Computational Science:
Computational Methods in Engineering

Matrix Terminology & Special Matrices

Outline

- Matrices & Parts of Matrices
- Special Matrices
- Size & Shape of Matrices
- Health of a Matrix

Matrices & Parts of Matrices

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Systems of Linear Equations

Systems of equations can be written in matrix form.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned} \quad \rightarrow \quad \underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \\ a_{n1} & a_{n2} & & a_{nn} \end{bmatrix}}_{\substack{[A] \\ \text{or} \\ \mathbf{A}}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{\substack{[x] \\ \text{or} \\ \mathbf{x}}} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}}_{\substack{[b] \\ \text{or} \\ \mathbf{b}}}$$

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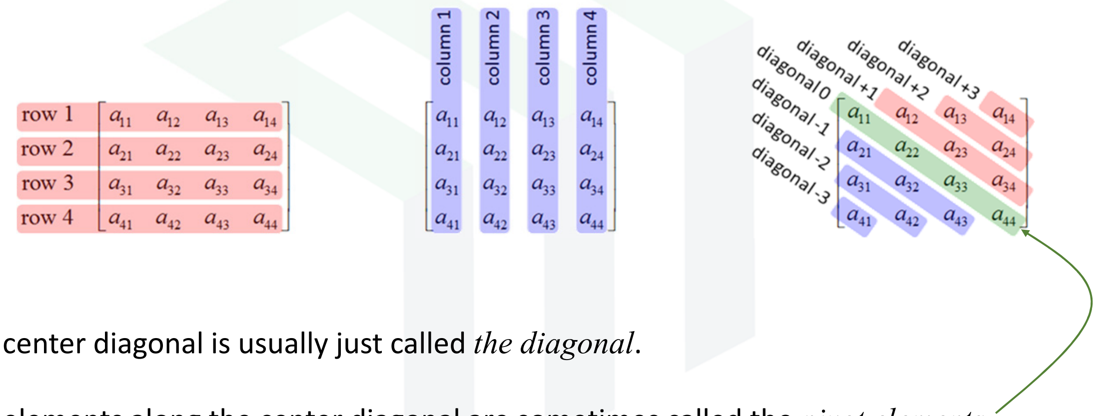
Rows, Columns, and Diagonals



The center diagonal is usually just called *the diagonal*.

The elements along the center diagonal are sometimes called the *pivot elements*.

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Special Matrices

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Special Matrices (1 of 2)

Symmetric Matrix

$$[A]=\begin{bmatrix}1 & 2 & 9 & 4\\ 2 & 6 & 5 & 8\\ 9 & 5 & 7 & 0\\ 4 & 8 & 0 & 3\end{bmatrix}$$

Diagonal Matrix

$$[A]=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 6 & 0 & 0\\ 0 & 0 & 7 & 0\\ 0 & 0 & 0 & 3\end{bmatrix}$$

Identity Matrix

$$[I]=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

Zero Matrix

$$[0]=\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$$

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Special Matrices (2 of 2)

Upper Triangular Matrix

$$[A]=\begin{bmatrix}1 & 2 & 9 & 4\\ 0 & 6 & 5 & 8\\ 0 & 0 & 7 & 5\\ 0 & 0 & 0 & 3\end{bmatrix}$$

Triangular matrices can be thought of as "almost" solved matrices. They are very fast to solve.

Lower Triangular Matrix

$$[A]=\begin{bmatrix}1 & 0 & 0 & 0\\ 2 & 6 & 0 & 0\\ 9 & 5 & 7 & 0\\ 4 & 8 & 1 & 3\end{bmatrix}$$

Banded Matrix

$$[A]=\begin{bmatrix}1 & 2 & 0 & 0\\ 4 & 6 & 5 & 0\\ 0 & 8 & 7 & 5\\ 0 & 0 & 10 & 3\end{bmatrix}$$

Bandwidth of 3

Vandermonde Matrix

$$\begin{bmatrix}1 & x_1 & x_1^2 & & x_1^N\\ 1 & x_2 & x_2^2 & \cdots & x_2^N\\ 1 & x_3 & x_3^2 & & x_3^N\\ & \vdots & & \ddots & \\ 1 & x_{N+1} & x_{N+1}^2 & & x_{N+1}^N\end{bmatrix}$$

Arises when curve fitting to polynomials. Usually ill-conditioned for large matrices.

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Block Matrices

Block matrices are "matrices of matrices."

$$[F]=\begin{bmatrix}[A] & [B]\\ [C] & [D]\end{bmatrix}=\begin{bmatrix}a_{11} & a_{12} & b_{11} & b_{12}\\ a_{21} & a_{22} & b_{21} & b_{22}\\ c_{11} & c_{12} & d_{11} & d_{12}\\ c_{21} & c_{22} & d_{21} & d_{22}\end{bmatrix}$$

$$[A]=\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}\qquad [B]=\begin{bmatrix}b_{11} & b_{12}\\ b_{21} & b_{22}\end{bmatrix}\qquad [C]=\begin{bmatrix}c_{11} & c_{12}\\ c_{21} & c_{22}\end{bmatrix}\qquad [D]=\begin{bmatrix}d_{11} & d_{12}\\ d_{21} & d_{22}\end{bmatrix}$$

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Sparse Matrices

Many matrices are composed of almost entirely zeros.

