Advanced Electromagnetics:
21st Century Electromagnetics

# Numerical Solution of Laplace's Equation

## Lecture Outline

- Intuitive interpretation of Laplace's equation $\nabla^2 u = 0$
- Numerical solution of Laplace's equation $\mathbf{Lu} = \mathbf{0}$
- Enclosed problems

# Intuitive Interpretation of Laplace's Equation

$$\nabla^2 u = 0$$

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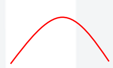
## Meaning of Laplace's Equation

Laplace's equation is

$$\nabla^2 u = 0$$

$\nabla^2$ is a 3D second-order derivative. ➡ A second-order derivative quantifies curvature. ➡ But the second-order derivative is set to zero.



Functions satisfying Laplace's equation vary linearly.

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# Laplace's Equation as a "Linear Number Filler Inner"

Given known values at certain points (e.g. physical boundary conditions), Laplace's equation calculates the numbers everywhere else so they vary linearly.

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | | | | |
| | | | | | | | | | | 9.0 | 9.0 | |
| | | | | | | | | | | 9.0 | 9.0 | |
| | | | | | | | | | | 9.0 | 9.0 | |
| | | | | | | | | | | 9.0 | 9.0 | |
| | | | | | | | | | | 9.0 | 9.0 | |
| | | | | | | | | | | 9.0 | 9.0 | |
| | | | | | | | | | | 9.0 | 9.0 | |
| | | | | | | | | | | | | |
| | -9.0 | -9.0 | -9.0 | | | | | | | | | |
| | -9.0 | -9.0 | -9.0 | | | | | | | | | |
| | -9.0 | -9.0 | -9.0 | | | | | | | | | |
| | | | | | | | | | | | | |

Map of known values (boundary values)

$$\nabla^2 u = 0$$

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.1 | 0.4 | 0.8 | 1.3 | 1.8 | 2.5 | 3.2 | 4.1 | 5.0 | 6.0 | 6.9 | 7.0 | 6.4 |
| -0.2 | 0.1 | 0.5 | 1.0 | 1.7 | 2.4 | 3.3 | 4.3 | 5.5 | 6.9 | 9.0 | 9.0 | 7.4 |
| -0.6 | -0.3 | 0.1 | 0.7 | 1.4 | 2.2 | 3.2 | 4.4 | 5.7 | 7.2 | 9.0 | 9.0 | 7.8 |
| -1.1 | -0.8 | -0.4 | 0.2 | 1.0 | 1.9 | 3.0 | 4.3 | 5.7 | 7.3 | 9.0 | 9.0 | 7.9 |
| -1.7 | -1.5 | -1.0 | -0.4 | 0.4 | 1.4 | 2.6 | 4.0 | 5.5 | 7.2 | 9.0 | 9.0 | 7.9 |
| -2.4 | -2.3 | -1.9 | -1.3 | -0.4 | 0.8 | 2.1 | 3.5 | 5.2 | 7.0 | 9.0 | 9.0 | 7.8 |
| -3.3 | -3.3 | -3.0 | -2.4 | -1.3 | -0.1 | 1.3 | 2.9 | 4.7 | 6.7 | 9.0 | 9.0 | 7.6 |
| -4.4 | -4.7 | -4.5 | -3.8 | -2.6 | -1.1 | 0.5 | 2.1 | 3.9 | 6.0 | 9.0 | 9.0 | 7.1 |
| -5.6 | -6.4 | -6.4 | -5.8 | -4.1 | -2.2 | -0.5 | 1.2 | 2.9 | 4.5 | 6.0 | 6.4 | 5.8 |
| -6.8 | -9.0 | -9.0 | -9.0 | -5.6 | -3.3 | -1.3 | 0.3 | 1.8 | 3.1 | 4.2 | 4.6 | 4.6 |
| -7.1 | -9.0 | -9.0 | -9.0 | -6.1 | -3.8 | -2.0 | -0.4 | 0.9 | 2.0 | 2.9 | 3.4 | 3.5 |
| -7.0 | -9.0 | -9.0 | -9.0 | -6.1 | -4.0 | -2.3 | -0.9 | 0.2 | 1.2 | 2.0 | 2.4 | 2.7 |
| -6.1 | -6.9 | -7.0 | -6.6 | -5.2 | -3.7 | -2.4 | -1.2 | -0.2 | 0.6 | 1.3 | 1.8 | 2.1 |

Laplace's equation is solved to fill in the values away from the boundaries.

