f_{2,3}}{\partial y} &\approx \frac{0 - f_{2,2}}{2\Delta y} \\ \frac{\partial f_{3,3}}{\partial y} &\approx \frac{0 - f_{3,2}}{2\Delta y} \\ \frac{\partial f_{4,3}}{\partial y} &\approx \frac{0 - f_{4,2}}{2\Delta y} \end{aligned}$$

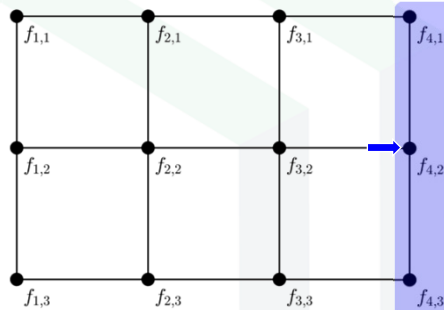
$[D_y]$ Derivative Matrix (8 of 14)



$$\frac{\partial f_{i,j}}{\partial y} \cong \frac{f_{i,j+1} - f_{i,j-1}}{2\Delta y}$$

$$\begin{aligned} \frac{\partial f_{1,1}}{\partial y} &\cong \frac{f_{1,2} - 0}{2\Delta y} \\ \frac{\partial f_{2,1}}{\partial y} &\cong \frac{f_{2,2} - 0}{2\Delta y} \\ \frac{\partial f_{3,1}}{\partial y} &\cong \frac{f_{3,2} - 0}{2\Delta y} \\ \frac{\partial f_{4,1}}{\partial y} &\cong \frac{f_{4,2} - 0}{2\Delta y} \\ \frac{\partial f_{1,2}}{\partial y} &\cong \frac{f_{1,3} - f_{1,1}}{2\Delta y} \\ \frac{\partial f_{2,2}}{\partial y} &\cong \frac{f_{2,3} - f_{2,1}}{2\Delta y} \\ \frac{\partial f_{3,2}}{\partial y} &\cong \frac{f_{3,3} - f_{3,1}}{2\Delta y} \\ \frac{\partial f_{4,2}}{\partial y} &\cong \frac{f_{4,3} - f_{4,1}}{2\Delta y} \\ \frac{\partial f_{1,3}}{\partial y} &\cong \frac{0 - f_{1,2}}{2\Delta y} \\ \frac{\partial f_{2,3}}{\partial y} &\cong \frac{0 - f_{2,2}}{2\Delta y} \\ \frac{\partial f_{3,3}}{\partial y} &\cong \frac{0 - f_{3,2}}{2\Delta y} \\ \frac{\partial f_{4,3}}{\partial y} &\cong \frac{0 - f_{4,2}}{2\Delta y} \end{aligned}$$

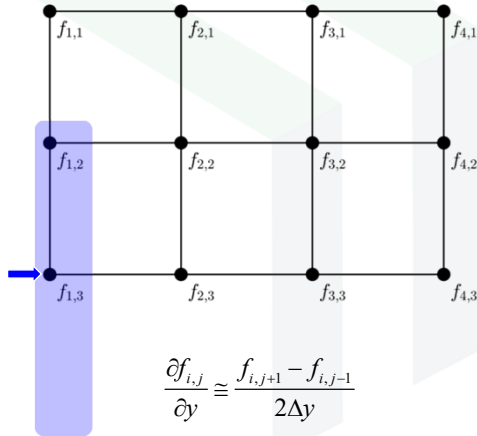
$[D_y]$ Derivative Matrix (9 of 14)



$$\frac{\partial f_{i,j}}{\partial y} \cong \frac{f_{i,j+1} - f_{i,j-1}}{2\Delta y}$$

$$\begin{aligned} \frac{\partial f_{1,1}}{\partial y} &\cong \frac{f_{1,2} - 0}{2\Delta y} \\ \frac{\partial f_{2,1}}{\partial y} &\cong \frac{f_{2,2} - 0}{2\Delta y} \\ \frac{\partial f_{3,1}}{\partial y} &\cong \frac{f_{3,2} - 0}{2\Delta y} \\ \frac{\partial f_{4,1}}{\partial y} &\cong \frac{f_{4,2} - 0}{2\Delta y} \\ \frac{\partial f_{1,2}}{\partial y} &\cong \frac{f_{1,3} - f_{1,1}}{2\Delta y} \\ \frac{\partial f_{2,2}}{\partial y} &\cong \frac{f_{2,3} - f_{2,1}}{2\Delta y} \\ \frac{\partial f_{3,2}}{\partial y} &\cong \frac{f_{3,3} - f_{3,1}}{2\Delta y} \\ \frac{\partial f_{4,2}}{\partial y} &\cong \frac{f_{4,3} - f_{4,1}}{2\Delta y} \\ \frac{\partial f_{1,3}}{\partial y} &\cong \frac{0 - f_{1,2}}{2\Delta y} \\ \frac{\partial f_{2,3}}{\partial y} &\cong \frac{0 - f_{2,2}}{2\Delta y} \\ \frac{\partial f_{3,3}}{\partial y} &\cong \frac{0 - f_{3,2}}{2\Delta y} \\ \frac{\partial f_{4,3}}{\partial y} &\cong \frac{0 - f_{4,2}}{2\Delta y} \end{aligned}$$

$[D_y]$ Derivative Matrix (10 of 14)



$$\frac{\partial f_{1,1}}{\partial y} \cong \frac{f_{1,2} - 0}{2\Delta y}$$

$$\frac{\partial f_{2,1}}{\partial y} \cong \frac{f_{2,2} - 0}{2\Delta y}$$

$$\frac{\partial f_{3,1}}{\partial y} \cong \frac{f_{3,2} - 0}{2\Delta y}$$

$$\frac{\partial f_{1,2}}{\partial y} \cong \frac{f_{1,3} - f_{1,1}}{2\Delta y}$$

$$\frac{\partial f_{2,2}}{\partial y} \cong \frac{f_{2,3} - f_{2,1}}{2\Delta y}$$

$$\frac{\partial f_{3,2}}{\partial y} \cong \frac{f_{3,3} - f_{3,1}}{2\Delta y}$$

