Computational Science:
Computational Methods in Engineering

Formulation of Slab Waveguide Analysis

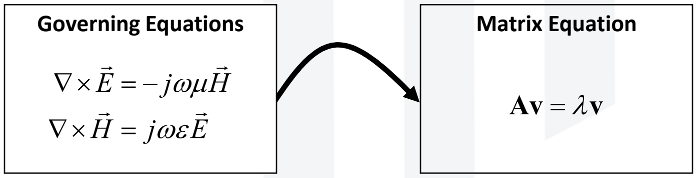
Outline

- Formulation
- The Solution

What is Formulation?

Formulation is the initial analytical work done before implementing a computer code.

Usually formulation starts with the governing equation(s) and ends with the matrix equation to be solved.



Governing Equations

$$\nabla \times \vec{E} = -j\omega\mu\vec{H}$$
$$\nabla \times \vec{H} = j\omega\varepsilon\vec{E}$$

Matrix Equation

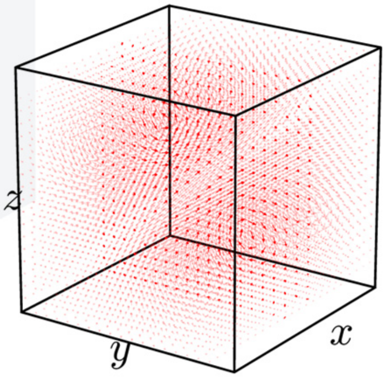
$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$$

Formulation

Governing Equations

Since this is an electrodynamics problem, start with Maxwell's curl equations.

$$\nabla \times \vec{E} = -j\omega\mu\vec{H}$$
$$\nabla \times \vec{H} = j\omega\varepsilon\vec{E}$$



Vector Curl

The curl of a vector is a measure of the vector field's tendency to circulate about an axis. The curl quantity is directly along this axis and the magnitude measures the strength of the circulation.

$$\nabla \times \vec{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) \hat{a}_x + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) \hat{a}_y + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \hat{a}_z$$

Expand Governing Equations (1 of 2)

Expand the first equation into its vector components.

$$\nabla \times \vec{E} = -j\omega\mu\vec{H}$$

$$\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} \right) \hat{a}_x + \left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} \right) \hat{a}_y + \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right) \hat{a}_z = -j\omega\mu\left(H_x\hat{a}_x + H_y\hat{a}_y + H_z\hat{a}_z \right)$$

$$\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} \right) \hat{a}_x + \left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} \right) \hat{a}_y + \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right) \hat{a}_z = -j\omega\mu H_x\hat{a}_x - j\omega\mu H_y\hat{a}_y - j\omega\mu H_z\hat{a}_z$$

Expand Governing Equations (1 of 2)

Expand the first equation into its vector components.

$$\nabla \times \vec{E} = -j\omega\mu\vec{H}$$

$$\left(\frac{\partial E_z}{\partial y}-\frac{\partial E_y}{\partial z}\right)\hat{a}_x+\left(\frac{\partial E_x}{\partial z}-\frac{\partial E_z}{\partial x}\right)\hat{a}_y+\left(\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}\right)\hat{a}_z=-j\omega\mu\left(H_x\hat{a}_x+H_y\hat{a}_y+H_z\hat{a}_z\right)$$

$$\left(\frac{\partial E_z}{\partial y}-\frac{\partial E_y}{\partial z}\right)\hat{a}_x+\left(\frac{\partial E_x}{\partial z}-\frac{\partial E_z}{\partial x}\right)\hat{a}_y+\left(\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}\right)\hat{a}_z=-j\omega\mu H_x\hat{a}_x-j\omega\mu H_y\hat{a}_y-j\omega\mu H_z\hat{a}_z$$

The vector components on each side must be equal.

$$\frac{\partial E_z}{\partial y}-\frac{\partial E_y}{\partial z}=-j\omega\mu H_x \qquad \frac{\partial E_x}{\partial z}-\frac{\partial E_z}{\partial x}=-j\omega\mu H_y \qquad \frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}=-j\omega\mu H_z$$

EMPossible

Expand Governing Equations (2 of 2)

There are now six coupled partial differential equations.

