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Computational Science:
Computational Methods in Engineering

Introduction to One-Dimensional Finite-Difference Method

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Outline

- Introduction & Problem Setup
- Conventional Finite-Difference Method
- Improved Finite-Difference Method

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Introduction & Problem Setup

3

The Finite-Difference Method

The finite-difference method is a way of obtaining a numerical solution to differential equations. It does not give a symbolic solution.

Governing Equation

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} + y = 0 \qquad 0 \le x \le 10$$

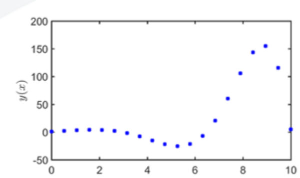
$$y(0) = 1,\ y(10) = 5$$

Matrix Equation

$$[A][y] = [b]$$

Numerical Solution

$$[y] = [A]^{-1}[b]$$



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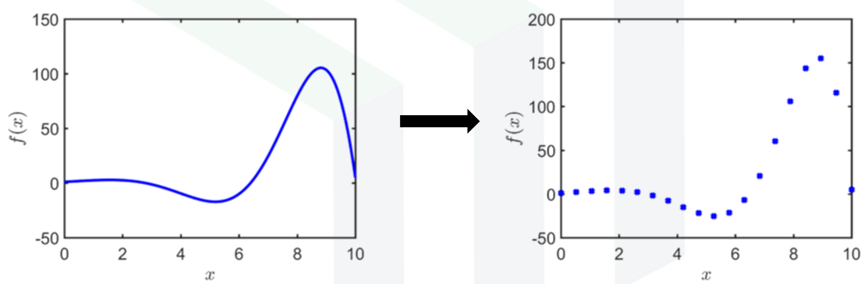
Functions are Discrete

To obtain a numerical solution using the finite-difference method, functions are stored as arrays of discrete points.



The function can be resolved more accurately using more points, but the solution will be more computationally intensive to obtain. This is a fundamental tradeoff.

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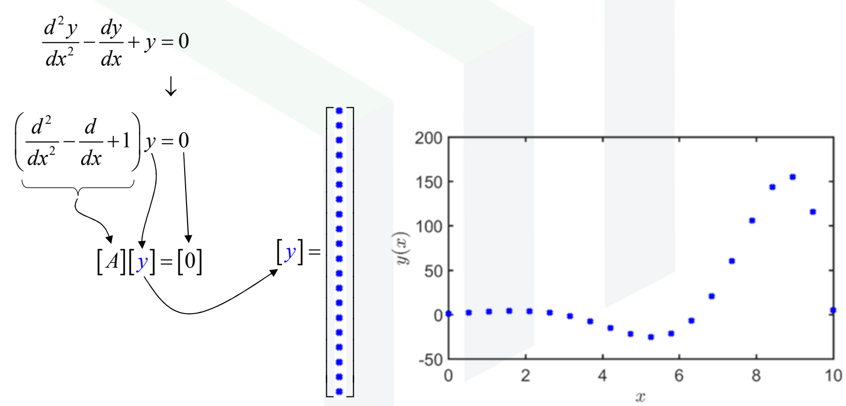
Discrete Functions are Stored as Column Vectors

Discrete functions are stored as a 1D array of numbers in a column vector.



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Conventional Finite-Difference Method

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Step 1 – Identify Governing Equation & Boundary Values

Governing Equation

$$\frac{d^2 y}{dx^2} - \frac{dy}{dx} + y = 0 \qquad 0 \le x \le 10$$

Boundary Values

$$y(0) = 1$$
$$y(10) = 5$$

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Step 2 – Approximate Derivatives with Finite-Differences (1 of 3)

First, let the function be discrete.

This allows the derivatives to be approximated with finite-differences.

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} + y = 0$$

$$\frac{y(x+\Delta x) - 2y(x) + y(x-\Delta x)}{\Delta x^2} - \frac{y(x) - y(x-\Delta x)}{\Delta x} + y(x) = 0$$

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Step 2 – Approximate Derivatives with Finite-Differences (2 of 3)

It is critical to ensure that each term in the finite-difference equation exists at the same point.

$$\underbrace{\frac{y(x+\Delta x) - 2y(x) + y(x-\Delta x)}{\Delta x^2}}_{\text{Exists at } x} - \underbrace{\frac{y(x) - y(x-\Delta x)}{\Delta x}}_{\text{Exists at } x - \Delta x/2} + y(x) = 0$$

Exists at x

Exists at x

This is not a healthy or stable formulation because not all of the terms exist at the same point.