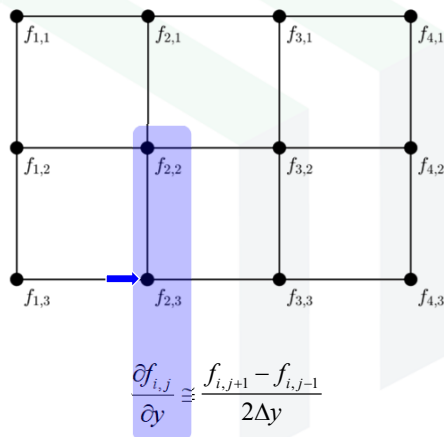
$$\frac{\partial f_{1,3}}{\partial y} \cong \frac{0 - f_{1,2}}{2\Delta y}$$

$$\frac{\partial f_{2,3}}{\partial y} \cong \frac{0 - f_{2,2}}{2\Delta y}$$

$$\frac{\partial f_{3,3}}{\partial y} \cong \frac{0 - f_{3,2}}{2\Delta y}$$

$$\frac{\partial f_{4,2}}{\partial y} \cong \frac{f_{4,3} - f_{4,1}}{2\Delta y}$$

$[D_y]$ Derivative Matrix (11 of 14)



$$\frac{\partial f_{1,1}}{\partial y} \cong \frac{f_{1,2} - 0}{2\Delta y}$$

$$\frac{\partial f_{2,1}}{\partial y} \cong \frac{f_{2,2} - 0}{2\Delta y}$$

$$\frac{\partial f_{3,1}}{\partial y} \cong \frac{f_{3,2} - 0}{2\Delta y}$$

$$\frac{\partial f_{1,2}}{\partial y} \cong \frac{f_{1,3} - f_{1,1}}{2\Delta y}$$

$$\frac{\partial f_{2,2}}{\partial y} \cong \frac{f_{2,3} - f_{2,1}}{2\Delta y}$$

$$\frac{\partial f_{3,2}}{\partial y} \cong \frac{f_{3,3} - f_{3,1}}{2\Delta y}$$

$$\frac{\partial f_{4,2}}{\partial y} \cong \frac{f_{4,3} - f_{4,1}}{2\Delta y}$$

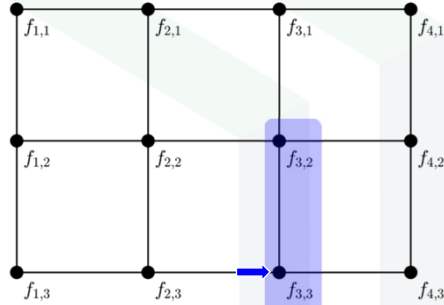
$$\frac{\partial f_{1,3}}{\partial y} \cong \frac{0 - f_{1,2}}{2\Delta y}$$

$$\frac{\partial f_{2,3}}{\partial y} \cong \frac{0 - f_{2,2}}{2\Delta y}$$

$$\frac{\partial f_{3,3}}{\partial y} \cong \frac{0 - f_{3,2}}{2\Delta y}$$

$$\frac{\partial f_{4,3}}{\partial y} \cong \frac{0 - f_{4,2}}{2\Delta y}$$

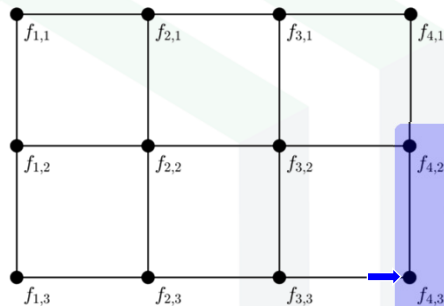
$[D_y]$ Derivative Matrix (12 of 14)



$$\frac{\partial f_{i,j}}{\partial y} \cong \frac{f_{i,j+1} - f_{i,j-1}}{2\Delta y}$$

- $\frac{\partial f_{1,1}}{\partial y} \cong \frac{f_{1,2} - 0}{2\Delta y}$
- $\frac{\partial f_{2,1}}{\partial y} \cong \frac{f_{2,2} - 0}{2\Delta y}$
- $\frac{\partial f_{3,1}}{\partial y} \cong \frac{f_{3,2} - 0}{2\Delta y}$
- $\frac{\partial f_{4,1}}{\partial y} \cong \frac{f_{4,2} - 0}{2\Delta y}$
- $\frac{\partial f_{1,2}}{\partial y} \cong \frac{f_{1,3} - f_{1,1}}{2\Delta y}$
- $\frac{\partial f_{2,2}}{\partial y} \cong \frac{f_{2,3} - f_{2,1}}{2\Delta y}$
- $\frac{\partial f_{3,2}}{\partial y} \cong \frac{f_{3,3} - f_{3,1}}{2\Delta y}$
- $\frac{\partial f_{4,2}}{\partial y} \cong \frac{f_{4,3} - f_{4,1}}{2\Delta y}$
- $\frac{\partial f_{1,3}}{\partial y} \cong \frac{0 - f_{1,2}}{2\Delta y}$
- $\frac{\partial f_{2,3}}{\partial y} \cong \frac{0 - f_{2,2}}{2\Delta y}$
- $\frac{\partial f_{3,3}}{\partial y} \cong \frac{0 - f_{3,2}}{2\Delta y}$
- $\frac{\partial f_{4,3}}{\partial y} \cong \frac{0 - f_{4,2}}{2\Delta y}$

$[D_y]$ Derivative Matrix (13 of 14)



$$\frac{\partial f_{i,j}}{\partial y} \cong \frac{f_{i,j+1} - f_{i,j-1}}{2\Delta y}$$

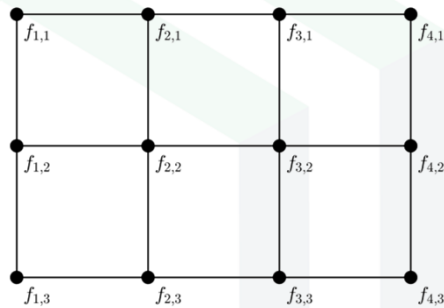
- $\frac{\partial f_{1,1}}{\partial y} \cong \frac{f_{1,2} - 0}{2\Delta y}$
- $\frac{\partial f_{2,1}}{\partial y} \cong \frac{f_{2,2} - 0}{2\Delta y}$
- $\frac{\partial f_{3,1}}{\partial y} \cong \frac{f_{3,2} - 0}{2\Delta y}$
- $\frac{\partial f_{4,1}}{\partial y} \cong \frac{f_{4,2} - 0}{2\Delta y}$
- $\frac{\partial f_{1,2}}{\partial y} \cong \frac{f_{1,3} - f_{1,1}}{2\Delta y}$
- $\frac{\partial f_{2,2}}{\partial y} \cong \frac{f_{2,3} - f_{2,1}}{2\Delta y}$
- $\frac{\partial f_{3,2}}{\partial y} \cong \frac{f_{3,3} - f_{3,1}}{2\Delta y}$
- $\frac{\partial f_{4,2}}{\partial y} \cong \frac{f_{4,3} - f_{4,1}}{2\Delta y}$
- $\frac{\partial f_{1,3}}{\partial y} \cong \frac{0 - f_{1,2}}{2\Delta y}$
- $\frac{\partial f_{2,3}}{\partial y} \cong \frac{0 - f_{2,2}}{2\Delta y}$
- $\frac{\partial f_{3,3}}{\partial y} \cong \frac{0 - f_{3,2}}{2\Delta y}$
- $\frac{\partial f_{4,3}}{\partial y} \cong \frac{0 - f_{4,2}}{2\Delta y}$