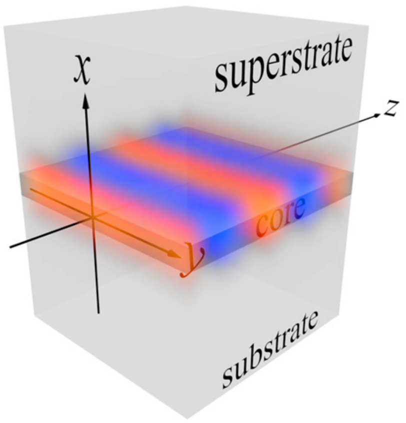
$$\nabla \times \vec{E} = -j\omega\mu\vec{H} \longrightarrow \begin{cases}\dfrac{\partial E_z}{\partial y}-\dfrac{\partial E_y}{\partial z}=-j\omega\mu H_x\\[2mm] \dfrac{\partial E_x}{\partial z}-\dfrac{\partial E_z}{\partial x}=-j\omega\mu H_y\\[2mm] \dfrac{\partial E_y}{\partial x}-\dfrac{\partial E_x}{\partial y}=-j\omega\mu H_z\end{cases}$$

$$\nabla \times \vec{H} = j\omega\varepsilon\vec{E} \longrightarrow \begin{cases}\dfrac{\partial H_z}{\partial y}-\dfrac{\partial H_y}{\partial z}=j\omega\varepsilon E_x\\[2mm] \dfrac{\partial H_x}{\partial z}-\dfrac{\partial H_z}{\partial x}=j\omega\varepsilon E_y\\[2mm] \dfrac{\partial H_y}{\partial x}-\dfrac{\partial H_x}{\partial y}=j\omega\varepsilon E_z\end{cases}$$

EMPossible

How to Reduce Dimensions

It is always good practice to minimize the number of dimensions utilized in a numerical analysis.



$\underline{x}$
Material changes as a function of x. The mode profile will change as a function of x. This dimension must be retained.

$\underline{y}$
Device is uniform. Wave does not propagate in this direction. Mode profile is uniform.

$$\frac{\partial}{\partial y} = 0$$

$\underline{z}$
Device is uniform. Wave propagates in this direction so wave phase is increasing.

$$\frac{\partial}{\partial z} = -j\beta$$

EMPossible

Apply $\partial/\partial y = 0$

Since nothing is changing in the y direction, any derivative with respect to y must be zero.

$$\begin{aligned}
\cancel{\frac{\partial E_z}{\partial y}} - \frac{\partial E_y}{\partial z} &= -j\omega\mu H_x & & & -\frac{\partial E_y}{\partial z} &= -j\omega\mu H_x \\
\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} &= -j\omega\mu H_y & &\Longrightarrow & \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} &= -j\omega\mu H_y \\
\frac{\partial E_y}{\partial x} - \cancel{\frac{\partial E_x}{\partial y}} &= -j\omega\mu H_z & & & \frac{\partial E_y}{\partial x} &= -j\omega\mu H_z
\end{aligned}$$

$$\begin{aligned}
\cancel{\frac{\partial H_z}{\partial y}} - \frac{\partial H_y}{\partial z} &= j\omega\varepsilon E_x & & & -\frac{\partial H_y}{\partial z} &= j\omega\varepsilon E_x \\
\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} &= j\omega\varepsilon E_y & &\Longrightarrow & \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} &= j\omega\varepsilon E_y \\
\frac{\partial H_y}{\partial x} - \cancel{\frac{\partial H_x}{\partial y}} &= j\omega\varepsilon E_z & & & \frac{\partial H_y}{\partial x} &= j\omega\varepsilon E_z
\end{aligned}$$

