Advanced Electromagnetics:
21st Century Electromagnetics

Form Invariance of Maxwell's Equations

Maxwell's Equations are Form Invariant

In ANY coordinate system, Maxwell's equations can be written as

Cartesian Coordinates

$$\nabla \times \vec{H} = j\omega[\varepsilon]\vec{E}$$
$$\nabla \times \vec{E} = -j\omega[\mu]\vec{H}$$

Cylindrical Coordinates

$$\nabla \times \vec{H} = j\omega[\varepsilon]\vec{E}$$
$$\nabla \times \vec{E} = -j\omega[\mu]\vec{H}$$

Spherical Coordinates

$$\nabla \times \vec{H} = j\omega[\varepsilon]\vec{E}$$
$$\nabla \times \vec{E} = -j\omega[\mu]\vec{H}$$

Martian Coordinates

$$\nabla \times \vec{H} = j\omega[\varepsilon]\vec{E}$$
$$\nabla \times \vec{E} = -j\omega[\mu]\vec{H}$$

Maxwell's equations can be transformed to a different coordinate system, but they still have the same form.

$$\nabla' \times \vec{H}' = j\omega[\varepsilon']\vec{E}'$$
$$\nabla' \times \vec{E}' = -j\omega[\mu']\vec{H}'$$

Important Consequence

The math associated with the coordinate transform can be "absorbed" completely into the material properties.

$$\nabla' \times \vec{H}' = j\omega[\varepsilon']\vec{E}' \qquad \nabla \times \vec{H} = j\omega[\varepsilon'']\vec{E}$$
$$\nabla' \times \vec{E}' = -j\omega[\mu']\vec{H}' \quad \Rightarrow \quad \nabla \times \vec{E} = -j\omega[\mu'']\vec{H}$$

Maxwell's equations are now expressed in the original coordinate system, but the fields behave as if they are in the transformed coordinates as long as they are embedded in a medium with permittivity $[\varepsilon'']$ and permeability $[\mu'']$.

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Absorbing the Coordinate Transformation into the Materials

Given the Jacobian $[J]$ describing the coordinate transformation, the material property tensors are related through

$$[\mu'] = \frac{[J][\mu][J]^T}{\det[J]} \qquad [\varepsilon'] = \frac{[J][\varepsilon][J]^T}{\det[J]}$$

Here, the constitutive parameters $[\mu]$ and $[\varepsilon]$ are interpretted as operations not functions.

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Proof of Form Invariance (1 of 3)

It is needed to show that the following transform is true.

$$\nabla \times \vec{E} = -j\omega[\mu]\vec{H} \quad \rightarrow \quad \nabla' \times \vec{E}' = -j\omega[\mu']\vec{H}'$$

Defining the coordinate transformation as $\vec{r}' = \vec{r}'(\vec{r})$, the functions can be transformed as

$$\vec{E}'(\vec{r}') = \left([J]^T\right)^{-1}\vec{E}(\vec{r})$$
$$\vec{H}'(\vec{r}') = \left([J]^T\right)^{-1}\vec{H}(\vec{r})$$
$$[\mu'(\vec{r}')] = \frac{[J][\mu(\vec{r})][J]^T}{\det[J]}$$

$\Rightarrow$

$$\vec{E}(\vec{r}) = [J]^T\vec{E}'(\vec{r}')$$
$$\vec{H}(\vec{r}) = [J]^T\vec{H}'(\vec{r}')$$
$$[\mu(\vec{r})] = \det[J]\left\{[J]^{-1}[\mu'(\vec{r}')]\left([J]^T\right)^{-1}\right\}$$

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Proof of Form Invariance (2 of 3)

Substitute the transforms into the original curl equation.

$$\nabla \times \vec{E} = -j\omega[\mu]\vec{H}$$

$\vec{E}(\vec{r}) = [J]^T\vec{E}'(\vec{r}')$

$[\mu(\vec{r})] = \det[J]\left\{[J]^{-1}[\mu'(\vec{r}')]\left([J]^T\right)^{-1}\right\}$

$\vec{H}(\vec{r}) = [J]^T\vec{H}'(\vec{r}')$

This becomes

$$\nabla \times \left\{[J]^T\vec{E}'\right\} = -j\omega\det[J]\left\{[J]^{-1}[\mu']\left([J]^T\right)^{-1}\right\}[J]^T\vec{H}'$$

$$\frac{[J](\nabla \times)[J]^T}{\det[J]}\vec{E}' = -j\omega[\mu']\vec{H}'$$

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Proof of Form Invariance (3 of 3)

Recall how "operations" are transformed using the Jacobian $[J]$.

$$\left[F'(\vec{r}')\right]=\frac{[J]\left[F(\vec{r})\right][J]^T}{\det[J]}$$

$\nabla' \times$

The group of terms around the curl operation indicates this is just the transformed curl operation.

$$\underbrace{\frac{[J](\nabla\times)[J]^T}{\det[J]}}_{\nabla'\times}\vec{E}'=-j\omega[\mu']\vec{H}' \quad\Longrightarrow\quad \nabla'\times\vec{E}'=-j\omega[\mu']\vec{H}'$$

This has the same form as the original equation.

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A Simple Example of the Proof

Start with Maxwell's curl equation.

$$\nabla\times\vec{E}(\vec{r})=-j\omega\left[\mu(\vec{r})\right]\vec{H}(\vec{r})$$

Define the following coordinate transform

$$\vec{r}'=a\vec{r}$$

The terms transform according to

$$\begin{aligned}\vec{E}(\vec{r})&\rightarrow\vec{E}'(\vec{r}')\\ \left[\mu(\vec{r})\right]&\rightarrow\left[\mu'(\vec{r}')\right]\\ \vec{H}(\vec{r})&\rightarrow\vec{H}'(\vec{r}')\end{aligned}$$

$$\nabla\times=\begin{bmatrix}0 & -\frac{\partial}{\partial z} & \frac{\partial}{\partial y}\\ \frac{\partial}{\partial z} & 0 & -\frac{\partial}{\partial x}\\ -\frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0\end{bmatrix}\rightarrow\nabla'\times=\frac{1}{a}\begin{bmatrix}0 & -\frac{\partial}{\partial z'} & \frac{\partial}{\partial y'}\\ \frac{\partial}{\partial z'} & 0 & -\frac{\partial}{\partial x'}\\ -\frac{\partial}{\partial y'} & \frac{\partial}{\partial x'} & 0\end{bmatrix}$$

The scale factor from the curl operation can be absorbed into the permeability.

$$\nabla\times\vec{E}(\vec{r})=-j\omega\left[a\mu(\vec{r})\right]\vec{H}(\vec{r})$$

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