Collocated Derivative Matrix

$$\left[D_y^2 \right]$$



$\left[D_y^2 \right]$ Derivative Matrix (1 of 14)



$$\frac{\partial^2 f_{i,j}}{\partial y^2} \cong \frac{f_{i,j+1} - 2f_{i,j} + f_{i,j-1}}{(\Delta y)^2}$$

$$\frac{\partial^2 f_{1,1}}{\partial y^2} \cong \frac{0 - 2f_{1,1} + f_{1,2}}{(\Delta y)^2}$$

$$\frac{\partial^2 f_{2,1}}{\partial y^2} \cong \frac{0 - 2f_{2,1} + f_{2,2}}{(\Delta y)^2}$$

$$\frac{\partial^2 f_{3,1}}{\partial y^2} \cong \frac{0 - 2f_{3,1} + f_{3,2}}{(\Delta y)^2}$$

$$\frac{\partial^2 f_{4,1}}{\partial y^2} \cong \frac{0 - 2f_{4,1} + f_{4,2}}{(\Delta y)^2}$$

$$\frac{\partial^2 f_{1,2}}{\partial y^2} \cong \frac{f_{1,1} - 2f_{1,2} + f_{1,3}}{(\Delta y)^2}$$

$$\frac{\partial^2 f_{2,2}}{\partial y^2} \cong \frac{f_{2,1} - 2f_{2,2} + f_{2,3}}{(\Delta y)^2}$$

$$\frac{\partial^2 f_{3,2}}{\partial y^2} \cong \frac{f_{3,1} - 2f_{3,2} + f_{3,3}}{(\Delta y)^2}$$

$$\frac{\partial^2 f_{4,2}}{\partial y^2} \cong \frac{f_{4,1} - 2f_{4,2} + f_{4,3}}{(\Delta y)^2}$$

$$\frac{\partial^2 f_{1,3}}{\partial y^2} \cong \frac{f_{1,2} - 2f_{1,3} + 0}{(\Delta y)^2}$$

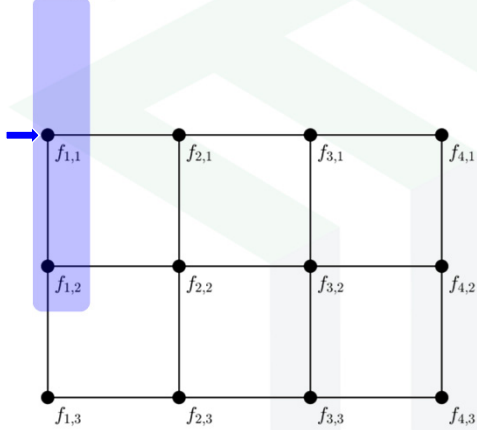
$$\frac{\partial^2 f_{2,3}}{\partial y^2} \cong \frac{f_{2,2} - 2f_{2,3} + 0}{(\Delta y)^2}$$

$$\frac{\partial^2 f_{3,3}}{\partial y^2} \cong \frac{f_{3,2} - 2f_{3,3} + 0}{(\Delta y)^2}$$

$$\frac{\partial^2 f_{4,3}}{\partial y^2} \cong \frac{f_{4,2} - 2f_{4,3} + 0}{(\Delta y)^2}$$



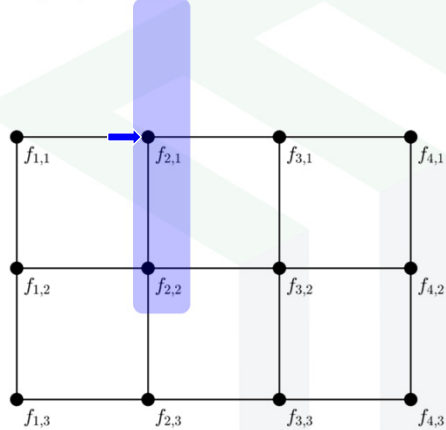
$[D_y^2]$ Derivative Matrix (2 of 14)



$$\frac{\partial^2 f_{i,j}}{\partial y^2} \cong \frac{f_{i,j+1} - 2f_{i,j} + f_{i,j-1}}{(\Delta y)^2}$$

$$\begin{aligned} \frac{\partial^2 f_{1,1}}{\partial y^2} &\cong \frac{0 - 2f_{1,1} + f_{1,2}}{(\Delta y)^2} \\ \frac{\partial^2 f_{2,1}}{\partial y^2} &\cong \frac{0 - 2f_{2,1} + f_{2,2}}{(\Delta y)^2} \\ \frac{\partial^2 f_{3,1}}{\partial y^2} &\cong \frac{0 - 2f_{3,1} + f_{3,2}}{(\Delta y)^2} \\ \frac{\partial^2 f_{4,1}}{\partial y^2} &\cong \frac{0 - 2f_{4,1} + f_{4,2}}{(\Delta y)^2} \\ \frac{\partial^2 f_{1,2}}{\partial y^2} &\cong \frac{f_{1,1} - 2f_{1,2} + f_{1,3}}{(\Delta y)^2} \\ \frac{\partial^2 f_{2,2}}{\partial y^2} &\cong \frac{f_{2,1} - 2f_{2,2} + f_{2,3}}{(\Delta y)^2} \\ \frac{\partial^2 f_{3,2}}{\partial y^2} &\cong \frac{f_{3,1} - 2f_{3,2} + f_{3,3}}{(\Delta y)^2} \\ \frac{\partial^2 f_{4,2}}{\partial y^2} &\cong \frac{f_{4,1} - 2f_{4,2} + f_{4,3}}{(\Delta y)^2} \\ \frac{\partial^2 f_{1,3}}{\partial y^2} &\cong \frac{f_{1,2} - 2f_{1,3} + 0}{(\Delta y)^2} \\ \frac{\partial^2 f_{2,3}}{\partial y^2} &\cong \frac{f_{2,2} - 2f_{2,3} + 0}{(\Delta y)^2} \\ \frac{\partial^2 f_{3,3}}{\partial y^2} &\cong \frac{f_{3,2} - 2f_{3,3} + 0}{(\Delta y)^2} \\ \frac{\partial^2 f_{4,3}}{\partial y^2} &\cong \frac{f_{4,2} - 2f_{4,3} + 0}{(\Delta y)^2} \end{aligned}$$

