Computational Science:
Computational Methods in Engineering

Formulation of Transmission Line Analysis

Outline

- Derivation of governing equations
- Finite-difference approximation of governing equations

Derivation of Governing Equations

Equations of Electrostatics

Any electromagnetic analysis begins with Maxwell's equations.

For electrostatic problems, start with the following.

$$\nabla \bullet \vec{D} = 0 \qquad \text{Eq. (1)}$$

$$\vec{D} = [\varepsilon]\vec{E} \qquad \text{Eq. (2)}$$

$$\vec{E} = -\nabla V \qquad \text{Eq. (3)}$$

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Expand Equations

$$\nabla \bullet \vec{D} = 0 \quad \rightarrow \quad \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{bmatrix} \begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = 0$$

$$\vec{D} = [\varepsilon]\vec{E} \quad \rightarrow \quad \begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \begin{bmatrix} \varepsilon_{xx} & 0 & 0 \\ 0 & \varepsilon_{yy} & 0 \\ 0 & 0 & \varepsilon_{zz} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}$$

$$\vec{E} = -\nabla V \quad \rightarrow \quad \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} = -\begin{bmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{bmatrix} V$$

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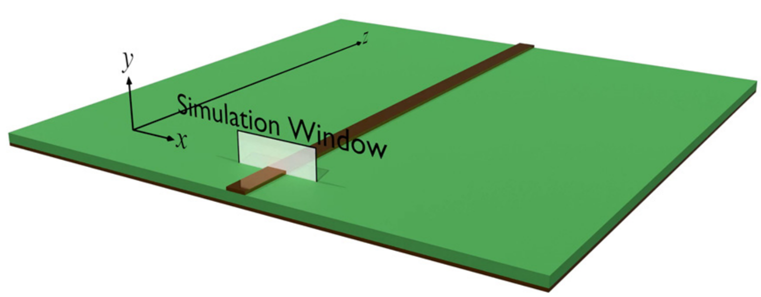
2D Analysis of Cross Section

It is desired to analyze a transmission line that is uniform in the direction that the wave propagates.

Let this direction be z.

Since the device is uniform in z, nothing changes in the z direction.

$$\frac{\partial}{\partial z} = 0$$



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Reduced Equations for 2D

$$\nabla \bullet \vec{D} = 0 \quad \rightarrow \quad \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \cancel{\frac{\partial}{\partial z}} \end{bmatrix} \begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = 0 \quad \rightarrow \quad \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \end{bmatrix} \begin{bmatrix} D_x \\ D_y \end{bmatrix} = 0 \qquad D_z = 0$$

$$\vec{D} = \varepsilon \vec{E} \quad \rightarrow \quad \begin{bmatrix} D_x \\ D_y \\ \cancel{D_z} \end{bmatrix} = \begin{bmatrix} \varepsilon_{xx} & 0 & 0 \\ 0 & \varepsilon_{yy} & 0 \\ 0 & 0 & \varepsilon_{zz} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ \cancel{E_z} \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} D_x \\ D_y \end{bmatrix} = \begin{bmatrix} \varepsilon_{xx} & 0 \\ 0 & \varepsilon_{yy} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \end{bmatrix} \qquad E_z = 0$$

$$\vec{E} = -\nabla V \quad \rightarrow \quad \begin{bmatrix} E_x \\ E_y \\ \cancel{E_z} \end{bmatrix} = -\begin{bmatrix} \partial/\partial x \\ \partial/\partial y \\ \cancel{\partial/\partial z} \end{bmatrix} V \quad \rightarrow \quad \begin{bmatrix} E_x \\ E_y \end{bmatrix} = -\begin{bmatrix} \partial/\partial x \\ \partial/\partial y \end{bmatrix} V$$

$$D_z = E_z = 0$$

This implies *transverse electromagnetic* (TEM) mode.