It is not an efficient use of memory to store all these zeros. Instead, only the non-zero elements are stored along with their positions in the matrix.

The opposite of a sparse matrix is a *full matrix*.

$$[A]=\begin{bmatrix}
2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & -1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & -1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & -1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & -1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & -1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2
\end{bmatrix}$$

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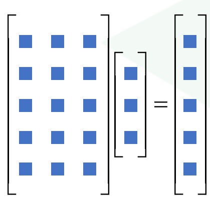
Size & Shape of Matrices

Matrix Problem Size

 # Equations > # Unknowns

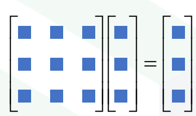


Usually occurs when the equations are derived from samples.

Solution is obtained as a *best fit* and is not exact.

Applications
- Curve fitting

 # Equations = # Unknowns

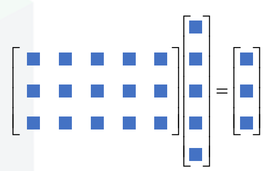


Most usual case.

Many standard algorithms exist to obtain an *exact* solution.

Applications
- Circuit theory
- Solving ODEs

 # Equations < # Unknowns



Usually occurs when little is known about the problem or solution.

Solution is obtained by *optimization* and is not exact.

Applications
- Topology optimization

Health of a Matrix

Health of a Matrix (1 of 3)

Is this system of equations solvable?

$$\begin{aligned} x+2y+z&=8 \\ x+2y+z&=8 \\ 3x-y+z&=4 \end{aligned} \qquad \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 3 & -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 8 \\ 8 \\ 4 \end{bmatrix}$$

No!
The 1st and 2nd equations are the same. The 2nd equation does not provide any new information to the problem.

$$\begin{aligned} x+2y+z&=8 \\ 2x+4y+2z&=16 \\ 3x-y+z&=4 \end{aligned} \qquad \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 3 & -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 8 \\ 16 \\ 4 \end{bmatrix}$$

No!
The 2nd equation is just 2× the 1st equation. The 2nd equation is still not providing any new information.

$$\begin{aligned} x+2y+z&=8 \\ 4x+y+2z&=12 \\ 3x-y+z&=4 \end{aligned} \qquad \begin{bmatrix} 1 & 2 & 1 \\ 4 & 1 & 2 \\ 3 & -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 8 \\ 12 \\ 4 \end{bmatrix}$$

No!
The 2nd equation is the sum of the 1st and 3rd equation, thus the 2nd equation still does not provide any new information.

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Health of a Matrix (2 of 3)

Is this system of equations solvable?

$$\begin{aligned} x+z&=8 \\ x+2z&=7 \\ 3x+z&=4 \end{aligned} \qquad \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 2 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 7 \\ 4 \end{bmatrix}$$

No!
None of these equations contain any information about y.

So how to know if a problem is solvable?

- All rows must be linearly independent – this ensures they each provide new information to the problem.
- No rows can be all zeros – This would not provide any information.
- No columns can be all zeros – This would be ignoring information from one of the unknowns.

$$\boxed{[A][x]=[b] \text{ is solvable if } \det[A] \neq 0}$$

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Health of a Matrix (3 of 3)

Is the following system of equations solvable?

$$\begin{aligned} x+2y+z&=8 \\ 1.0001x+2y+z&=8.0001 \\ 3x-y+z&=4 \end{aligned} \qquad \begin{bmatrix} 1 & 2 & 1 \\ 1.0001 & 2 & 1 \\ 3 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 8.0001 \\ 4 \end{bmatrix}$$

Technically yes, but perhaps the solution to be somewhat “touchy” and unstable. This is an *ill-conditioned* matrix.

Condition Number of a Matrix

The condition number $\kappa(\mathbf{A})$ of matrix $\mathbf{A}$ is a measure of how much an answer will change given small changes in the matrix $\mathbf{A}$.

Matrices with high condition numbers are less stable. Small changes in the element values of $\mathbf{A}$ will result in large changes in the elements of $\mathbf{x}$.

$$\kappa(\mathbf{A}) = \left| \frac{\sigma_{\max}(\mathbf{A})}{\sigma_{\min}(\mathbf{A})} \right|$$

$\sigma_{\min}(\mathbf{A}) \equiv$ smallest singular value of $\mathbf{A}$

$\sigma_{\max}(\mathbf{A}) \equiv$ largest singular value of $\mathbf{A}$

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Example: Condition Number

What is the condition number?

$$\mathbf{A}=\begin{bmatrix}1 & 2 & 1\\ 1 & 2 & 1\\ 3 & -1 & 1\end{bmatrix} \qquad \kappa(\mathbf{A})=5.84\times10^{16}$$

$$\mathbf{A}=\begin{bmatrix}1 & 2 & 1\\ 1 & 0 & 1\\ 3 & -1 & 1\end{bmatrix} \qquad \kappa(\mathbf{A})=7.76$$

$$\mathbf{A}=\begin{bmatrix}1 & 2 & 1\\ 1.0001 & 2 & 1\\ 3 & -1 & 1\end{bmatrix} \qquad \kappa(\mathbf{A})=1.4\times10^{5}$$

$$\mathbf{A}=\begin{bmatrix}1 & 2 & 1\\ 1.01 & 2 & 1\\ 3 & -1 & 1\end{bmatrix} \qquad \kappa(\mathbf{A})=1.4\times10^{3}$$