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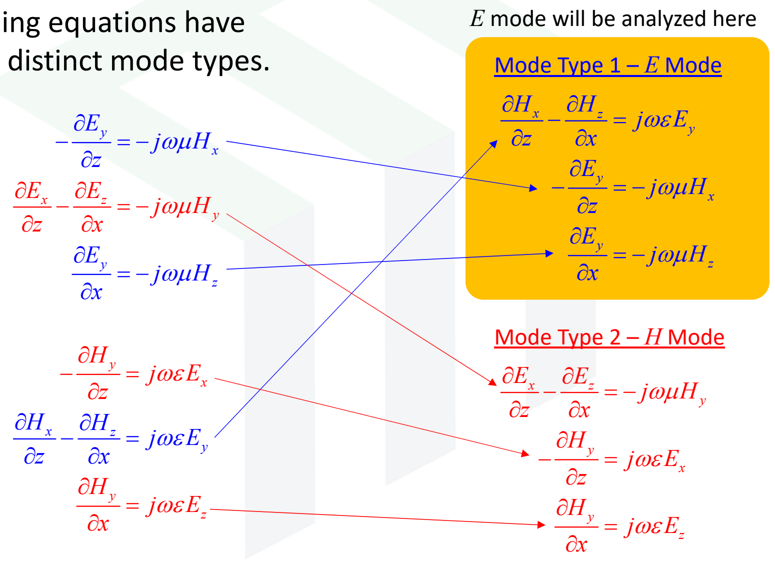
Two Distinct Mode Types

The revised governing equations have separated into two distinct mode types.



$$-\frac{\partial E_y}{\partial z} = -j\omega\mu H_x$$

$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -j\omega\mu H_y$$

$$\frac{\partial E_y}{\partial x} = -j\omega\mu H_z$$

$$-\frac{\partial H_y}{\partial z} = j\omega\varepsilon E_x$$

$$\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = j\omega\varepsilon E_y$$

$$\frac{\partial H_y}{\partial x} = j\omega\varepsilon E_z$$

E mode will be analyzed here

Mode Type 1 – *E* Mode

$$\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = j\omega\varepsilon E_y$$

$$-\frac{\partial E_y}{\partial z} = -j\omega\mu H_x$$

$$\frac{\partial E_y}{\partial x} = -j\omega\mu H_z$$

Mode Type 2 – *H* Mode

$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -j\omega\mu H_y$$

$$-\frac{\partial H_y}{\partial z} = j\omega\varepsilon E_x$$

$$\frac{\partial H_y}{\partial x} = j\omega\varepsilon E_z$$

What About $\partial/\partial z$?

The guided mode has the following mathematical form

$$\vec{E}(x,y,z) = \vec{A}(x)e^{-j\beta z}$$

Calculate the partial derivative with respect to z and see what happens.

$$\frac{\partial}{\partial z}\vec{E}(x,y,z) = \frac{\partial}{\partial z}\left[\vec{A}(x)e^{-j\beta z}\right] = \vec{A}(x)\frac{\partial}{\partial z}e^{-j\beta z} + \cancel{e^{-j\beta z}\frac{\partial}{\partial z}\vec{A}(x)} = -j\beta\underbrace{\vec{A}(x)e^{-j\beta z}}_{\vec{E}(x,y,z)}$$

$$= -j\beta\vec{E}(x,y,z)$$

It can be concluded that for this slab waveguide analysis,

$$\frac{\partial}{\partial z} = -j\beta$$

1D Governing Equations

The equations for the E mode were

$$\frac{\partial}{\partial z}H_x - \frac{\partial H_z}{\partial x} = j\omega\varepsilon E_y$$
$$-\frac{\partial}{\partial z}E_y = -j\omega\mu H_x$$
$$\frac{\partial E_y}{\partial x} = -j\omega\mu H_z$$

Now replace $\partial/\partial z$ with $-j\beta z$.

$$-j\beta H_x - \frac{dH_z}{dx} = j\omega\varepsilon E_y$$
$$j\beta E_y = -j\omega\mu H_x$$
$$\frac{dE_y}{dx} = -j\omega\mu H_z$$

The partial derivative has become an ordinary derivative because there is only one independent variable remaining…x.

