Advanced Electromagnetics:
21st Century Electromagnetics

Numerical Solution of Laplace’s Equation

Lecture Outline
• Intuitive interpretation of Laplace’s equation $\nabla^2 u = 0$
• Numerical solution of Laplace’s equation $Lu = 0$
• Enclosed problems
Intuitive Interpretation of Laplace’s Equation

\[ \nabla^2 u = 0 \]

Meaning of Laplace’s Equation

Laplace’s equation is

\[ \nabla^2 u = 0 \]

- \( \nabla^2 \) is a 3D second-order derivative.
- A second-order derivative quantifies curvature.
- But the second-order derivative is set to zero.

Functions satisfying Laplace’s equation vary linearly.
Laplace’s Equation as a “Linear Number Filler Inner”

Given known values at certain points (e.g. physical boundary conditions), Laplace’s equation calculates the numbers everywhere else so they vary linearly.

Numerical Solution of Laplace’s Equation

\[
Lu = 0
\]
Numerical Solution of Laplace’s Equation (1 of 3)

Step 1 – Use the finite-difference method to express Laplace’s equation in matrix form.

\[ \nabla^2 u(x, y) = 0 \rightarrow \frac{\partial^2 u(x, y)}{\partial x^2} + \frac{\partial^2 u(x, y)}{\partial y^2} = 0 \rightarrow D_x^2 u + D_y^2 u = 0 \rightarrow Lu = 0 \]

\[ L = D_x^2 + D_y^2 \]

Step 2 – Build a function \( b(x, y) \) containing the boundary values.

The function \( b(x, y) \) contains the values of \( u(x, y) \) in the locations where \( u(x, y) \) is known. The other points in \( b(x, y) \) can be set to anything because they will be ignored, but zero is a convenient choice.

Numerical Solution of Laplace’s Equation (2 of 3)

Step 3 – Build a diagonal force matrix \( F \).

\[ F = \text{diag}\left[ f(x, y) \right] \]

The array \( f(x,y) \) contains 1’s at the positions where \( u(x, y) \) is to be forced. It contains 0’s everywhere else. It should be set to 1’s at the locations of the boundary values.

Step 4 – Incorporate the boundary values into Laplace’s equation.

\[ L' = F + (I - F)L \]

\[ b' = Fb \]

Note: both \( F \) and \( I \) should be stored as sparse matrices!
Numerical Solution of Laplace’s Equation (3 of 3)

Step 5 – Solve Laplace’s equation

\[ u = (L')^{-1} b' \]

The function \( u(x, y) \) has all of the forced values, but also contains all of the numbers in between.

Enclosed Problems
Enclosed Problems

Sometimes a solution is only needed that is perfectly enclosed by the boundary values.

Map Function $M(x, y)$

Step 1 – Make a map $M(x, y)$ of where Laplace’s equation is to be solved and include the locations of the boundary.
Reducing the Matrix Equation

Step 2 – Eliminate the rows and columns in the matrix equation \( Lu = b \) that correspond to points that do not have to be calculated.

\[
\begin{bmatrix}
L' & u = b'
\end{bmatrix}
\]

\[
\begin{bmatrix}
L'' & u'' = b''
\end{bmatrix}
\]

\[ \% \text{ REDUCE LAPLACE'S EQUATION} \]
\[ \text{ind} = \text{find}(M(:)); \]
\[ L = L(\text{ind}, \text{ind}); \]
\[ b = b(\text{ind}); \]

Solve and Expand Solution

Step 3 – Solve Laplace’s equation.

\[ u'' = (L'')^{-1} b'' \]

Step 4 – Insert solution back into grid.

\[ \% \text{ SOLVE LAPLACE'S EQUATION} \]
\[ u = L \backslash b; \]

\[ \% \text{ INSERT SOLUTION BACK INTO GRID} \]
\[ U = \text{zeros}(Nx, Ny); \]
\[ U(\text{ind}) = u; \]