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Final Governing Equations

$$\begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \end{bmatrix} \begin{bmatrix} D_x \\ D_y \end{bmatrix} = 0$$

$$\begin{bmatrix} D_x \\ D_y \end{bmatrix} = \begin{bmatrix} \varepsilon_{xx} & 0 \\ 0 & \varepsilon_{yy} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \end{bmatrix}$$

$$\begin{bmatrix} E_x \\ E_y \end{bmatrix} = -\begin{bmatrix} \partial/\partial x \\ \partial/\partial y \end{bmatrix} V$$

$$\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} = 0$$

$$D_x = \varepsilon_{xx} E_x$$
$$D_y = \varepsilon_{yy} E_y$$

$$E_x = -\frac{\partial V}{\partial x}$$
$$E_y = -\frac{\partial V}{\partial y}$$

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Finite-Difference Approximation of Governing Equations

Grid Strategy (1 of 2)

The manner in which the functions are staggered across the grid comes from the governing equations themselves.

From the constitutive equations, observe that D_x and ε_{xx} should lie on the same points as E_x.

$$D_x = \varepsilon_{xx} E_x \quad \Rightarrow \quad D_x^{i,j} = \varepsilon_0 \varepsilon_{xx}^{i,j} E_x^{i,j}$$

Similarly, D_y and ε_{yy} should lie on the same points as E_y.

$$D_y = \varepsilon_{yy} E_y \quad \Rightarrow \quad D_y^{i,j} = \varepsilon_0 \varepsilon_{yy}^{i,j} E_y^{i,j}$$

Thus, the constitutive relations do not suggest any staggering is needed.

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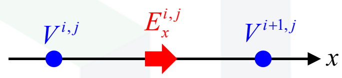
Grid Strategy (2 of 2)

Next, inspect the equations relating $\vec{E}$ and V.

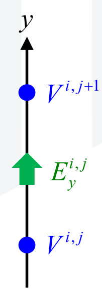
Observe that V will need to be staggered around E_x in the x direction.

$$E_x = -\frac{\partial V}{\partial x} \Rightarrow E_x^{i,j} = -\frac{V^{i+1,j} - V^{i,j}}{\Delta x}$$



Observe that V will need to be staggered around E_y in the y direction.

$$E_y = -\frac{\partial V}{\partial y} \Rightarrow E_y^{i,j} = -\frac{V^{i,j+1} - V^{i,j}}{\Delta y}$$



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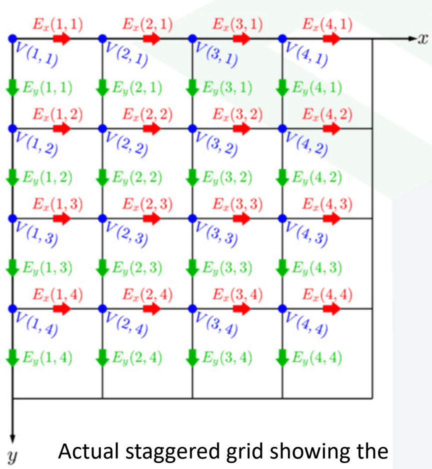
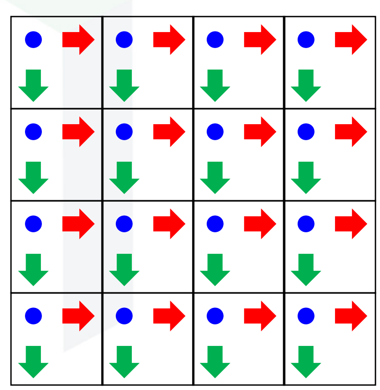
Grid Strategy (3 of 3)

Putting all of this together, the staggered grid is…



Actual staggered grid showing the true position of the function values.



An alternative view of the staggered grid that more clearly conveys which cell each function value resides in.

