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Computational Science:
Computational Methods in Engineering

LU Decomposition

Determining the Upper Triangular Matrix $[U]$

Start with

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \quad \rightarrow \quad [A][x]=[b]$$

At some point during Gauss elimination, this system of equations was converted to an upper-triangular matrix $[U]$.

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \quad \rightarrow \quad [U][x]=[d]$$

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Determining the Lower Triangular Matrix $[L]$

There exists a lower-triangular matrix $[L]$ such that

$$[A]=[L][U]$$

This equation is why the method is called LU decomposition.

$$[L]=\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$$

Recall these special terms from Gauss elimination.

Now substitute this decomposition into the original equation.

$$[A][x]=[b]$$

$$[A][x]-[b]=[0]$$

$$[L][U][x]-[b]=[0]$$

$$[L]\left([U][x]-[L]^{-1}[b]\right)=[0]$$

$$[L]\left([U][x]-[d]\right)=[0] \qquad \text{where } [d]=[L]^{-1}[b]$$

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Algorithm

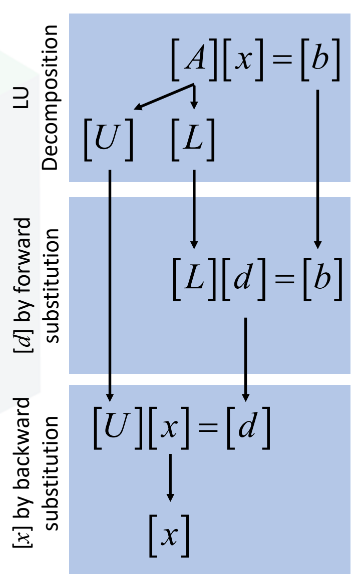
Step 1 – Decompose $[A]$ into $[L]$ and $[U]$.

a) Use GE to calculate $[U]$
b) Store l terms during GE
c) Build $[L]$ from l terms.
d) store $[L]$ and $[U]$ together as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ l_{21} & a'_{22} & a'_{23} \\ l_{31} & l_{32} & a''_{33} \end{bmatrix}$$

Step 2 – Solve $[L][d]=[b]$ using fast backward substitution.

Step 3 – Solve $[U][x]=[d]$ using fast backward substitution.



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