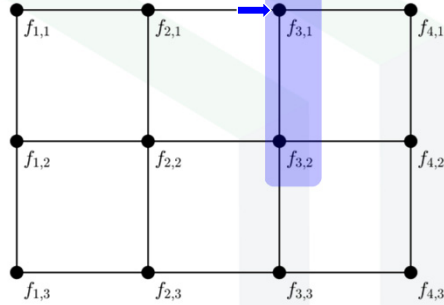
$[D_y^2]$ Derivative Matrix (3 of 14)



$$\frac{\partial^2 f_{i,j}}{\partial y^2} \cong \frac{f_{i,j+1} - 2f_{i,j} + f_{i,j-1}}{(\Delta y)^2}$$

$$\begin{aligned} \frac{\partial^2 f_{1,1}}{\partial y^2} &\cong \frac{0 - 2f_{1,1} + f_{1,2}}{(\Delta y)^2} \\ \frac{\partial^2 f_{2,1}}{\partial y^2} &\cong \frac{0 - 2f_{2,1} + f_{2,2}}{(\Delta y)^2} \\ \frac{\partial^2 f_{3,1}}{\partial y^2} &\cong \frac{0 - 2f_{3,1} + f_{3,2}}{(\Delta y)^2} \\ \frac{\partial^2 f_{4,1}}{\partial y^2} &\cong \frac{0 - 2f_{4,1} + f_{4,2}}{(\Delta y)^2} \\ \frac{\partial^2 f_{1,2}}{\partial y^2} &\cong \frac{f_{1,1} - 2f_{1,2} + f_{1,3}}{(\Delta y)^2} \\ \frac{\partial^2 f_{2,2}}{\partial y^2} &\cong \frac{f_{2,1} - 2f_{2,2} + f_{2,3}}{(\Delta y)^2} \\ \frac{\partial^2 f_{3,2}}{\partial y^2} &\cong \frac{f_{3,1} - 2f_{3,2} + f_{3,3}}{(\Delta y)^2} \\ \frac{\partial^2 f_{4,2}}{\partial y^2} &\cong \frac{f_{4,1} - 2f_{4,2} + f_{4,3}}{(\Delta y)^2} \\ \frac{\partial^2 f_{1,3}}{\partial y^2} &\cong \frac{f_{1,2} - 2f_{1,3} + 0}{(\Delta y)^2} \\ \frac{\partial^2 f_{2,3}}{\partial y^2} &\cong \frac{f_{2,2} - 2f_{2,3} + 0}{(\Delta y)^2} \\ \frac{\partial^2 f_{3,3}}{\partial y^2} &\cong \frac{f_{3,2} - 2f_{3,3} + 0}{(\Delta y)^2} \\ \frac{\partial^2 f_{4,3}}{\partial y^2} &\cong \frac{f_{4,2} - 2f_{4,3} + 0}{(\Delta y)^2} \end{aligned}$$

$[D_y^2]$ Derivative Matrix (4 of 14)



$$\frac{\partial^2 f_{i,j}}{\partial y^2} \cong \frac{f_{i,j+1} - 2f_{i,j} + f_{i,j-1}}{(\Delta y)^2}$$

$$\frac{\partial^2 f_{1,1}}{\partial y^2} \cong \frac{0 - 2f_{1,1} + f_{1,2}}{(\Delta y)^2}$$

$$\frac{\partial^2 f_{2,1}}{\partial y^2} \cong \frac{0 - 2f_{2,1} + f_{2,2}}{(\Delta y)^2}$$

$$\frac{\partial^2 f_{3,1}}{\partial y^2} \cong \frac{0 - 2f_{3,1} + f_{3,2}}{(\Delta y)^2}$$

$$\frac{\partial^2 f_{4,1}}{\partial y^2} \cong \frac{0 - 2f_{4,1} + f_{4,2}}{(\Delta y)^2}$$

$$\frac{\partial^2 f_{1,2}}{\partial y^2} \cong \frac{f_{1,1} - 2f_{1,2} + f_{1,3}}{(\Delta y)^2}$$

$$\frac{\partial^2 f_{2,2}}{\partial y^2} \cong \frac{f_{2,1} - 2f_{2,2} + f_{2,3}}{(\Delta y)^2}$$

$$\frac{\partial^2 f_{3,2}}{\partial y^2} \cong \frac{f_{3,1} - 2f_{3,2} + f_{3,3}}{(\Delta y)^2}$$

$$\frac{\partial^2 f_{4,2}}{\partial y^2} \cong \frac{f_{4,1} - 2f_{4,2} + f_{4,3}}{(\Delta y)^2}$$

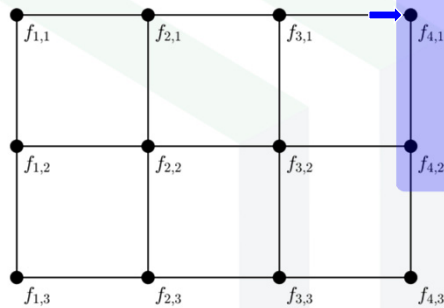
$$\frac{\partial^2 f_{1,3}}{\partial y^2} \cong \frac{f_{1,2} - 2f_{1,3} + 0}{(\Delta y)^2}$$

$$\frac{\partial^2 f_{2,3}}{\partial y^2} \cong \frac{f_{2,2} - 2f_{2,3} + 0}{(\Delta y)^2}$$

$$\frac{\partial^2 f_{3,3}}{\partial y^2} \cong \frac{f_{3,2} - 2f_{3,3} + 0}{(\Delta y)^2}$$

$$\frac{\partial^2 f_{4,3}}{\partial y^2} \cong \frac{f_{4,2} - 2f_{4,3} + 0}{(\Delta y)^2}$$