Normalize the Grid Coordinate(s)

Before converting the equations to matrix form, the spatial coordinate x should be normalized to put it in terms of the free space wavelength λ_0 in some manner.

$$\tilde{x} \overset{???}{=} \frac{x}{\lambda_0}$$

It will be mathematically convenient to normalize x by multiplying by the free space wave number k_0 instead of dividing by λ_0.

$$\tilde{x} = k_0 x \qquad k_0 = \frac{2\pi}{\lambda_0}$$

However, multiplying by k_0 also scales x by a factor of 2π.

Observe that multiplying by k_0 is still dividing by λ_0.

Normalizing Maxwell’s Equations

Start with the following equation,

$$-j\beta H_x - \frac{dH_z}{dx} = j\omega\varepsilon E_y$$

and replace x with $\tilde{x}/k_0$.

$$-j\beta H_x - k_0\frac{dH_z}{d\tilde{x}} = j\omega\varepsilon E_y$$

Next, divide both sides of the equation by k_0.

$$-j\frac{\beta}{k_0}H_x - \frac{dH_z}{d\tilde{x}} = \frac{j\omega\varepsilon}{k_0}E_y$$

Recognizing that $\beta = k_0 n_{\text{eff}}$, the equation becomes

$$-jn_{\text{eff}}H_x - \frac{dH_z}{d\tilde{x}} = \frac{j\omega\varepsilon}{k_0}E_y = \frac{j\omega\varepsilon_0\varepsilon_r}{\omega\sqrt{\mu_0\varepsilon_0}}E_y = j\sqrt{\frac{\varepsilon_0}{\mu_0}}\varepsilon_r E_y$$

Normalize All E Mode Equations

Applying the normalizations to all three equations gives

$$-jn_{\text{eff}}H_x - \frac{dH_z}{d\tilde{x}} = j\sqrt{\frac{\varepsilon_0}{\mu_0}}\varepsilon_r E_y$$

$$jn_{\text{eff}}E_y = -j\sqrt{\frac{\mu_0}{\varepsilon_0}}\mu_r H_x$$

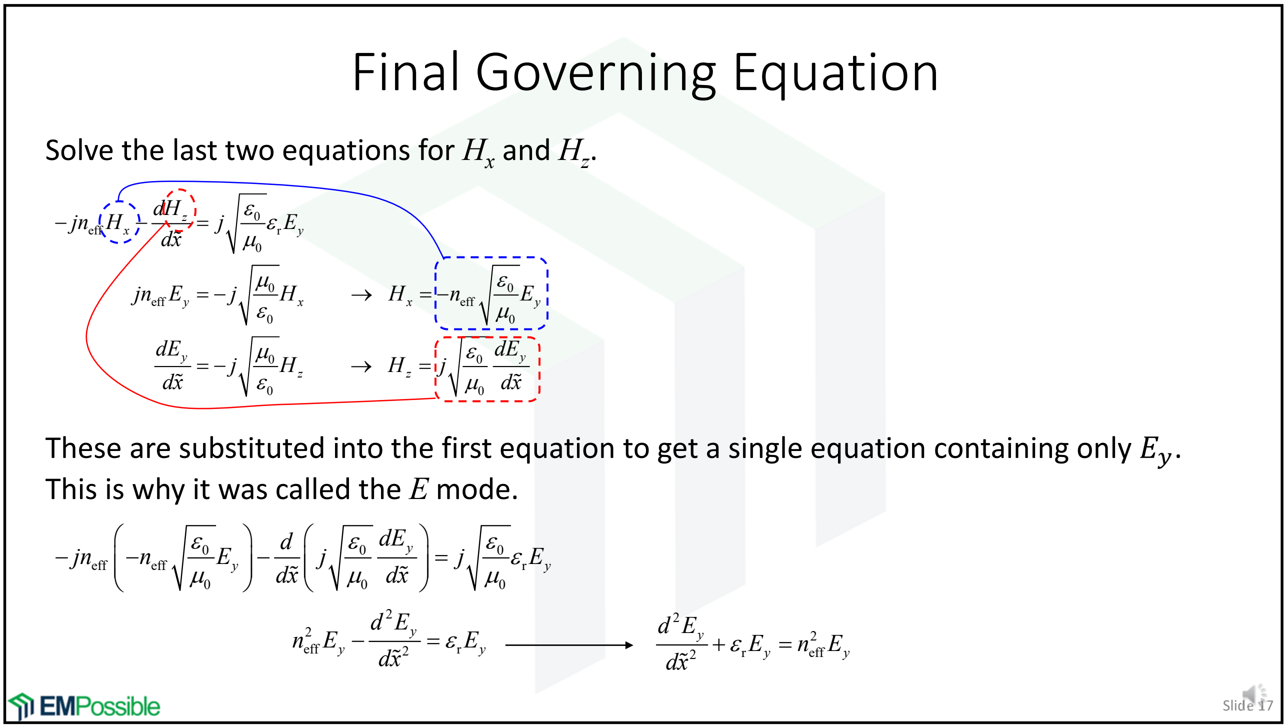
$$\frac{dE_y}{d\tilde{x}} = -j\sqrt{\frac{\mu_0}{\varepsilon_0}}\mu_r H_z$$

