Electromagnetics:
Microwave Engineering

Introduction to Smith Charts
Lecture Outline

• Smith chart is a polar plot of reflection coefficient $\Gamma$
• Normalized impedance $z$
• Derivation of the Smith chart
• Final notes
Smith Chart is a Polar Plot of Reflection
The Smith chart is based on a polar plot of the voltage reflection coefficient $\Gamma$. The outer boundary corresponds to $|\Gamma| = 1$. The reflection coefficient in any passive system must be $|\Gamma| \leq 1$.

$$\Gamma = |\Gamma| e^{j\theta}$$

$|\Gamma|$ = radius on Smith chart

$\theta$ = angle measured CCW from right side of chart
Normalized Impedance $z$
Normalized Impedance $z$

All impedances on the Smith Chart are normalized. This is usually done with respect to the characteristic impedance of the transmission line $Z_0$.

$$z = \frac{Z}{Z_0}$$
The reflection coefficient $\Gamma$ from a load can be written in terms of normalized impedances.

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{Z_L / Z_0 - Z_0 / Z_0}{Z_L / Z_0 + Z_0 / Z_0} = \frac{z_L - 1}{z_L + 1}$$

$Z_L \equiv \text{Load impedance}$

$z_L \equiv \text{Normalized load impedance}$
Derivation of the Smith Chart
Solve for Normalized Load Impedance $z_L$

Solve the previous equation for normalized load impedance $z_L$ to get

$$\Gamma = \frac{z_L - 1}{z_L + 1}$$

$$z_L = \frac{1 + \Gamma}{1 - \Gamma}$$
Real & Imaginary Parts of Normalized Load Impedance

The normalized load impedance $z_L$ and reflection coefficient $\Gamma$ can be written in terms of real and imaginary parts.

$$z_L = r_L + jx_L \quad \Gamma = \Gamma_r + j\Gamma_i$$

Substituting these into the normalized load impedance equation yields

$$z_L = \frac{1+\Gamma}{1-\Gamma}$$

$$r_L + jx_L = \frac{1+(\Gamma_r + j\Gamma_i)}{1-(\Gamma_r + j\Gamma_i)}$$

$$r_L + jx_L = \frac{(1+\Gamma_r) + j\Gamma_i}{(1-\Gamma_r) - j\Gamma_i}$$
Solve for $r_L$ and $x_L$

Solve the previous equation for $r_L$ and $x_L$ by setting the real and imaginary parts equal.

\[
r_L + jx_L = \frac{(1+\Gamma_r) + j\Gamma_i}{(1-\Gamma_r) - j\Gamma_i}
\]

\[
= \frac{(1+\Gamma_r) + j\Gamma_i}{(1-\Gamma_r) - j\Gamma_i} \cdot \frac{(1-\Gamma_r) + j\Gamma_i}{(1-\Gamma_r) + j\Gamma_i}
\]

\[
= \frac{(1+\Gamma_r)(1-\Gamma_r) + j(1+\Gamma_r)\Gamma_i + j\Gamma_i(1-\Gamma_r) - \Gamma_i^2}{(1-\Gamma_r)^2 + \Gamma_i^2}
\]

\[
= \frac{1-\Gamma_r^2 + j\Gamma_i + j\Gamma_r\Gamma_i + j\Gamma_i - j\Gamma_r\Gamma_i - \Gamma_i^2}{(1-\Gamma_r)^2 + \Gamma_i^2}
\]

\[
= \frac{1-\Gamma_r^2 - \Gamma_i^2 + j2\Gamma_i}{(1-\Gamma_r)^2 + \Gamma_i^2}
\]

\[
= \frac{1-\Gamma_r^2 - \Gamma_i^2}{(1-\Gamma_r)^2 + \Gamma_i^2} + j \frac{2\Gamma_i}{(1-\Gamma_r)^2 + \Gamma_i^2}
\]

\[
r_L = \frac{1-\Gamma_r^2 - \Gamma_i^2}{(1-\Gamma_r)^2 + \Gamma_i^2}
\]

\[
x_L = \frac{2\Gamma_i}{(1-\Gamma_r)^2 + \Gamma_i^2}
\]
Rearrange Equation Containing $r_L$

Rearrange the equation for $r_L$ so that it has the form of a circle.