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Step 2 – Approximate Derivatives with Finite-Differences (3 of 3)

This is the correct finite-difference equation.

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} + y = 0$$

$$\downarrow$$

$$\frac{y(x+\Delta x) - 2y(x) + y(x-\Delta x)}{\Delta x^2} - \frac{y(x+\Delta x) - y(x-\Delta x)}{2\Delta x} + y(x) = 0$$

All terms exist at x.

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Step 3 – Write Finite-Difference Equation Using Array Indices

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} + y = 0$$

$$\downarrow$$

$$\frac{y(x+\Delta x) - 2y(x) + y(x-\Delta x)}{\Delta x^2} - \frac{y(x+\Delta x) - y(x-\Delta x)}{2\Delta x} + y(x) = 0$$

$$\downarrow$$

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{\Delta x^2} - \frac{y_{i+1} - y_{i-1}}{2\Delta x} + y_i = 0$$

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Step 4 – Rearrange Finite-Difference Equation

The finite-difference equation is rearranged so as to collect the y terms.

$$\frac{y_{i+1}-2y_i+y_{i-1}}{\Delta x^2}-\frac{y_{i+1}-y_{i-1}}{2\Delta x}+y_i=0$$

$$\downarrow$$

$$\frac{1}{\Delta x^2}y_{i+1}-\frac{2}{\Delta x^2}y_i+\frac{1}{\Delta x^2}y_{i-1}-\frac{1}{2\Delta x}y_{i+1}+\frac{1}{2\Delta x}y_{i-1}+y_i=0$$

$$\downarrow$$

$$\left(\frac{1}{\Delta x^2}+\frac{1}{2\Delta x}\right)y_{i-1}+\left(1-\frac{2}{\Delta x^2}\right)y_i+\left(\frac{1}{\Delta x^2}-\frac{1}{2\Delta x}\right)y_{i+1}=0$$

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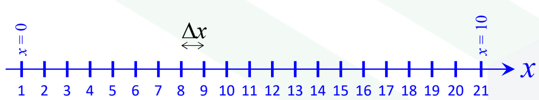
Step 5 – Setup Grid

Solve this problem using 21 points.



The grid spacing Δx is then

$$\Delta x=\frac{x_b-x_a}{N-1}=\frac{10-0}{21-1}=0.5$$

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Step 6 – Revise Finite-Difference Equation

Substituting Δx = 0.5 into the finite-difference equation gives

$$\left(\frac{1}{\Delta x^2}+\frac{1}{2\Delta x}\right)y_{i-1}+\left(1-\frac{2}{\Delta x^2}\right)y_i+\left(\frac{1}{\Delta x^2}-\frac{1}{2\Delta x}\right)y_{i+1}=0$$

$$\downarrow$$

$$\left(\frac{1}{0.5^2}+\frac{1}{2\cdot 0.5}\right)y_{i-1}+\left(1-\frac{2}{0.5^2}\right)y_i+\left(\frac{1}{0.5^2}-\frac{1}{2\cdot 0.5}\right)y_{i+1}=0$$

$$\downarrow$$

$$5y_{i-1}-7y_i+3y_{i+1}=0$$

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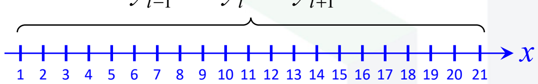
Step 7 – Write Finite-Difference Equation at Each Point on Grid

