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Computational Science:
Computational Methods in Engineering

Gauss-Jordan Method

What is the Gauss-Jordan Method?

The Gauss-Jordan method is a technique to solve

$$\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix}\begin{bmatrix} \\ x \\ \\ \end{bmatrix}=\begin{bmatrix} \\ b \\ \\ \end{bmatrix}$$

or to calculate matrix inverses.

$$\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix}^{-1}$$

It is an excellent technique for solving these problems by hand!

Step 1

Start with a matrix problem $[A][x]=[b]$.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

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Step 2

Construct an *augmented matrix* from $[A]$ and $[b]$.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\begin{bmatrix} [A] & [b] \end{bmatrix} \rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{bmatrix}$$

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Step 3

Normalize the first row by dividing by the diagonal element a_{11}.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & a_{12}/a_{11} & a_{13}/a_{11} & b_1/a_{11} \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & a'_{12} & a'_{13} & b'_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{bmatrix}$$

$$a'_{12} = \frac{a_{12}}{a_{11}} \qquad a'_{13} = \frac{a_{13}}{a_{11}} \qquad b'_1 = \frac{b_1}{a_{11}}$$

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Step 4

Eliminate x_1 from all other rows.

$$(\text{New Row 2}) = (\text{Old Row 2}) - a_{21}(\text{Row 1})$$
$$(\text{New Row 3}) = (\text{Old Row 3}) - a_{31}(\text{Row 1})$$

$$\begin{bmatrix} 1 & a'_{12} & a'_{13} & b'_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & a'_{12} & a'_{13} & b'_1 \\ 0 & a'_{22} & a'_{23} & b'_2 \\ 0 & a'_{32} & a'_{33} & b'_3 \end{bmatrix}$$

New Row 2

$$a'_{22} = a_{22} - a_{21}a'_{12}$$
$$a'_{23} = a_{23} - a_{21}a'_{13}$$
$$b'_2 = b_2 - a_{21}b'_1$$

New Row 3

$$a'_{32} = a_{32} - a_{31}a'_{12}$$
$$a'_{33} = a_{33} - a_{31}a'_{13}$$
$$b'_3 = b_3 - a_{31}b'_1$$

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Step 5

Normalize the second row by dividing by the diagonal element a'_{22}.

$$\begin{bmatrix} 1 & a'_{12} & a'_{13} & b'_1 \\ 0 & a'_{22} & a'_{23} & b'_2 \\ 0 & a'_{32} & a'_{33} & b'_3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & a'_{12} & a'_{13} & b'_1 \\ 0 & 1 & a'_{23}/a'_{22} & b'_2/a'_{22} \\ 0 & a'_{32} & a'_{33} & b'_3 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & a'_{12} & a'_{13} & b'_1 \\ 0 & 1 & a''_{23} & b''_2 \\ 0 & a'_{32} & a'_{33} & b'_3 \end{bmatrix}$$

$$a''_{23} = a'_{23}/a'_{22} \qquad b''_2 = b'_2/a'_{22}$$

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Step 6

Eliminate x_2 from all other rows.

$$(\text{New Row 1}) = (\text{Old Row 1}) - a'_{12}(\text{Row 2})$$
$$(\text{New Row 3}) = (\text{Old Row 3}) - a'_{32}(\text{Row 2})$$

$$\begin{bmatrix} 1 & a'_{12} & a'_{13} & b'_1 \\ 0 & 1 & a''_{23} & b''_2 \\ 0 & a'_{32} & a'_{33} & b'_3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & a''_{13} & b''_1 \\ 0 & 1 & a''_{23} & b''_2 \\ 0 & 0 & a''_{33} & b''_3 \end{bmatrix}$$

New Row 1

$$a''_{13} = a'_{13} - a'_{12}a''_{23}$$
$$b''_1 = b'_1 - a'_{12}b''_2$$

New Row 3

$$a''_{33} = a'_{33} - a'_{32}a''_{23}$$
$$b''_3 = b'_3 - a'_{32}b''_2$$

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Step 7

Normalize the third row by dividing by the diagonal element a''_{33} .

$$\begin{bmatrix} 1 & 0 & a''_{13} & b''_1 \\ 0 & 1 & a''_{23} & b''_2 \\ 0 & 0 & a''_{33} & b''_3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & a''_{13} & b''_1 \\ 0 & 1 & a''_{23} & b''_2 \\ 0 & 0 & 1 & b''_3/a''_{33} \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & a''_{13} & b''_1 \\ 0 & 1 & a''_{23} & b''_2 \\ 0 & 0 & 1 & b'''_3 \end{bmatrix}$$

$$b'''_3 = b''_3/a''_{33}$$

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Step 8

Eliminate x_3 from all other rows.