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Matrix Form of Governing Equations

There are five coupled equations. These can immediately be written in matrix form as:

$$\frac{\partial D_x}{\partial x}+\frac{\partial D_y}{\partial y}=0 \qquad \frac{D_x^{i,j}-D_x^{i-1,j}}{\Delta x}+\frac{D_y^{i,j}-D_y^{i,j-1}}{\Delta y}=0 \qquad \mathbf{D}_x^e\mathbf{d}_x+\mathbf{D}_y^e\mathbf{d}_y=\mathbf{0}$$

$$\begin{aligned}D_x&=\varepsilon_{xx}E_x\\ D_y&=\varepsilon_{yy}E_y\end{aligned} \quad \Rightarrow \quad \begin{aligned}D_x^{i,j}&=\varepsilon_0\varepsilon_{xx}^{i,j}E_x^{i,j}\\ D_y^{i,j}&=\varepsilon_0\varepsilon_{yy}^{i,j}E_y^{i,j}\end{aligned} \quad \Rightarrow \quad \begin{aligned}\mathbf{d}_x&=\varepsilon_0\boldsymbol{\varepsilon}_{xx}\mathbf{e}_x\\ \mathbf{d}_y&=\varepsilon_0\boldsymbol{\varepsilon}_{yy}\mathbf{e}_y\end{aligned}$$

$$\begin{aligned}E_x&=-\frac{\partial V}{\partial x}\\ E_y&=-\frac{\partial V}{\partial y}\end{aligned} \qquad \begin{aligned}E_x^{i,j}&=-\frac{V^{i+1,j}-V^{i,j}}{\Delta x}\\ E_y^{i,j}&=-\frac{V^{i,j+1}-V^{i,j}}{\Delta y}\end{aligned} \qquad \begin{aligned}\mathbf{e}_x&=-\mathbf{D}_x^v\mathbf{v}\\ \mathbf{e}_y&=-\mathbf{D}_y^v\mathbf{v}\end{aligned}$$

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Block Matrix Form

Write the matrix equations in block matrix form.

$$\begin{bmatrix}\frac{\partial}{\partial x} & \frac{\partial}{\partial y}\end{bmatrix}\begin{bmatrix}D_x\\ D_y\end{bmatrix}=0 \qquad \begin{bmatrix}\mathbf{D}_x^e & \mathbf{D}_y^e\end{bmatrix}\begin{bmatrix}\mathbf{d}_x\\ \mathbf{d}_y\end{bmatrix}=\mathbf{0}$$

$$\begin{bmatrix}D_x\\ D_y\end{bmatrix}=\begin{bmatrix}\varepsilon_{xx} & 0\\ 0 & \varepsilon_{yy}\end{bmatrix}\begin{bmatrix}E_x\\ E_y\end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix}\mathbf{d}_x\\ \mathbf{d}_y\end{bmatrix}=\varepsilon_0\begin{bmatrix}\boldsymbol{\varepsilon}_{xx} & \mathbf{0}\\ \mathbf{0} & \boldsymbol{\varepsilon}_{yy}\end{bmatrix}\begin{bmatrix}\mathbf{e}_x\\ \mathbf{e}_y\end{bmatrix}$$

$$\begin{bmatrix}E_x\\ E_y\end{bmatrix}=-\begin{bmatrix}\partial/\partial x\\ \partial/\partial y\end{bmatrix}V \qquad \begin{bmatrix}\mathbf{e}_x\\ \mathbf{e}_y\end{bmatrix}=-\begin{bmatrix}\mathbf{D}_x^v\\ \mathbf{D}_y^v\end{bmatrix}\mathbf{v}$$

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Eliminate $\vec{D}$ Field

Eliminate the $\vec{D}$ field by substituting the second equation into the first.