$$5y_{i-1}-7y_i+3y_{i+1}=0$$



$$\begin{aligned}
5y_0 - 7y_1 + 3y_2 &= 0\\
5y_1 - 7y_2 + 3y_3 &= 0\\
5y_2 - 7y_3 + 3y_4 &= 0\\
5y_3 - 7y_4 + 3y_5 &= 0\\
5y_4 - 7y_5 + 3y_6 &= 0\\
5y_5 - 7y_6 + 3y_7 &= 0\\
5y_6 - 7y_7 + 3y_8 &= 0\\
5y_7 - 7y_8 + 3y_9 &= 0\\
5y_8 - 7y_9 + 3y_{10} &= 0\\
5y_9 - 7y_{10} + 3y_{11} &= 0\\
5y_{10} - 7y_{11} + 3y_{12} &= 0\\
5y_{11} - 7y_{12} + 3y_{13} &= 0\\
5y_{12} - 7y_{13} + 3y_{14} &= 0\\
5y_{13} - 7y_{14} + 3y_{15} &= 0\\
5y_{14} - 7y_{15} + 3y_{16} &= 0\\
5y_{15} - 7y_{16} + 3y_{17} &= 0\\
5y_{16} - 7y_{17} + 3y_{18} &= 0\\
5y_{17} - 7y_{18} + 3y_{19} &= 0\\
5y_{18} - 7y_{19} + 3y_{20} &= 0\\
5y_{19} - 7y_{20} + 3y_{21} &= 0\\
5y_{20} - 7y_{21} + 3y_{22} &= 0
\end{aligned}$$

These terms exist outside of the grid.

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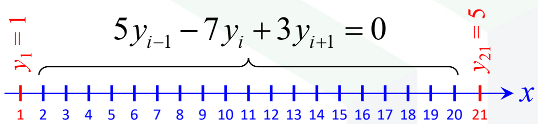
Step 8 – Incorporate Boundary Values



$$
\begin{aligned}
y_1 &= 1 \\
5y_1 - 7y_2 + 3y_3 &= 0 \\
5y_2 - 7y_3 + 3y_4 &= 0 \\
5y_3 - 7y_4 + 3y_5 &= 0 \\
5y_4 - 7y_5 + 3y_6 &= 0 \\
5y_5 - 7y_6 + 3y_7 &= 0 \\
5y_6 - 7y_7 + 3y_8 &= 0 \\
5y_7 - 7y_8 + 3y_9 &= 0 \\
5y_8 - 7y_9 + 3y_{10} &= 0 \\
5y_9 - 7y_{10} + 3y_{11} &= 0 \\
5y_{10} - 7y_{11} + 3y_{12} &= 0 \\
5y_{11} - 7y_{12} + 3y_{13} &= 0 \\
5y_{12} - 7y_{13} + 3y_{14} &= 0 \\
5y_{13} - 7y_{14} + 3y_{15} &= 0 \\
5y_{14} - 7y_{15} + 3y_{16} &= 0 \\
5y_{15} - 7y_{16} + 3y_{17} &= 0 \\
5y_{16} - 7y_{17} + 3y_{18} &= 0 \\
5y_{17} - 7y_{18} + 3y_{19} &= 0 \\
5y_{18} - 7y_{19} + 3y_{20} &= 0 \\
5y_{19} - 7y_{20} + 3y_{21} &= 0 \\
y_{21} &= 5
\end{aligned}
$$

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Step 8 – Write Set of Equations as a Single Matrix Equation

$$
\underbrace{\left[\begin{array}{ccccccccccccccccccccc}
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
5&-7&3&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&5&-7&3&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&5&-7&3&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&5&-7&3&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&5&-7&3&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&5&-7&3&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&5&-7&3&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&5&-7&3&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&5&-7&3&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&5&-7&3&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&5&-7&3&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&5&-7&3&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&5&-7&3&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&5&-7&3&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&5&-7&3&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&5&-7&3&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&5&-7&3&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&5&-7&3&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&5&-7&3 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1
\end{array}\right]}_{[A]}
\underbrace{\left[\begin{array}{c}
y_1\\y_2\\y_3\\y_4\\y_5\\y_6\\y_7\\y_8\\y_9\\y_{10}\\y_{11}\\y_{12}\\y_{13}\\y_{14}\\y_{15}\\y_{16}\\y_{17}\\y_{18}\\y_{19}\\y_{20}\\y_{21}
\end{array}\right]}_{[y]}
=
\underbrace{\left[\begin{array}{c}
1\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\5
\end{array}\right]}_{[b]}
$$

