Computational Science:
Computational Methods in Engineering

Gauss-Seidel Method

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The Jacobi method required $[A]$ to be diagonally dominant, which restricts what the method can be used to solve.

The Gauss-Seidel method is an alternative to the Jacobi method to overcome this limitation.

Formulation (1 of 4)

The matrix $[A]$ can be decomposed into the sum of a lower triangular matrix $[L']$ and an upper triangular matrix $[U']$.

$$[A]=[L']+[U']$$

$$[A]=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & \vdots & \ddots & \\ a_{N1} & a_{N2} & & a_{NN} \end{bmatrix} \quad [L']=\begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \\ a_{N1} & a_{N2} & & a_{NN} \end{bmatrix} \quad [U']=\begin{bmatrix} 0 & a_{12} & \cdots & a_{1N} \\ 0 & 0 & \cdots & a_{2N} \\ \vdots & \vdots & \ddots & \\ 0 & 0 & & 0 \end{bmatrix}$$

Note: $[L']$ and $[U']$ here are not the same $[L]$ and $[U]$ from LU decomposition.

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Formulation (2 of 4)

The goal is to solve $[A][x]=[b]$. Given that $[A]=[L']+[U']$, this becomes

$$[A][x]=[b]$$

$$\left([L']+[U']\right)[x]=[b]$$

$$[L'][x]+[U'][x]=[b]$$

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Formulation (3 of 4)

From experience with triangular matrices, it is known that $[L'][x]=[b]$ is very fast and efficient to solve for $[x]$ using forward-substitution.

Rearrange the matrix equation to take advantage of this.

$$[L'][x]+[U'][x]=[b]$$

$$[L'][x]=[b]-[U'][x]$$

$$[x]=\underbrace{[L']^{-1}\left([b]-[U'][x]\right)}_{\text{This can be solved very fast!}}$$

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Formulation (4 of 4)

The update equation is derived from the last expression.

$$[x]=[L']^{-1}\left([b]-[U'][x]\right)$$

$$\downarrow$$

$$\boxed{[x]_{i+1}=[L']^{-1}\left([b]-[U'][x]_i\right)}$$

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