$[D_y^2]$ Derivative Matrix (5 of 14)



$$\frac{\partial^2 f_{i,j}}{\partial y^2} \cong \frac{f_{i,j+1} - 2f_{i,j} + f_{i,j-1}}{(\Delta y)^2}$$

$$\frac{\partial^2 f_{1,1}}{\partial y^2} \cong \frac{0 - 2f_{1,1} + f_{1,2}}{(\Delta y)^2}$$

$$\frac{\partial^2 f_{2,1}}{\partial y^2} \cong \frac{0 - 2f_{2,1} + f_{2,2}}{(\Delta y)^2}$$

$$\frac{\partial^2 f_{3,1}}{\partial y^2} \cong \frac{0 - 2f_{3,1} + f_{3,2}}{(\Delta y)^2}$$

$$\frac{\partial^2 f_{4,1}}{\partial y^2} \cong \frac{0 - 2f_{4,1} + f_{4,2}}{(\Delta y)^2}$$

$$\frac{\partial^2 f_{1,2}}{\partial y^2} \cong \frac{f_{1,1} - 2f_{1,2} + f_{1,3}}{(\Delta y)^2}$$

$$\frac{\partial^2 f_{2,2}}{\partial y^2} \cong \frac{f_{2,1} - 2f_{2,2} + f_{2,3}}{(\Delta y)^2}$$

$$\frac{\partial^2 f_{3,2}}{\partial y^2} \cong \frac{f_{3,1} - 2f_{3,2} + f_{3,3}}{(\Delta y)^2}$$

$$\frac{\partial^2 f_{4,2}}{\partial y^2} \cong \frac{f_{4,1} - 2f_{4,2} + f_{4,3}}{(\Delta y)^2}$$

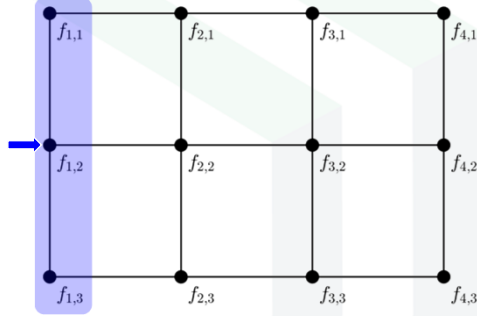
$$\frac{\partial^2 f_{1,3}}{\partial y^2} \cong \frac{f_{1,2} - 2f_{1,3} + 0}{(\Delta y)^2}$$

$$\frac{\partial^2 f_{2,3}}{\partial y^2} \cong \frac{f_{2,2} - 2f_{2,3} + 0}{(\Delta y)^2}$$

$$\frac{\partial^2 f_{3,3}}{\partial y^2} \cong \frac{f_{3,2} - 2f_{3,3} + 0}{(\Delta y)^2}$$

$$\frac{\partial^2 f_{4,3}}{\partial y^2} \cong \frac{f_{4,2} - 2f_{4,3} + 0}{(\Delta y)^2}$$

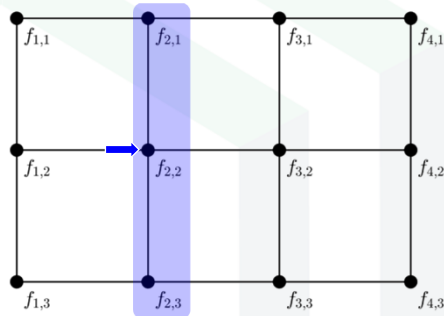
$[D_y^2]$ Derivative Matrix (6 of 14)



$$\frac{\partial^2 f_{i,j}}{\partial y^2} \cong \frac{f_{i,j+1} - 2f_{i,j} + f_{i,j-1}}{(\Delta y)^2}$$

$$\begin{aligned} \frac{\partial^2 f_{1,1}}{\partial y^2} &\cong \frac{0 - 2f_{1,1} + f_{1,2}}{(\Delta y)^2} \\ \frac{\partial^2 f_{2,1}}{\partial y^2} &\cong \frac{0 - 2f_{2,1} + f_{2,2}}{(\Delta y)^2} \\ \frac{\partial^2 f_{3,1}}{\partial y^2} &\cong \frac{0 - 2f_{3,1} + f_{3,2}}{(\Delta y)^2} \\ \frac{\partial^2 f_{4,1}}{\partial y^2} &\cong \frac{0 - 2f_{4,1} + f_{4,2}}{(\Delta y)^2} \\ \frac{\partial^2 f_{1,2}}{\partial y^2} &\cong \frac{f_{1,1} - 2f_{1,2} + f_{1,3}}{(\Delta y)^2} \\ \frac{\partial^2 f_{2,2}}{\partial y^2} &\cong \frac{f_{2,1} - 2f_{2,2} + f_{2,3}}{(\Delta y)^2} \\ \frac{\partial^2 f_{3,2}}{\partial y^2} &\cong \frac{f_{3,1} - 2f_{3,2} + f_{3,3}}{(\Delta y)^2} \\ \frac{\partial^2 f_{4,2}}{\partial y^2} &\cong \frac{f_{4,1} - 2f_{4,2} + f_{4,3}}{(\Delta y)^2} \\ \frac{\partial^2 f_{1,3}}{\partial y^2} &\cong \frac{f_{1,2} - 2f_{1,3} + 0}{(\Delta y)^2} \\ \frac{\partial^2 f_{2,3}}{\partial y^2} &\cong \frac{f_{2,2} - 2f_{2,3} + 0}{(\Delta y)^2} \\ \frac{\partial^2 f_{3,3}}{\partial y^2} &\cong \frac{f_{3,2} - 2f_{3,3} + 0}{(\Delta y)^2} \\ \frac{\partial^2 f_{4,3}}{\partial y^2} &\cong \frac{f_{4,2} - 2f_{4,3} + 0}{(\Delta y)^2} \end{aligned}$$

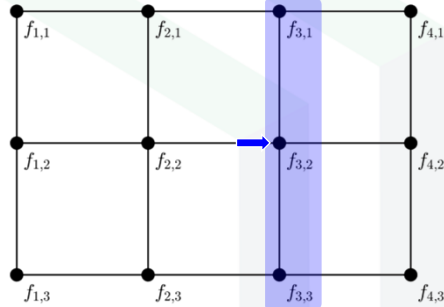
$[D_y^2]$ Derivative Matrix (7 of 14)



$$\frac{\partial^2 f_{i,j}}{\partial y^2} \cong \frac{f_{i,j+1} - 2f_{i,j} + f_{i,j-1}}{(\Delta y)^2}$$