$\leftarrow$

$$
\begin{aligned}
y_1 &= 1 \\
5y_1 - 7y_2 + 3y_3 &= 0 \\
5y_2 - 7y_3 + 3y_4 &= 0 \\
5y_3 - 7y_4 + 3y_5 &= 0 \\
5y_4 - 7y_5 + 3y_6 &= 0 \\
5y_5 - 7y_6 + 3y_7 &= 0 \\
5y_6 - 7y_7 + 3y_8 &= 0 \\
5y_7 - 7y_8 + 3y_9 &= 0 \\
5y_8 - 7y_9 + 3y_{10} &= 0 \\
5y_9 - 7y_{10} + 3y_{11} &= 0 \\
5y_{10} - 7y_{11} + 3y_{12} &= 0 \\
5y_{11} - 7y_{12} + 3y_{13} &= 0 \\
5y_{12} - 7y_{13} + 3y_{14} &= 0 \\
5y_{13} - 7y_{14} + 3y_{15} &= 0 \\
5y_{14} - 7y_{15} + 3y_{16} &= 0 \\
5y_{15} - 7y_{16} + 3y_{17} &= 0 \\
5y_{16} - 7y_{17} + 3y_{18} &= 0 \\
5y_{17} - 7y_{18} + 3y_{19} &= 0 \\
5y_{18} - 7y_{19} + 3y_{20} &= 0 \\
5y_{19} - 7y_{20} + 3y_{21} &= 0 \\
y_{21} &= 5
\end{aligned}
$$

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Step 9 – Solve Matrix Equation

$$
\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_7 \\ y_8 \\ y_9 \\ y_{10} \\ y_{11} \\ y_{12} \\ y_{13} \\ y_{14} \\ y_{15} \\ y_{16} \\ y_{17} \\ y_{18} \\ y_{19} \\ y_{20} \\ y_{21} \end{bmatrix}
=
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
5 & -7 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 5 & -7 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 5 & -7 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 5 & -7 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 5 & -7 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 5 & -7 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 5 & -7 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & -7 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & -7 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & -7 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & -7 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & -7 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & -7 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & -7 & 3 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & -7 & 3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & -7 & 3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & -7 & 3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & -7 & 3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & -7 & 3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}^{-1}
\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 5 \end{bmatrix}
=
\begin{bmatrix} 1.0000 \\ 2.0497 \\ 3.1160 \\ 3.8545 \\ 3.8005 \\ 2.4436 \\ -0.6324 \\ -5.5482 \\ -11.8919 \\ -18.5007 \\ -23.3484 \\ -23.6453 \\ -16.2582 \\ 1.4730 \\ 30.5339 \\ 68.7909 \\ 109.6222 \\ 141.1336 \\ 146.6082 \\ 106.8630 \\ 5.0000 \end{bmatrix}
$$

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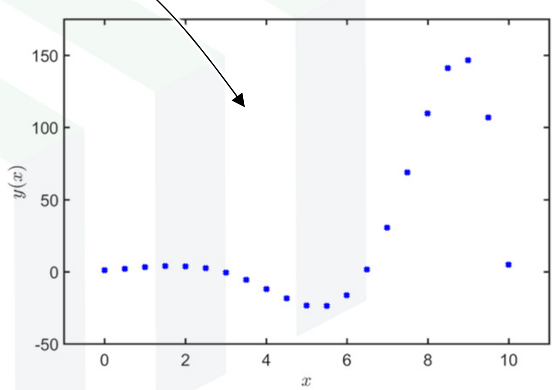
Step 10 – Plot the Result

$$
\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_7 \\ y_8 \\ y_9 \\ y_{10} \\ y_{11} \\ y_{12} \\ y_{13} \\ y_{14} \\ y_{15} \\ y_{16} \\ y_{17} \\ y_{18} \\ y_{19} \\ y_{20} \\ y_{21} \end{bmatrix}
=
\begin{bmatrix} 1.0000 \\ 2.0497 \\ 3.1160 \\ 3.8545 \\ 3.8005 \\ 2.4436 \\ -0.6324 \\ -5.5482 \\ -11.8919 \\ -18.5007 \\ -23.3484 \\ -23.6453 \\ -16.2582 \\ 1.4730 \\ 30.5339 \\ 68.7909 \\ 109.6222 \\ 141.1336 \\ 146.6082 \\ 106.8630 \\ 5.0000 \end{bmatrix}
$$