$$r_L = \frac{1 - \Gamma_r^2 - \Gamma_i^2}{(1 - \Gamma_r)^2 + \Gamma_i^2}$$

$$\left(1 - \Gamma_r\right)^2 + \Gamma_i^2 = \frac{1 - \Gamma_r^2 - \Gamma_i^2}{r_L}$$

$$\left(1 - \Gamma_r\right)^2 + \Gamma_i^2 - \frac{1}{r_L} + \frac{\Gamma_r^2}{r_L} + \frac{\Gamma_i^2}{r_L} = 0$$

$$-2\Gamma_r + \frac{\Gamma_r^2}{r_L} + \Gamma_i^2 + \frac{\Gamma_i^2}{r_L} + 1 - \frac{1}{r_L} = 0$$

$$-2r_L\Gamma_r + r_L\Gamma_r^2 + \Gamma_i^2 + r_L\Gamma_i^2 + r_L - 1 = 0$$

$$-2r_L\Gamma_r + (r_L + 1)\Gamma_r^2 + (r_L + 1)\Gamma_i^2 + r_L - 1 = 0$$

$$\Gamma_r^2 - 2\frac{r_L\Gamma_r}{r_L + 1} + \Gamma_i^2 + \frac{r_L - 1}{r_L + 1} = 0$$

(can be factored)
Rearrange Equation Containing $r_L$

Rearrange the equation for $r_L$ so that it has the form of a circle.

$$\left( \Gamma_r - \frac{r_L}{r_L + 1} \right)^2 - \left( \frac{r_L}{r_L + 1} \right)^2 + \Gamma_i^2 + \frac{r_L - 1}{r_L + 1} = 0$$

$$\left( \Gamma_r - \frac{r_L}{r_L + 1} \right)^2 + \Gamma_i^2 = \left( \frac{r_L}{r_L + 1} \right)^2 - \frac{r_L - 1}{r_L + 1}$$

$$\left( \Gamma_r - \frac{r_L}{r_L + 1} \right)^2 + \Gamma_i^2 = \frac{r_L^2}{(r_L + 1)^2} - \frac{(r_L - 1)(r_L + 1)}{(r_L + 1)^2}$$

$$\left( \Gamma_r - \frac{r_L}{r_L + 1} \right)^2 + \Gamma_i^2 = \frac{r_L^2}{(r_L + 1)^2} - \frac{r_L^2 - 1}{(r_L + 1)^2}$$

$$(\Gamma_r - \frac{r_L}{r_L + 1})^2 + \Gamma_i^2 = \frac{1}{(r_L + 1)^2}$$
Rearrange the equation for $x_L$ so that it has the form of a circle.

\[
x_L = \frac{2\Gamma_i}{(1 - \Gamma_r)^2 + \Gamma_i^2}
\]

\[
(1 - \Gamma_r)^2 + \Gamma_i^2 = \frac{2\Gamma_i}{x_L}
\]

swap terms

\[
(1 - \Gamma_r)^2 + \Gamma_i^2 - \frac{2}{x_L} \Gamma_i = 0
\]

\[
(\Gamma_r - 1)^2 + \left(\Gamma_i - \frac{1}{x_L}\right)^2 - \frac{1}{x_L^2} = 0
\]
Two Families of Circles

**Constant Resistance Circles**

\[
\left( \Gamma_r - \frac{r_L}{r_L + 1} \right)^2 + \Gamma_i^2 = \left( \frac{1}{1 + r_L} \right)^2
\]

These have centers at

\[
\Gamma_r = \frac{r_L}{r_L + 1}, \quad \Gamma_i = 0
\]

Radii

\[
\frac{1}{1 + r_L}
\]

**Constant Reactance Circles**

\[
(\Gamma_r - 1)^2 + \left( \Gamma_i - \frac{1}{x_L} \right)^2 = \left( \frac{1}{x_L} \right)^2
\]

These have centers at

\[
\Gamma_r = 1, \quad \Gamma_i = \frac{1}{x_L}
\]

Radii

\[
\frac{1}{x_L}
\]
Putting It All Together

Ignore what is outside the $|\Gamma| = 1$ circle.

Do not draw the constant $|\Gamma|$ circles.

This is the Smith chart!
Alternate Way of Visualizing the Smith Chart

- Circles of constant resistance
- Circles of constant reactance
- Reactance Regions

- Open circuit
- Short circuit
The 3D Smith Chart unifies passive and active circuit design.