$$\begin{aligned} \frac{\partial^2 f_{1,1}}{\partial y^2} &\cong \frac{0 - 2f_{1,1} + f_{1,2}}{(\Delta y)^2} \\ \frac{\partial^2 f_{2,1}}{\partial y^2} &\cong \frac{0 - 2f_{2,1} + f_{2,2}}{(\Delta y)^2} \\ \frac{\partial^2 f_{3,1}}{\partial y^2} &\cong \frac{0 - 2f_{3,1} + f_{3,2}}{(\Delta y)^2} \\ \frac{\partial^2 f_{4,1}}{\partial y^2} &\cong \frac{0 - 2f_{4,1} + f_{4,2}}{(\Delta y)^2} \\ \frac{\partial^2 f_{1,2}}{\partial y^2} &\cong \frac{f_{1,1} - 2f_{1,2} + f_{1,3}}{(\Delta y)^2} \\ \frac{\partial^2 f_{2,2}}{\partial y^2} &\cong \frac{f_{2,1} - 2f_{2,2} + f_{2,3}}{(\Delta y)^2} \\ \frac{\partial^2 f_{3,2}}{\partial y^2} &\cong \frac{f_{3,1} - 2f_{3,2} + f_{3,3}}{(\Delta y)^2} \\ \frac{\partial^2 f_{4,2}}{\partial y^2} &\cong \frac{f_{4,1} - 2f_{4,2} + f_{4,3}}{(\Delta y)^2} \\ \frac{\partial^2 f_{1,3}}{\partial y^2} &\cong \frac{f_{1,2} - 2f_{1,3} + 0}{(\Delta y)^2} \\ \frac{\partial^2 f_{2,3}}{\partial y^2} &\cong \frac{f_{2,2} - 2f_{2,3} + 0}{(\Delta y)^2} \\ \frac{\partial^2 f_{3,3}}{\partial y^2} &\cong \frac{f_{3,2} - 2f_{3,3} + 0}{(\Delta y)^2} \\ \frac{\partial^2 f_{4,3}}{\partial y^2} &\cong \frac{f_{4,2} - 2f_{4,3} + 0}{(\Delta y)^2} \end{aligned}$$

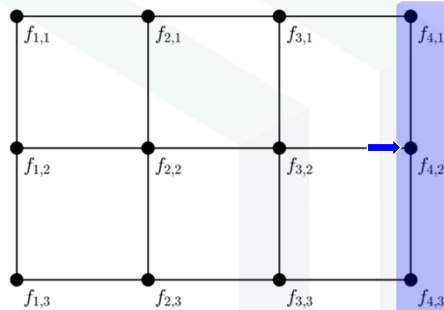
$[D_y^2]$ Derivative Matrix (8 of 14)



$$\frac{\partial^2 f_{i,j}}{\partial y^2} \cong \frac{f_{i,j+1} - 2f_{i,j} + f_{i,j-1}}{(\Delta y)^2}$$

$$\begin{aligned} \frac{\partial^2 f_{1,1}}{\partial y^2} &\cong \frac{0 - 2f_{1,1} + f_{1,2}}{(\Delta y)^2} \\ \frac{\partial^2 f_{2,1}}{\partial y^2} &\cong \frac{0 - 2f_{2,1} + f_{2,2}}{(\Delta y)^2} \\ \frac{\partial^2 f_{3,1}}{\partial y^2} &\cong \frac{0 - 2f_{3,1} + f_{3,2}}{(\Delta y)^2} \\ \frac{\partial^2 f_{4,1}}{\partial y^2} &\cong \frac{0 - 2f_{4,1} + f_{4,2}}{(\Delta y)^2} \\ \frac{\partial^2 f_{1,2}}{\partial y^2} &\cong \frac{f_{1,1} - 2f_{1,2} + f_{1,3}}{(\Delta y)^2} \\ \frac{\partial^2 f_{2,2}}{\partial y^2} &\cong \frac{f_{2,1} - 2f_{2,2} + f_{2,3}}{(\Delta y)^2} \\ \frac{\partial^2 f_{3,2}}{\partial y^2} &\cong \frac{f_{3,1} - 2f_{3,2} + f_{3,3}}{(\Delta y)^2} \\ \frac{\partial^2 f_{4,2}}{\partial y^2} &\cong \frac{f_{4,1} - 2f_{4,2} + f_{4,3}}{(\Delta y)^2} \\ \frac{\partial^2 f_{1,3}}{\partial y^2} &\cong \frac{f_{1,2} - 2f_{1,3} + 0}{(\Delta y)^2} \\ \frac{\partial^2 f_{2,3}}{\partial y^2} &\cong \frac{f_{2,2} - 2f_{2,3} + 0}{(\Delta y)^2} \\ \frac{\partial^2 f_{3,3}}{\partial y^2} &\cong \frac{f_{3,2} - 2f_{3,3} + 0}{(\Delta y)^2} \\ \frac{\partial^2 f_{4,3}}{\partial y^2} &\cong \frac{f_{4,2} - 2f_{4,3} + 0}{(\Delta y)^2} \end{aligned}$$

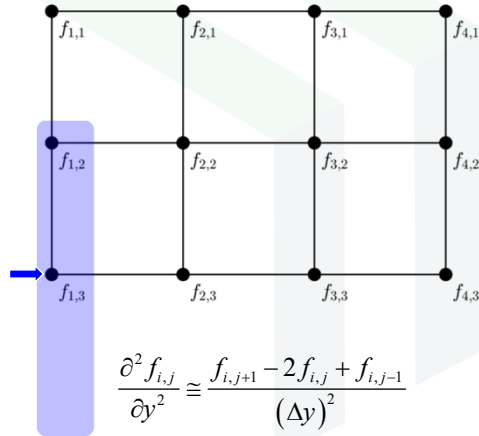
$[D_y^2]$ Derivative Matrix (9 of 14)



$$\frac{\partial^2 f_{i,j}}{\partial y^2} \cong \frac{f_{i,j+1} - 2f_{i,j} + f_{i,j-1}}{(\Delta y)^2}$$

$$\begin{aligned} \frac{\partial^2 f_{1,1}}{\partial y^2} &\cong \frac{0 - 2f_{1,1} + f_{1,2}}{(\Delta y)^2} \\ \frac{\partial^2 f_{2,1}}{\partial y^2} &\cong \frac{0 - 2f_{2,1} + f_{2,2}}{(\Delta y)^2} \\ \frac{\partial^2 f_{3,1}}{\partial y^2} &\cong \frac{0 - 2f_{3,1} + f_{3,2}}{(\Delta y)^2} \\ \frac{\partial^2 f_{4,1}}{\partial y^2} &\cong \frac{0 - 2f_{4,1} + f_{4,2}}{(\Delta y)^2} \\ \frac{\partial^2 f_{1,2}}{\partial y^2} &\cong \frac{f_{1,1} - 2f_{1,2} + f_{1,3}}{(\Delta y)^2} \\ \frac{\partial^2 f_{2,2}}{\partial y^2} &\cong \frac{f_{2,1} - 2f_{2,2} + f_{2,3}}{(\Delta y)^2} \\ \frac{\partial^2 f_{3,2}}{\partial y^2} &\cong \frac{f_{3,1} - 2f_{3,2} + f_{3,3}}{(\Delta y)^2} \\ \frac{\partial^2 f_{4,2}}{\partial y^2} &\cong \frac{f_{4,1} - 2f_{4,2} + f_{4,3}}{(\Delta y)^2} \\ \frac{\partial^2 f_{1,3}}{\partial y^2} &\cong \frac{f_{1,2} - 2f_{1,3} + 0}{(\Delta y)^2} \\ \frac{\partial^2 f_{2,3}}{\partial y^2} &\cong \frac{f_{2,2} - 2f_{2,3} + 0}{(\Delta y)^2} \\ \frac{\partial^2 f_{3,3}}{\partial y^2} &\cong \frac{f_{3,2} - 2f_{3,3} + 0}{(\Delta y)^2} \\ \frac{\partial^2 f_{4,3}}{\partial y^2} &\cong \frac{f_{4,2} - 2f_{4,3} + 0}{(\Delta y)^2} \end{aligned}$$