Last, at optical frequencies, the magnetic response is negligible so $\mu_r = 1$.

$$-jn_{\text{eff}}H_x - \frac{dH_z}{d\tilde{x}} = j\sqrt{\frac{\varepsilon_0}{\mu_0}}\varepsilon_r E_y$$

$$jn_{\text{eff}}E_y = -j\sqrt{\frac{\mu_0}{\varepsilon_0}}H_x$$

$$\frac{dE_y}{d\tilde{x}} = -j\sqrt{\frac{\mu_0}{\varepsilon_0}}H_z$$



Final Governing Equation

Solve the last two equations for H_x and H_z.

$$-jn_{\text{eff}}\boxed{H_x}-\boxed{\frac{dH_z}{d\tilde{x}}}=j\sqrt{\frac{\varepsilon_0}{\mu_0}}\varepsilon_r E_y$$

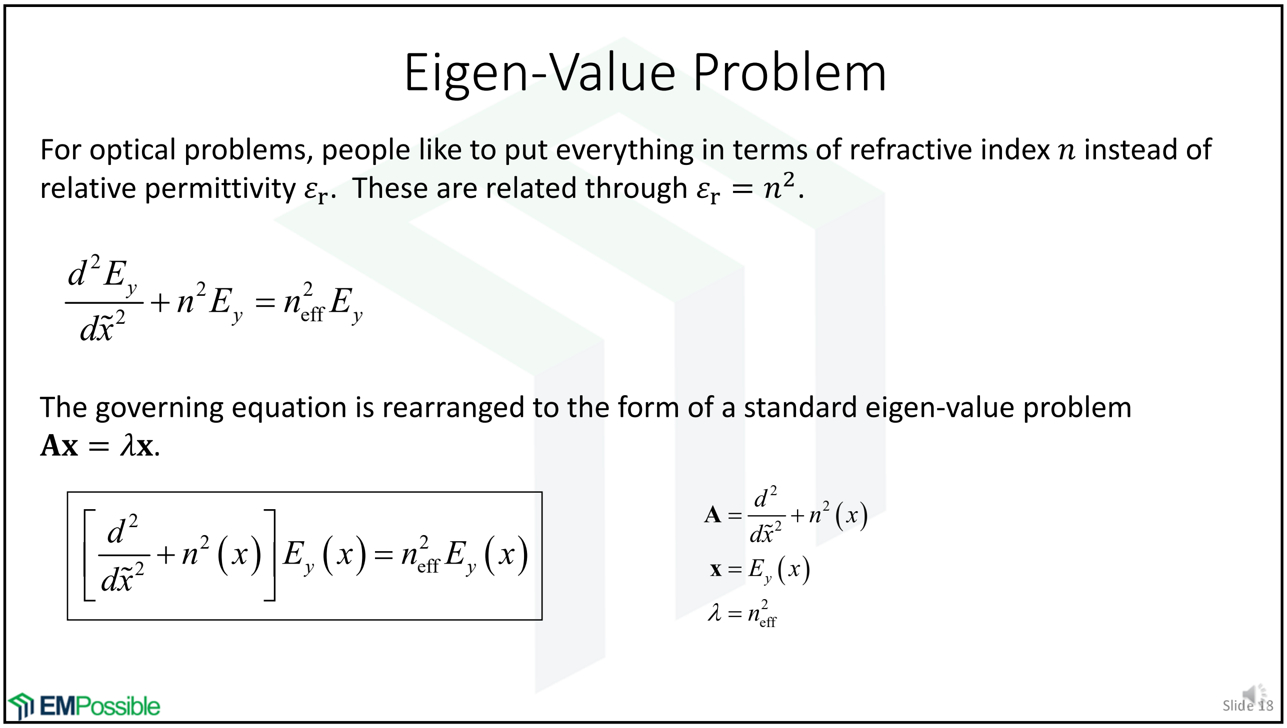
$$jn_{\text{eff}}E_y=-j\sqrt{\frac{\mu_0}{\varepsilon_0}}H_x \quad\rightarrow\quad H_x=\boxed{-n_{\text{eff}}\sqrt{\frac{\varepsilon_0}{\mu_0}}E_y}$$