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Improved Finite-Difference Method

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Step 1 – Identify Governing Equation & Boundary Values

Governing Equation

$$\frac{d^2 y}{dx^2} - \frac{dy}{dx} + y = 0 \qquad 0 \le x \le 10$$

Boundary Values

$$y(0) = 1$$
$$y(10) = 5$$

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Step 2 – Write Equation in Matrix Form Going Term-by-Term

$$\frac{d^2}{dx^2}y(x) - \frac{d}{dx}y(x) + y(x) = 0$$

$$\downarrow$$

$$\left[\mathrm{D}_x^2\right][y] - [\mathrm{D}_x][y] + [y] = [0]$$

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Step 3 – Factor Out $[y]$ To Put in Standard Form

$$\left[\mathrm{D}_x^2\right][y] - [\mathrm{D}_x][y] + [y] = [0]$$

$$\downarrow$$

$$\left(\left[\mathrm{D}_x^2\right] - [\mathrm{D}_x] + [I]\right)[y] = [0]$$

$$\downarrow$$

$$[A][y] = [0] \quad \text{standard form}$$

$$[A] = \left[\mathrm{D}_x^2\right] - [\mathrm{D}_x] + [I]$$

```
A = DX2 - DX + I;
```

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Step 4 – Incorporate Boundary Values (1 of 4)

Start with direct matrix equation.

$$[A][y]=[0]$$

$$\begin{bmatrix}
-7 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
5 & -7 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 5 & -7 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 5 & -7 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 5 & -7 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 5 & -7 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 5 & -7 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 5 & -7 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & -7 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & -7 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & -7 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & -7 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & -7 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & -7 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & -7 & 3 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & -7 & 3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & -7 & 3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & -7 & 3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & -7 & 3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & -7 & 3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & -7
\end{bmatrix}
\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_7 \\ y_8 \\ y_9 \\ y_{10} \\ y_{11} \\ y_{12} \\ y_{13} \\ y_{14} \\ y_{15} \\ y_{16} \\ y_{17} \\ y_{18} \\ y_{19} \\ y_{20} \\ y_{21} \end{bmatrix}
=
\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

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Step 4 – Incorporate Boundary Values (2 of 4)

Zero out rows in $[A]$ that correspond to the boundary points. For this case, these are the rows for y_1 and y_{21}.

$$\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
5 & -7 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 5 & -7 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 5 & -7 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 5 & -7 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 5 & -7 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 5 & -7 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 5 & -7 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & -7 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & -7 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & -7 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & -7 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & -7 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & -7 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & -7 & 3 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & -7 & 3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & -7 & 3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & -7 & 3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & -7 & 3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & -7 & 3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_7 \\ y_8 \\ y_9 \\ y_{10} \\ y_{11} \\ y_{12} \\ y_{13} \\ y_{14} \\ y_{15} \\ y_{16} \\ y_{17} \\ y_{18} \\ y_{19} \\ y_{20} \\ y_{21} \end{bmatrix}
=
\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

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Step 4 – Incorporate Boundary Values (3 of 4)

Insert 1's in the diagonal position of the rows in $[A]$ where zeros were inserted.

$$
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
5 & -7 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 5 & -7 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 5 & -7 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 5 & -7 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 5 & -7 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 5 & -7 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 5 & -7 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & -7 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & -7 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & -7 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & -7 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & -7 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & -7 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & -7 & 3 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & -7 & 3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & -7 & 3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & -7 & 3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & -7 & 3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & -7 & 3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_7 \\ y_8 \\ y_9 \\ y_{10} \\ y_{11} \\ y_{12} \\ y_{13} \\ y_{14} \\ y_{15} \\ y_{16} \\ y_{17} \\ y_{18} \\ y_{19} \\ y_{20} \\ y_{21}
\end{bmatrix}
=
\begin{bmatrix}
0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0
\end{bmatrix}
$$