$[D_y^2]$ Derivative Matrix (10 of 14)



$$\frac{\partial^2 f_{1,1}}{\partial y^2} \cong \frac{0 - 2f_{1,1} + f_{1,2}}{(\Delta y)^2}$$

$$\frac{\partial^2 f_{2,1}}{\partial y^2} \cong \frac{0 - 2f_{2,1} + f_{2,2}}{(\Delta y)^2}$$

$$\frac{\partial^2 f_{3,1}}{\partial y^2} \cong \frac{0 - 2f_{3,1} + f_{3,2}}{(\Delta y)^2}$$

$$\frac{\partial^2 f_{4,1}}{\partial y^2} \cong \frac{0 - 2f_{4,1} + f_{4,2}}{(\Delta y)^2}$$

$$\frac{\partial^2 f_{1,2}}{\partial y^2} \cong \frac{f_{1,1} - 2f_{1,2} + f_{1,3}}{(\Delta y)^2}$$

$$\frac{\partial^2 f_{2,2}}{\partial y^2} \cong \frac{f_{2,1} - 2f_{2,2} + f_{2,3}}{(\Delta y)^2}$$

$$\frac{\partial^2 f_{3,2}}{\partial y^2} \cong \frac{f_{3,1} - 2f_{3,2} + f_{3,3}}{(\Delta y)^2}$$

$$\frac{\partial^2 f_{4,2}}{\partial y^2} \cong \frac{f_{4,1} - 2f_{4,2} + f_{4,3}}{(\Delta y)^2}$$

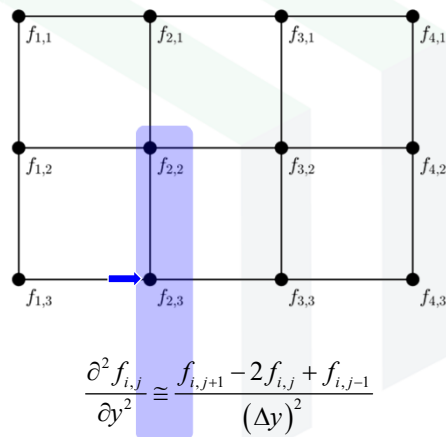
$$\frac{\partial^2 f_{1,3}}{\partial y^2} \cong \frac{f_{1,2} - 2f_{1,3} + 0}{(\Delta y)^2}$$

$$\frac{\partial^2 f_{2,3}}{\partial y^2} \cong \frac{f_{2,2} - 2f_{2,3} + 0}{(\Delta y)^2}$$

$$\frac{\partial^2 f_{3,3}}{\partial y^2} \cong \frac{f_{3,2} - 2f_{3,3} + 0}{(\Delta y)^2}$$

$$\frac{\partial^2 f_{4,3}}{\partial y^2} \cong \frac{f_{4,2} - 2f_{4,3} + 0}{(\Delta y)^2}$$

$[D_y^2]$ Derivative Matrix (11 of 14)



$$\frac{\partial^2 f_{1,1}}{\partial y^2} \cong \frac{0 - 2f_{1,1} + f_{1,2}}{(\Delta y)^2}$$

$$\frac{\partial^2 f_{2,1}}{\partial y^2} \cong \frac{0 - 2f_{2,1} + f_{2,2}}{(\Delta y)^2}$$

$$\frac{\partial^2 f_{3,1}}{\partial y^2} \cong \frac{0 - 2f_{3,1} + f_{3,2}}{(\Delta y)^2}$$

$$\frac{\partial^2 f_{4,1}}{\partial y^2} \cong \frac{0 - 2f_{4,1} + f_{4,2}}{(\Delta y)^2}$$

$$\frac{\partial^2 f_{1,2}}{\partial y^2} \cong \frac{f_{1,1} - 2f_{1,2} + f_{1,3}}{(\Delta y)^2}$$

$$\frac{\partial^2 f_{2,2}}{\partial y^2} \cong \frac{f_{2,1} - 2f_{2,2} + f_{2,3}}{(\Delta y)^2}$$

$$\frac{\partial^2 f_{3,2}}{\partial y^2} \cong \frac{f_{3,1} - 2f_{3,2} + f_{3,3}}{(\Delta y)^2}$$

$$\frac{\partial^2 f_{4,2}}{\partial y^2} \cong \frac{f_{4,1} - 2f_{4,2} + f_{4,3}}{(\Delta y)^2}$$

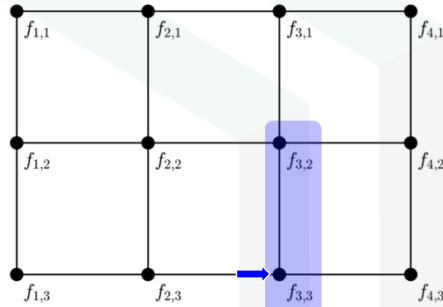
$$\frac{\partial^2 f_{1,3}}{\partial y^2} \cong \frac{f_{1,2} - 2f_{1,3} + 0}{(\Delta y)^2}$$

$$\frac{\partial^2 f_{2,3}}{\partial y^2} \cong \frac{f_{2,2} - 2f_{2,3} + 0}{(\Delta y)^2}$$

$$\frac{\partial^2 f_{3,3}}{\partial y^2} \cong \frac{f_{3,2} - 2f_{3,3} + 0}{(\Delta y)^2}$$

$$\frac{\partial^2 f_{4,3}}{\partial y^2} \cong \frac{f_{4,2} - 2f_{4,3} + 0}{(\Delta y)^2}$$

