Computational Science:
Computational Methods in Engineering

Jacobi Iteration Method

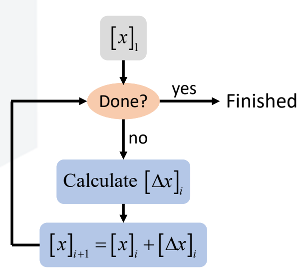
What is the Jacobi Iteration Method?

The Gauss-Jordan method was a direct solution of $[A][x]=[b]$.

This can be inefficient for large matrices, especially when a good initial guess $[x]$ is known.

An iterative algorithm can be devised that improves the initial guess every iteration.

The method only converges for *diagonally dominant* matrices.
The algorithm is very picky about this!



Diagonally Dominant Matrices

A square matrix is said to be diagonally dominant if for each row, the magnitude of the diagonal element is greater than or equal to the sum of the magnitudes of all other elements in that row.

$$|a_{ii}| \geq \sum_{j \neq i} |a_{ij}|$$

Examples

$$[A] = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Not diagonally dominant. ☹

$$[A] = \begin{bmatrix} 3 & 1 & 1 \\ 0 & 1 & 7 \\ 2 & 1 & 4 \end{bmatrix}$$

Not diagonally dominant. ☹

$$[A] = \begin{bmatrix} -4 & 2 & 1 \\ -1 & -3 & 1 \\ -3 & -2 & -8 \end{bmatrix}$$

Diagonally dominant! ☺

$$[A] = \begin{bmatrix} 2 & 0 & 1 \\ -1 & 2 & 2 \\ 5 & -4 & -8 \end{bmatrix}$$

Not diagonally dominant. ☹

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Formulation (1 of 2)

The matrix problem is

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Expand this into its component equations.

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

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Formulation (2 of 2)

Solve the first equation for x_1, the second equation for x_2, and the third equation for x_3.

These are the equations that will be iterated.

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \quad\rightarrow\quad x_1 = \frac{b_1 - a_{12}x_2 - a_{13}x_3}{a_{11}}$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \quad\rightarrow\quad x_2 = \frac{b_2 - a_{21}x_1 - a_{23}x_3}{a_{22}}$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \quad\rightarrow\quad x_3 = \frac{b_3 - a_{31}x_1 - a_{32}x_2}{a_{33}}$$

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Implementation (1 of 2)

Step 1 – Define Problem

$$[A] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad [b] = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Step 2 – Come up with an initial guess for $[x]$.

$$x_1^{(1)},\ x_2^{(1)},\ \text{and}\ x_3^{(1)}$$

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Implementation (2 of 2)

Step 3 – Iterate $[x]$ until convergence

a) Calculate new values of $[x]$

$$x_1^{(i+1)} = \frac{b_1 - a_{12}x_2^{(i)} - a_{13}x_3^{(i)}}{a_{11}} \qquad x_2^{(i+1)} = \frac{b_2 - a_{21}x_1^{(i)} - a_{23}x_3^{(i)}}{a_{22}} \qquad x_3^{(i+1)} = \frac{b_3 - a_{31}x_1^{(i)} - a_{32}x_2^{(i)}}{a_{33}}$$

b) Check how much the values have changed

$$\varepsilon_1^{(i+1)} = \left|\frac{x_1^{(i+1)} - x_1^{(i)}}{x_1^{(i+1)}}\right| \qquad \varepsilon_2^{(i+1)} = \left|\frac{x_2^{(i+1)} - x_2^{(i)}}{x_2^{(i+1)}}\right| \qquad \varepsilon_3^{(i+1)} = \left|\frac{x_3^{(i+1)} - x_3^{(i)}}{x_3^{(i+1)}}\right|$$

c) Continue to iterate until all values of ε are sufficiently small.

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Matrix Formulation (1 of 2)

Rearrange the update equations this way

$$x_1 = \frac{b_1 - a_{12}x_2 - a_{13}x_3}{a_{11}} \quad \rightarrow \quad x_1 = \frac{1}{a_{11}}\left[b_1 - \left(a_{12}x_2 + a_{13}x_3\right)\right]$$

$$x_2 = \frac{b_2 - a_{21}x_1 - a_{23}x_3}{a_{22}} \quad \rightarrow \quad x_2 = \frac{1}{a_{22}}\left[b_2 - \left(a_{21}x_1 + a_{23}x_3\right)\right]$$

$$x_3 = \frac{b_3 - a_{31}x_1 - a_{32}x_2}{a_{33}} \quad \rightarrow \quad x_3 = \frac{1}{a_{33}}\left[b_3 - \left(a_{31}x_1 + a_{32}x_2\right)\right]$$

By inspecting these equations, the update equation can be written in matrix form as

$$[x]_{i+1} = \left(\operatorname{diag}[A]\right)^{-1} \cdot \left\{[b] - \left([A] - \operatorname{diag}[A]\right)[x]_i\right\}$$

$$\operatorname{diag}[A] = \begin{bmatrix} a_{11} & & & \\ & a_{22} & & \\ & & \ddots & \\ & & & a_{MM} \end{bmatrix}$$

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Matrix Formulation (2 of 2)

For simplicity, let

$$[D] = \text{diag}[A]$$

Rearrange the "update equation" so that it makes more intuitive sense.

$$\begin{aligned} [x]_{i+1} &= [D]^{-1} \cdot \left\{ [b] - \left([A] - [D] \right) [x]_i \right\} \\ &= [D]^{-1}[b] - [D]^{-1}[A][x]_i + [D]^{-1}[D][x]_i \\ &= \underbrace{[x]_i}_{\text{Previous solution}} + \underbrace{[D]^{-1}\left([b] - [A][x]_i \right)}_{\text{Improvement on } [x]_i} \end{aligned}$$

Improvement on $[x]_i$

Previous solution

Block Diagram of Jacobi Method