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# Numerical Solution of Laplace's Equation

# Lu = 0

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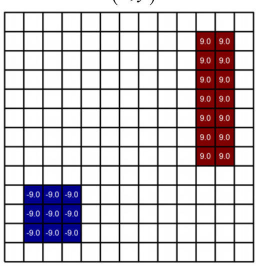
# Numerical Solution of Laplace's Equation (1 of 3)

Step 1 – Use the finite-difference method to express Laplace's equation in matrix form.

$$\nabla^2 u(x,y)=0 \quad \rightarrow \quad \frac{\partial^2 u(x,y)}{\partial x^2}+\frac{\partial^2 u(x,y)}{\partial y^2}=0 \quad \rightarrow \quad \mathbf{D}_x^2\mathbf{u}+\mathbf{D}_y^2\mathbf{u}=\mathbf{0} \quad \rightarrow \quad \begin{gathered}\mathbf{Lu}=\mathbf{0}\\ \mathbf{L}=\mathbf{D}_x^2+\mathbf{D}_y^2\end{gathered}$$

Step 2 – Build a function $b(x,y)$ containing the boundary values.

$b(x,y)$



The function $b(x,y)$ contains the values of $u(x,y)$ in the locations where $u(x,y)$ is known. The other points in $b(x,y)$ can be set to anything because they will be ignored, but zero is a convenient choice.

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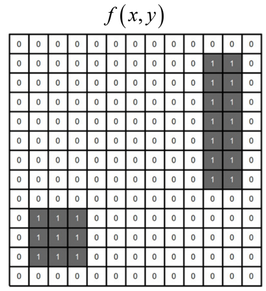
# Numerical Solution of Laplace's Equation (2 of 3)

Step 3 – Build a diagonal force matrix $\mathbf{F}$.

$$\mathbf{F}=\operatorname{diag}\left[f(x,y)\right]$$

The array $f(x,y)$ contains 1's at the positions where $u(x,y)$ is to be forced. It contains 0's everywhere else. It should be set to 1's at the locations of the boundary values.

$f(x,y)$



Step 4 – Incorporate the boundary values into Laplace's equation.

Note: both $\mathbf{F}$ and $\mathbf{I}$ should be stored as sparse matrices!

$$\mathbf{L}'=\mathbf{F}+(\mathbf{I}-\mathbf{F})\mathbf{L}$$

$$\mathbf{b}'=\mathbf{F}\mathbf{b}$$

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# Numerical Solution of Laplace's Equation (3 of 3)

Step 5 – Solve Laplace's equation

$$\mathbf{u} = \left(\mathbf{L}'\right)^{-1} \mathbf{b}'$$

The function $u(x,y)$ has all of the forced values, but also contains all of the numbers in between.

$u(x,y)$

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.1 | 0.4 | 0.8 | 1.3 | 1.8 | 2.5 | 3.2 | 4.1 | 5.0 | 6.0 | 6.9 | 7.0 | 6.4 |
| -0.2 | 0.1 | 0.5 | 1.0 | 1.7 | 2.4 | 3.3 | 4.3 | 5.5 | 6.9 | 9.0 | 9.0 | 7.4 |
| -0.6 | -0.3 | 0.1 | 0.7 | 1.4 | 2.2 | 3.2 | 4.4 | 5.7 | 7.2 | 9.0 | 9.0 | 7.8 |
| -1.1 | -0.8 | -0.4 | 0.2 | 1.0 | 1.9 | 3.0 | 4.3 | 5.7 | 7.3 | 9.0 | 9.0 | 7.9 |
| -1.7 | -1.5 | -1.0 | -0.4 | 0.4 | 1.4 | 2.6 | 4.0 | 5.5 | 7.2 | 9.0 | 9.0 | 7.9 |
| -2.4 | -2.3 | -1.9 | -1.3 | -0.4 | 0.8 | 2.1 | 3.5 | 5.2 | 7.0 | 9.0 | 9.0 | 7.8 |
| -3.3 | -3.3 | -3.0 | -2.4 | -1.3 | -0.1 | 1.3 | 2.9 | 4.7 | 6.7 | 9.0 | 9.0 | 7.6 |
| -4.4 | -4.7 | -4.5 | -3.8 | -2.6 | -1.1 | 0.5 | 2.1 | 3.9 | 6.0 | 9.0 | 9.0 | 7.1 |
| -5.6 | -6.4 | -6.4 | -5.8 | -4.1 | -2.2 | -0.5 | 1.2 | 2.9 | 4.5 | 6.0 | 6.4 | 5.8 |
| -6.8 | -9.0 | -9.0 | -9.0 | -5.6 | -3.3 | -1.3 | 0.3 | 1.8 | 3.1 | 4.2 | 4.6 | 4.6 |
| -7.1 | -9.0 | -9.0 | -9.0 | -6.1 | -3.8 | -2.0 | -0.4 | 0.9 | 2.0 | 2.9 | 3.4 | 3.5 |
| -7.0 | -9.0 | -9.0 | -9.0 | -6.1 | -4.0 | -2.3 | -0.9 | 0.2 | 1.2 | 2.0 | 2.4 | 2.7 |
| -6.1 | -6.9 | -7.0 | -6.6 | -5.2 | -3.7 | -2.4 | -1.2 | -0.2 | 0.6 | 1.3 | 1.8 | 2.1 |