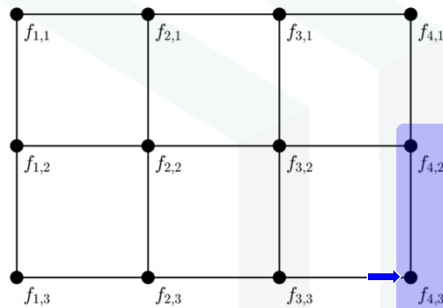
$[D_y^2]$ Derivative Matrix (12 of 14)



$$\frac{\partial^2 f_{i,j}}{\partial y^2} \cong \frac{f_{i,j+1} - 2f_{i,j} + f_{i,j-1}}{(\Delta y)^2}$$

$$\begin{aligned} \frac{\partial^2 f_{1,1}}{\partial y^2} &\cong \frac{0 - 2f_{1,1} + f_{1,2}}{(\Delta y)^2} \\ \frac{\partial^2 f_{2,1}}{\partial y^2} &\cong \frac{0 - 2f_{2,1} + f_{2,2}}{(\Delta y)^2} \\ \frac{\partial^2 f_{3,1}}{\partial y^2} &\cong \frac{0 - 2f_{3,1} + f_{3,2}}{(\Delta y)^2} \\ \frac{\partial^2 f_{4,1}}{\partial y^2} &\cong \frac{0 - 2f_{4,1} + f_{4,2}}{(\Delta y)^2} \\ \frac{\partial^2 f_{1,2}}{\partial y^2} &\cong \frac{f_{1,1} - 2f_{1,2} + f_{1,3}}{(\Delta y)^2} \\ \frac{\partial^2 f_{2,2}}{\partial y^2} &\cong \frac{f_{2,1} - 2f_{2,2} + f_{2,3}}{(\Delta y)^2} \\ \frac{\partial^2 f_{3,2}}{\partial y^2} &\cong \frac{f_{3,1} - 2f_{3,2} + f_{3,3}}{(\Delta y)^2} \\ \frac{\partial^2 f_{4,2}}{\partial y^2} &\cong \frac{f_{4,1} - 2f_{4,2} + f_{4,3}}{(\Delta y)^2} \\ \frac{\partial^2 f_{1,3}}{\partial y^2} &\cong \frac{f_{1,2} - 2f_{1,3} + 0}{(\Delta y)^2} \\ \frac{\partial^2 f_{2,3}}{\partial y^2} &\cong \frac{f_{2,2} - 2f_{2,3} + 0}{(\Delta y)^2} \\ \frac{\partial^2 f_{3,3}}{\partial y^2} &\cong \frac{f_{3,2} - 2f_{3,3} + 0}{(\Delta y)^2} \\ \frac{\partial^2 f_{4,3}}{\partial y^2} &\cong \frac{f_{4,2} - 2f_{4,3} + 0}{(\Delta y)^2} \end{aligned}$$

$[D_y^2]$ Derivative Matrix (13 of 14)



$$\frac{\partial^2 f_{i,j}}{\partial y^2} \cong \frac{f_{i,j+1} - 2f_{i,j} + f_{i,j-1}}{(\Delta y)^2}$$

$$\begin{aligned} \frac{\partial^2 f_{1,1}}{\partial y^2} &\cong \frac{0 - 2f_{1,1} + f_{1,2}}{(\Delta y)^2} \\ \frac{\partial^2 f_{2,1}}{\partial y^2} &\cong \frac{0 - 2f_{2,1} + f_{2,2}}{(\Delta y)^2} \\ \frac{\partial^2 f_{3,1}}{\partial y^2} &\cong \frac{0 - 2f_{3,1} + f_{3,2}}{(\Delta y)^2} \\ \frac{\partial^2 f_{4,1}}{\partial y^2} &\cong \frac{0 - 2f_{4,1} + f_{4,2}}{(\Delta y)^2} \\ \frac{\partial^2 f_{1,2}}{\partial y^2} &\cong \frac{f_{1,1} - 2f_{1,2} + f_{1,3}}{(\Delta y)^2} \\ \frac{\partial^2 f_{2,2}}{\partial y^2} &\cong \frac{f_{2,1} - 2f_{2,2} + f_{2,3}}{(\Delta y)^2} \\ \frac{\partial^2 f_{3,2}}{\partial y^2} &\cong \frac{f_{3,1} - 2f_{3,2} + f_{3,3}}{(\Delta y)^2} \\ \frac{\partial^2 f_{4,2}}{\partial y^2} &\cong \frac{f_{4,1} - 2f_{4,2} + f_{4,3}}{(\Delta y)^2} \\ \frac{\partial^2 f_{1,3}}{\partial y^2} &\cong \frac{f_{1,2} - 2f_{1,3} + 0}{(\Delta y)^2} \\ \frac{\partial^2 f_{2,3}}{\partial y^2} &\cong \frac{f_{2,2} - 2f_{2,3} + 0}{(\Delta y)^2} \\ \frac{\partial^2 f_{3,3}}{\partial y^2} &\cong \frac{f_{3,2} - 2f_{3,3} + 0}{(\Delta y)^2} \\ \frac{\partial^2 f_{4,3}}{\partial y^2} &\cong \frac{f_{4,2} - 2f_{4,3} + 0}{(\Delta y)^2} \end{aligned}$$

Final Notes for Derivative Matrices

67

Placing Diagonals in Sparse Matrices in MATLAB

```

M = 6;
Z = sparse(M,M);
d = ones(M,1);
A = spdiags(d,0,Z);

```

⇒

```

A =
[ 1  0  0  0  0  0 ]
[ 0  1  0  0  0  0 ]
[ 0  0  1  0  0  0 ]
[ 0  0  0  1  0  0 ]
[ 0  0  0  0  1  0 ]
[ 0  0  0  0  0  1 ]

```

```

M = 6;
Z = sparse(M,M);
d = ones(M,1);
A = spdiags(-d,-1,Z);
A = spdiags(+d,+1,A);

```

⇒

```

A =
[ 0  1  0  0  0  0 ]
[ -1  0  1  0  0  0 ]
[ 0  -1  0  1  0  0 ]
[ 0  0  -1  0  1  0 ]
[ 0  0  0  -1  0  1 ]
[ 0  0  0  0  -1  0 ]

```

68

The Rule Using Dirichlet Boundary Conditions

$$\text{If } N_x = 1, \text{ then } [D_x] = [D_x^2] = \dots = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}$$

$$\text{If } N_y = 1, \text{ then } [D_y] = [D_y^2] = \dots = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}$$

These are all sparse matrices of all zeros. 