$$\begin{bmatrix} \mathbf{d}_x \\ \mathbf{d}_y \end{bmatrix} = \varepsilon_0 \begin{bmatrix} \boldsymbol{\varepsilon}_{xx} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\varepsilon}_{yy} \end{bmatrix} \begin{bmatrix} \mathbf{e}_x \\ \mathbf{e}_y \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{D}_x^e & \mathbf{D}_y^e \end{bmatrix} \begin{bmatrix} \mathbf{d}_x \\ \mathbf{d}_y \end{bmatrix} = \mathbf{0}$$

$$\begin{bmatrix} \mathbf{D}_x^e & \mathbf{D}_y^e \end{bmatrix} \left(\varepsilon_0 \begin{bmatrix} \boldsymbol{\varepsilon}_{xx} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\varepsilon}_{yy} \end{bmatrix} \begin{bmatrix} \mathbf{e}_x \\ \mathbf{e}_y \end{bmatrix} \right) = \mathbf{0}$$

$$\begin{bmatrix} \mathbf{D}_x^e & \mathbf{D}_y^e \end{bmatrix} \begin{bmatrix} \boldsymbol{\varepsilon}_{xx} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\varepsilon}_{yy} \end{bmatrix} \begin{bmatrix} \mathbf{e}_x \\ \mathbf{e}_y \end{bmatrix} = \mathbf{0}$$

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Eliminate $\vec{E}$ Field

The $\vec{E}$ field can be eliminated in a similar way.

$$\begin{bmatrix} \mathbf{e}_x \\ \mathbf{e}_y \end{bmatrix} = -\begin{bmatrix} \mathbf{D}_x^v \\ \mathbf{D}_y^v \end{bmatrix} \mathbf{v}$$

$$\begin{bmatrix} \mathbf{D}_x^e & \mathbf{D}_y^e \end{bmatrix} \begin{bmatrix} \boldsymbol{\varepsilon}_{xx} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\varepsilon}_{yy} \end{bmatrix} \begin{bmatrix} \mathbf{e}_x \\ \mathbf{e}_y \end{bmatrix} = \mathbf{0}$$

$$\begin{bmatrix} \mathbf{D}_x^e & \mathbf{D}_y^e \end{bmatrix} \begin{bmatrix} \boldsymbol{\varepsilon}_{xx} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\varepsilon}_{yy} \end{bmatrix} \left(-\begin{bmatrix} \mathbf{D}_x^v \\ \mathbf{D}_y^v \end{bmatrix} \mathbf{v} \right) = \mathbf{0}$$

$$\begin{bmatrix} \mathbf{D}_x^e & \mathbf{D}_y^e \end{bmatrix} \begin{bmatrix} \boldsymbol{\varepsilon}_{xx} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\varepsilon}_{yy} \end{bmatrix} \begin{bmatrix} \mathbf{D}_x^v \\ \mathbf{D}_y^v \end{bmatrix} \mathbf{v} = \mathbf{0}$$

There is now a single scalar differential equation in matrix form.

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Final Matrix Equations

The final matrix equation is the matrix form of the inhomogeneous Laplace's equation.

Inhomogeneous Laplace's Equation

$$\nabla \bullet \left[\varepsilon_{\mathrm{r}} \left(\nabla V \right) \right] = 0 \quad \Rightarrow \quad \begin{bmatrix} \mathbf{D}_x^e & \mathbf{D}_y^e \end{bmatrix} \begin{bmatrix} \boldsymbol{\varepsilon}_{xx} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\varepsilon}_{yy} \end{bmatrix} \begin{bmatrix} \mathbf{D}_x^v \\ \mathbf{D}_y^v \end{bmatrix} \mathbf{v} = \mathbf{0}$$

Homogeneous Laplace's Equation

The block matrix equation of the homogeneous Laplace's equation is easily written from the above result because $\varepsilon_{\mathrm{r}} = 1$ everywhere.

$$\nabla^2 V_{\mathrm{h}} = 0 \quad \Rightarrow \quad \begin{bmatrix} \mathbf{D}_x^e & \mathbf{D}_y^e \end{bmatrix} \begin{bmatrix} \mathbf{D}_x^v \\ \mathbf{D}_y^v \end{bmatrix} \mathbf{v} = \mathbf{0}$$

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