$$\frac{dE_y}{d\tilde{x}}=-j\sqrt{\frac{\mu_0}{\varepsilon_0}}H_z \quad\rightarrow\quad H_z=\boxed{j\sqrt{\frac{\varepsilon_0}{\mu_0}}\frac{dE_y}{d\tilde{x}}}$$

These are substituted into the first equation to get a single equation containing only E_y. This is why it was called the E mode.

$$-jn_{\text{eff}}\left(-n_{\text{eff}}\sqrt{\frac{\varepsilon_0}{\mu_0}}E_y\right)-\frac{d}{d\tilde{x}}\left(j\sqrt{\frac{\varepsilon_0}{\mu_0}}\frac{dE_y}{d\tilde{x}}\right)=j\sqrt{\frac{\varepsilon_0}{\mu_0}}\varepsilon_r E_y$$

$$n_{\text{eff}}^2E_y-\frac{d^2E_y}{d\tilde{x}^2}=\varepsilon_r E_y \longrightarrow \frac{d^2E_y}{d\tilde{x}^2}+\varepsilon_r E_y=n_{\text{eff}}^2E_y$$



Eigen-Value Problem

For optical problems, people like to put everything in terms of refractive index n instead of relative permittivity ε_r. These are related through $\varepsilon_r = n^2$.

$$\frac{d^2E_y}{d\tilde{x}^2}+n^2E_y=n_{\text{eff}}^2E_y$$

The governing equation is rearranged to the form of a standard eigen-value problem $\mathbf{Ax}=\lambda\mathbf{x}$.

$$\boxed{\left[\frac{d^2}{d\tilde{x}^2}+n^2(x)\right]E_y(x)=n_{\text{eff}}^2E_y(x)}$$

$$\mathbf{A}=\frac{d^2}{d\tilde{x}^2}+n^2(x)$$

$$\mathbf{x}=E_y(x)$$

$$\lambda=n_{\text{eff}}^2$$

Matrix Form

Go term-by-term to write the equation in matrix form.

$$\left[\frac{d^2}{d\tilde{x}^2}+n^2(x)\right]E_y(x)=n_{\text{eff}}^2E_y(x)$$

$$\left(\mathbf{D}_{\tilde{x}}^2+\mathbf{n}^2\right)\mathbf{e}_y=n_{\text{eff}}^2\mathbf{e}_y$$

or

$$\left(\mathbf{D}_{\tilde{x}}^2+\boldsymbol{\varepsilon}\right)\mathbf{e}_y=n_{\text{eff}}^2\mathbf{e}_y$$