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# Enclosed Problems

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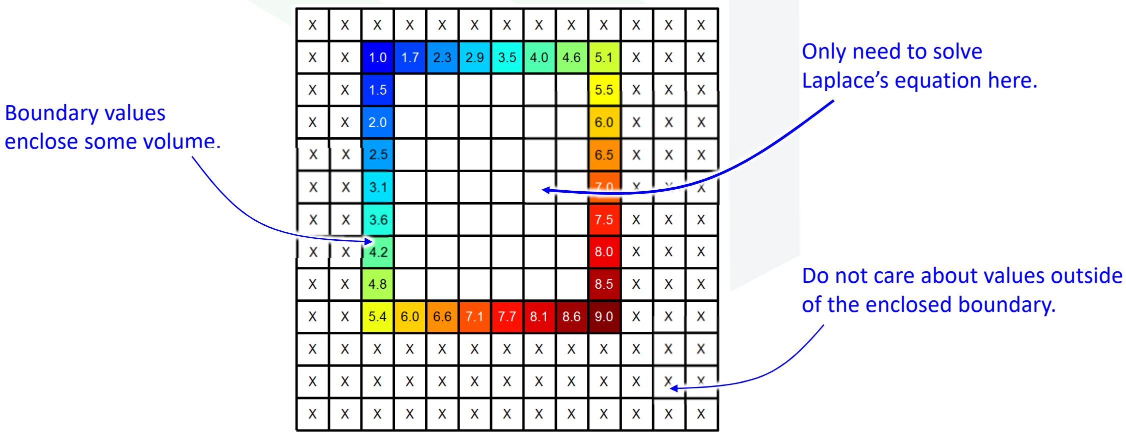
# Enclosed Problems

Sometimes a solution is only needed that is perfectly enclosed by the boundary values.

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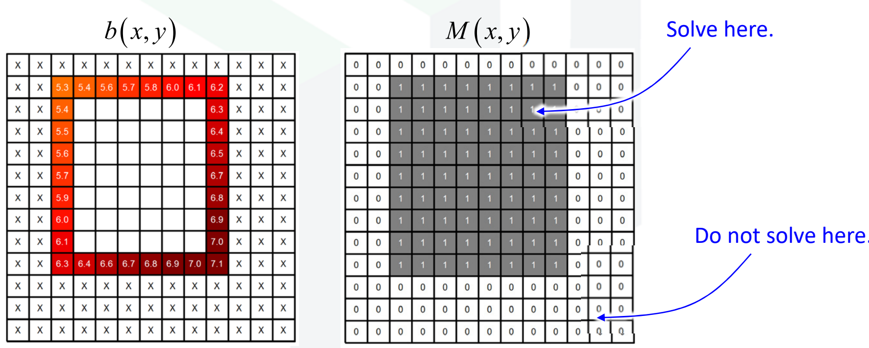
# Map Function $M(x,y)$

Step 1 – Make a map $M(x,y)$ of where Laplace's equation is to be solved and include the locations of the boundary.