$$(\text{New Row 1}) = (\text{Old Row 1}) - a''_{13}(\text{Row 3})$$
$$(\text{New Row 2}) = (\text{Old Row 2}) - a''_{23}(\text{Row 3})$$

$$\begin{bmatrix} 1 & 0 & a''_{13} & b''_1 \\ 0 & 1 & a''_{23} & b''_2 \\ 0 & 0 & 1 & b'''_3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & b'''_1 \\ 0 & 1 & 0 & b'''_2 \\ 0 & 0 & 1 & b'''_3 \end{bmatrix}$$

New Row 1

$$b'''_1 = b''_1 - a''_{13} b'''_3$$

New Row 2

$$b'''_2 = b''_2 - a''_{23} b'''_3$$

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Step 9

Extract solution from augmented matrix.

$$\begin{bmatrix} 1 & 0 & 0 & b_1''' \\ 0 & 1 & 0 & b_2''' \\ 0 & 0 & 1 & b_3''' \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1''' \\ b_2''' \\ b_3''' \end{bmatrix}$$

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Example

$[A] = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 1 \\ 1 & 1 & 0 \end{bmatrix} \quad [b] = \begin{bmatrix} 16 \\ 12 \\ 2 \end{bmatrix}$ Step 1 – Define problem

$[A \quad b] = \begin{bmatrix} 1 & 2 & 3 & 16 \\ 0 & 4 & 1 & 12 \\ 1 & 1 & 0 & 2 \end{bmatrix}$ Step 2 – Form augmented matrix

$\begin{bmatrix} 1 & 2 & 3 & 16 \\ 0 & 4 & 1 & 12 \\ 1 & 1 & 0 & 2 \end{bmatrix}$ Step 3 – Normalize row 1

$\begin{bmatrix} 1 & 2 & 3 & 16 \\ 0 & 4 & 1 & 12 \\ 0 & -1 & -3 & -14 \end{bmatrix}$ Step 4 – Eliminate x_1 from rows 2 and 3

$\begin{bmatrix} 1 & 2 & 3 & 16 \\ 0 & 1 & 0.25 & 3 \\ 0 & -1 & -3 & -14 \end{bmatrix}$ Step 5 – Normalize row 2

$\begin{bmatrix} 1 & 0 & 2.5 & 10 \\ 0 & 1 & 0.25 & 3 \\ 0 & 0 & -2.75 & -11 \end{bmatrix}$ Step 6 – Eliminate x_2 from rows 1 and 3

$\begin{bmatrix} 1 & 0 & 2.5 & 10 \\ 0 & 1 & 0.25 & 3 \\ 0 & 0 & 1 & 4 \end{bmatrix}$ Step 7 – Normalize row 3

$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 4 \end{bmatrix}$ Step 8 – Eliminate x_3 from rows 1 and 2

$[x] = \begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix}$ Step 9 – Extract answer from augmented matrix

* Be sure to extract the answer from the augmented matrix. The augmented matrix is not the final answer!

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Algorithm for Any Size Matrix

1. Define $[A]$ and $[b]$
2. Construct augmented matrix

   $$[U] = \big[[A] \ \ [b] \big]$$

   ```
   U = [ A b ];
   ```

3. Iterate through all rows (m)
   a) Normalize mth row by dividing by diagonal element.

      $$[U]_{\text{row } m} = [U]_{\text{row } m} \div a_{mm}$$

      ```
      U(m,:) = U(m,:)/U(m,m);
      ```

   b) Iterate through all other rows (r), skipping the mth row
      i. Eliminate x_m from row r

         $$[U]_{\text{row } r} = [U]_{\text{row } r} - a_{rm} \cdot [U]_{\text{row } m}$$

         ```
         U(r,:) = U(r,:) - U(r,m)*U(m,:);
         ```

4. Extract $[x]$ from augmented matrix

   $$[x] = [U]_{\text{column } M+1}$$

   ```
   x = U(:,M+1);
   ```

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How to Find Matrix Inverses

1. Define $[A]$
2. Construct augmented matrix

   $$[U] = \big[[A] \ \ [I] \big]$$

3. Perform Gauss-Jordan method (iterate through all rows)
4. Extract $[A]^{-1}$ from the final augmented matrix

   $$[U'] = \big[[I] \ \ [A]^{-1} \big]$$

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