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Step 4 – Incorporate Boundary Values (4 of 4)

Insert boundary values in column vector $[b]$.

$$
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
5 & -7 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 5 & -7 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 5 & -7 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 5 & -7 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 5 & -7 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 5 & -7 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 5 & -7 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & -7 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & -7 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & -7 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & -7 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & -7 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & -7 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & -7 & 3 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & -7 & 3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & -7 & 3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & -7 & 3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & -7 & 3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & -7 & 3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_7 \\ y_8 \\ y_9 \\ y_{10} \\ y_{11} \\ y_{12} \\ y_{13} \\ y_{14} \\ y_{15} \\ y_{16} \\ y_{17} \\ y_{18} \\ y_{19} \\ y_{20} \\ y_{21}
\end{bmatrix}
=
\begin{bmatrix}
1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 5
\end{bmatrix}
$$

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Step 5 – Solve Matrix Equation

$$
\begin{bmatrix} y_1\\ y_2\\ y_3\\ y_4\\ y_5\\ y_6\\ y_7\\ y_8\\ y_9\\ y_{10}\\ y_{11}\\ y_{12}\\ y_{13}\\ y_{14}\\ y_{15}\\ y_{16}\\ y_{17}\\ y_{18}\\ y_{19}\\ y_{20}\\ y_{21} \end{bmatrix}
=
\begin{bmatrix}
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
5&-7&3&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&5&-7&3&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&5&-7&3&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&5&-7&3&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&5&-7&3&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&5&-7&3&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&5&-7&3&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&5&-7&3&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&5&-7&3&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&5&-7&3&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&5&-7&3&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&5&-7&3&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&5&-7&3&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&5&-7&3&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&5&-7&3&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&5&-7&3&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&5&-7&3&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&5&-7&3&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&5&-7&3\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1
\end{bmatrix}^{-1}
\begin{bmatrix} 1\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\5 \end{bmatrix}
=
\begin{bmatrix} 1.0000\\ 2.0497\\ 3.1160\\ 3.8545\\ 3.8005\\ 2.4436\\ -0.6324\\ -5.5482\\ -11.8919\\ -18.5007\\ -23.3484\\ -23.6453\\ -16.2582\\ 1.4730\\ 30.5339\\ 68.7909\\ 109.6222\\ 141.1336\\ 146.6082\\ 106.8630\\ 5.0000 \end{bmatrix}
$$

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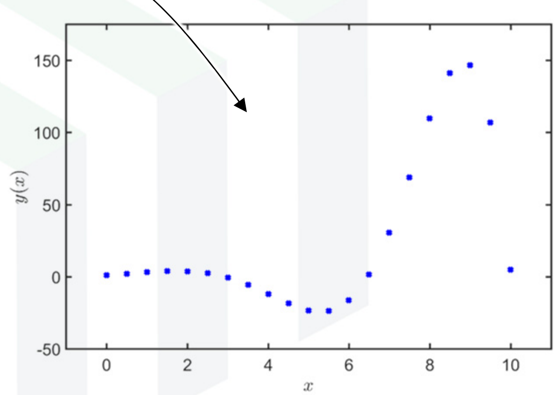
Step 6 – Plot the Result

$$
\begin{bmatrix} y_1\\ y_2\\ y_3\\ y_4\\ y_5\\ y_6\\ y_7\\ y_8\\ y_9\\ y_{10}\\ y_{11}\\ y_{12}\\ y_{13}\\ y_{14}\\ y_{15}\\ y_{16}\\ y_{17}\\ y_{18}\\ y_{19}\\ y_{20}\\ y_{21} \end{bmatrix}
=
\begin{bmatrix} 1.0000\\ 2.0497\\ 3.1160\\ 3.8545\\ 3.8005\\ 2.4436\\ -0.6324\\ -5.5482\\ -11.8919\\ -18.5007\\ -23.3484\\ -23.6453\\ -16.2582\\ 1.4730\\ 30.5339\\ 68.7909\\ 109.6222\\ 141.1336\\ 146.6082\\ 106.8630\\ 5.0000 \end{bmatrix}
$$



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