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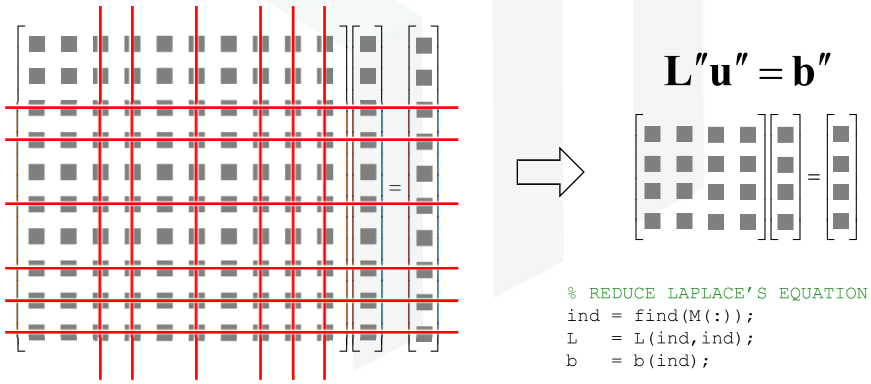
# Reducing the Matrix Equation

Step 2 – Eliminate the rows and columns in the matrix equation $\mathbf{Lu} = \mathbf{b}$ that correspond to points that do not have to be calculated.

$$\mathbf{L}'\mathbf{u} = \mathbf{b}'$$

$$\mathbf{L}''\mathbf{u}'' = \mathbf{b}''$$



```
% REDUCE LAPLACE'S EQUATION
ind = find(M(:));
L   = L(ind,ind);
b   = b(ind);
```

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# Solve and Expand Solution

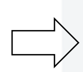
Step 3 – Solve Laplace's equation.

$$\mathbf{u}'' = \left(\mathbf{L}''\right)^{-1}\mathbf{b}''$$

```
% SOLVE LAPLACE'S EQUATION
u = L\b;
```

Step 4 – Insert solution back into grid.

```
% INSERT SOLUTION BACK INTO GRID
U = zeros(Nx,Ny);
U(ind) = u;
```



| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| X | X | X | X | X | X | X | X | X | X | X | X | X |
| X | X | 1.0 | 1.7 | 2.3 | 2.9 | 3.5 | 4.0 | 4.6 | 5.1 | X | X | X |
| X | X | 1.5 | | | | | | | 5.5 | X | X | X |
| X | X | 2.0 | | | | | | | 6.0 | X | X | X |
| X | X | 2.5 | | | | | | | 6.5 | X | X | X |
| X | X | 3.1 | | | | | | | 7.0 | X | X | X |
| X | X | 3.6 | | | | | | | 7.5 | X | X | X |
| X | X | 4.2 | | | | | | | 8.0 | X | X | X |
| X | X | 4.8 | | | | | | | 8.5 | X | X | X |
| X | X | 5.4 | 6.0 | 6.6 | 7.1 | 7.7 | 8.1 | 8.6 | 9.0 | X | X | X |
| X | X | X | X | X | X | X | X | X | X | X | X | X |
| X | X | X | X | X | X | X | X | X | X | X | X | X |
| X | X | X | X | X | X | X | X | X | X | X | X | X |

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| X | X | X | X | X | X | X | X | X | X | X | X | X |
| X | X | 1.0 | 1.7 | 2.3 | 2.9 | 3.5 | 4.0 | 4.6 | 5.1 | X | X | X |
| X | X | 1.5 | 2.1 | 2.8 | 3.4 | 3.9 | 4.5 | 5.0 | 5.5 | X | X | X |
| X | X | 2.0 | 2.6 | 3.2 | 3.8 | 4.4 | 5.0 | 5.5 | 6.0 | X | X | X |
| X | X | 2.5 | 3.1 | 3.8 | 4.3 | 4.9 | 5.4 | 6.0 | 6.5 | X | X | X |
| X | X | 3.1 | 3.7 | 4.3 | 4.9 | 5.4 | 6.0 | 6.5 | 7.0 | X | X | X |
| X | X | 3.6 | 4.2 | 4.8 | 5.4 | 5.9 | 6.5 | 7.0 | 7.5 | X | X | X |
| X | X | 4.2 | 4.8 | 5.4 | 6.0 | 6.5 | 7.0 | 7.5 | 8.0 | X | X | X |
| X | X | 4.8 | 5.4 | 6.0 | 6.5 | 7.1 | 7.6 | 8.0 | 8.5 | X | X | X |
| X | X | 5.4 | 6.0 | 6.6 | 7.1 | 7.7 | 8.1 | 8.6 | 9.0 | X | X | X |
| X | X | X | X | X | X | X | X | X | X | X | X | X |
| X | X | X | X | X | X | X | X | X | X | X | X | X |
| X | X | X | X | X | X | X | X | X | X | X | X | X |

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