Eigen Matrix

Eigen Value

EMPossible

Solution

Solving the Eigen-Value Problem

$$\left(\mathbf{D}_{\tilde{x}}^2+\boldsymbol{\varepsilon}\right)\mathbf{e}_y = n_{\text{eff}}^2\mathbf{e}_y \quad \rightarrow \quad \begin{matrix}\mathbf{V} \equiv \text{Eigen-vector matrix} \\ \boldsymbol{\lambda} \equiv \text{Eigen-value matrix}\end{matrix}$$

$$\mathbf{V} = \left[\begin{bmatrix} e_y^{(1)}(1) \\ e_y^{(1)}(2) \\ e_y^{(1)}(3) \\ \vdots \\ e_y^{(1)}(N_x-1) \\ e_y^{(1)}(N_x) \end{bmatrix} \begin{bmatrix} e_y^{(2)}(1) \\ e_y^{(2)}(2) \\ e_y^{(2)}(3) \\ \vdots \\ e_y^{(2)}(N_x-1) \\ e_y^{(2)}(N_x) \end{bmatrix} \cdots \begin{bmatrix} e_y^{(M)}(1) \\ e_y^{(M)}(2) \\ e_y^{(M)}(3) \\ \vdots \\ e_y^{(M)}(N_x-1) \\ e_y^{(M)}(N_x) \end{bmatrix}\right]$$

M = # modes
Usually $M = N_x$

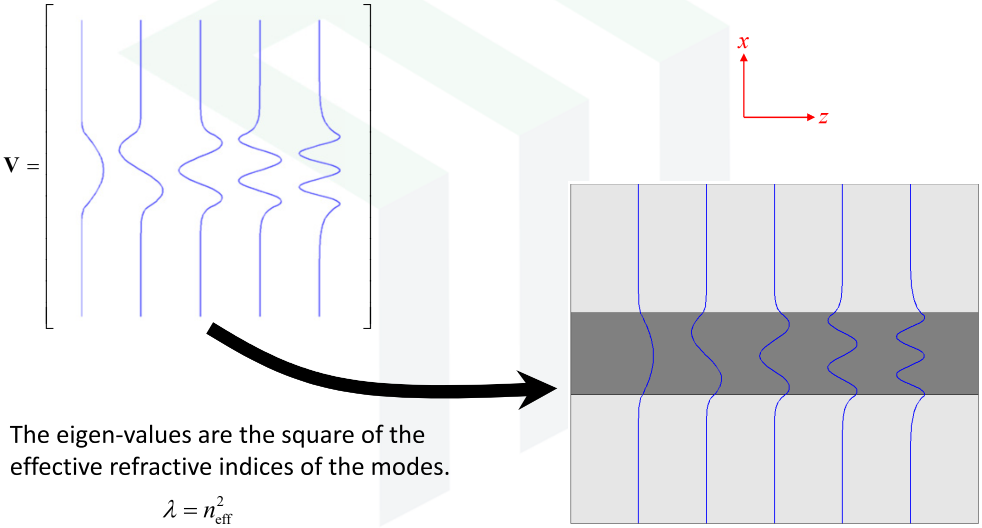
$$\boldsymbol{\lambda} = \begin{bmatrix} \left(n_{\text{eff}}^{(1)}\right)^2 & & & \\ & \left(n_{\text{eff}}^{(2)}\right)^2 & & \\ & & \ddots & \\ & & & \left(n_{\text{eff}}^{(M)}\right)^2 \end{bmatrix}$$

Eigen-vectors and eigen-values come in pairs.

Do not mix up their pairing!

Visualizing the Solution

The columns of the eigen-vector matrix are pictures of the modes.



The eigen-values are the square of the effective refractive indices of the modes.

$$\lambda = n_{\text{eff}}^2$$

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